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ABOUT AN INEQUALITY BY ELDENIZ HESENOV-II

By Marin Chirciu-Romania

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1) In $\Delta ABC, I$ –incenter,R_a, R_b, R_c –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$\sqrt[3]{\left(\frac{a}{R_a}\right)^2}+\sqrt[3]{\left(\frac{b}{R_b}\right)^2}+\sqrt[3]{\left(\frac{c}{R_c}\right)^2}\le 5$$

Proposed by Eldeniz Hesenov-Georgia

Solution by Marin Chirciu-Romania

Lemma 1. 2) In $\Delta ABC, I$ –incenter, R_a –circumradii of ΔBIC, then:

$$R_a=2R\sin\frac{A}{2}$$

Proof. Using identity $F=\frac{abc}{4R}$ in ΔBIC, we get:

$$R_a=\frac{IB\cdot IC\cdot BC}{4F_{\Delta IBC}}=\frac{IB\cdot IC\cdot a}{4\frac{IB\cdot IC\cdot \sin(BIC)}{2}}=\frac{a}{2\cos\frac{A}{2}}=\frac{2R\cdot \sin A}{2\cos\frac{A}{2}}=$$

$$=\frac{2R\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}}=2R\cdot\sin\frac{A}{2}$$

Lemma 2. 3) In $\Delta ABC, I$ –incenter, R_a –circumradii of ΔBIC, then:

$$\sum\sqrt[3]{\left(\frac{a}{R_a}\right)^2}=\sum\sqrt[3]{4\cos^2\frac{A}{2}}$$

Proof. Using identity: $R_a=2r\cdot\sin\frac{A}{2}$, we get:

$$\sqrt[3]{\left(\frac{a}{R_a}\right)^2}=\sqrt[3]{\left(\frac{2RsinA}{2Rsin\frac{A}{2}}\right)^2}=\sqrt[3]{\left(\frac{sinA}{sin\frac{A}{2}}\right)^2}=\sqrt[3]{\left(\frac{2sin\frac{A}{2}cos\frac{A}{2}}{sin\frac{A}{2}}\right)^2}=\sqrt[3]{4cos^2\frac{A}{2}}$$

$$=\sqrt[3]{4\frac{s(s-a)}{bc}}$$

Let's get back to the main problem.

Using Lemma and AM-GM, we get:

$$\sum\sqrt[3]{\left(\frac{a}{R_a}\right)^2}=\sum\sqrt[3]{4cos^2\frac{A}{2}}=\frac{1}{\sqrt[3]{9}}\sum\sqrt[3]{4cos^2\frac{A}{2}\cdot 3\cdot 3}\le$$

$$\le\frac{1}{\sqrt[3]{9}}\sum\frac{4cos^2\frac{A}{2}+3+3}{3}=\frac{2}{3\sqrt[3]{9}}\sum\left(2cos^2\frac{A}{2}+3\right)=$$

$$=\frac{2}{3\sqrt[3]{9}}\left(2\sum cos^2\frac{A}{2}+9\right)=\frac{2}{3\sqrt[3]{9}}\left(2\frac{4R+r}{2R}+9\right)=\frac{2}{3\sqrt[3]{9}}\left(13+\frac{r}{R}\right)$$

From $R\ge 2r(Euler)$, we have:

$$\frac{2}{3\sqrt[3]{9}}\left(13+\frac{r}{R}\right)\ge\frac{2}{3\sqrt[3]{9}}\left(13+\frac{1}{2}\right)=\frac{2}{3\sqrt[3]{9}}\frac{27}{2}=3\sqrt[3]{3}<5$$

Remark. Let's find reverse inequality:

4) In $\Delta ABC, I$ –incenter,R_a, R_b, R_c –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$\sqrt[3]{\left(\frac{a}{R_a}\right)^2}+\sqrt[3]{\left(\frac{b}{R_b}\right)^2}+\sqrt[3]{\left(\frac{c}{R_c}\right)^2}\ge 9\sqrt[3]{\left(\frac{2r}{R}\right)^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Lemma and AM-GM inequality, we get:

$$\sum\sqrt[3]{\left(\frac{a}{R_a}\right)^2}=\sum\sqrt[3]{4cos^2\frac{A}{2}}\ge 3\sqrt[3]{64\prod cos^2\frac{A}{2}}=12\sqrt[3]{\prod cos^2\frac{A}{2}}=$$

$$=12\sqrt[3]{\left(\frac{s}{4R}\right)^2}\overset{\substack{Mitrinovic\\ s^2\ge 27r^2}}{\ge}12\sqrt[3]{\frac{27r^2}{16R^2}}=9\sqrt[3]{\left(\frac{2r}{R}\right)^2}.$$

Equality holds if and only if triangle is equilateral.

5) In $\Delta ABC, I$ −incenter,R_a, R_b, R_c −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$9\sqrt[3]{\left(\frac{2r}{R}\right)^2}\le\sqrt[3]{\left(\frac{a}{R_a}\right)^2}+\sqrt[3]{\left(\frac{b}{R_b}\right)^2}+\sqrt[3]{\left(\frac{c}{R_c}\right)^2}\le\frac{2}{3\sqrt[3]{9}}\left(13+\frac{r}{R}\right)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

For RHS, using Lemma and AM-GM inequality, we have:

$$\sum\sqrt[3]{\left(\frac{a}{R_a}\right)^2}=\sum\sqrt[3]{4cos^2\frac{A}{2}}=\frac{1}{\sqrt[3]{9}}\sum\sqrt[3]{4cos^2\frac{A}{2}\cdot 3\cdot 3}\le$$

$$\le\frac{1}{\sqrt[3]{9}}\sum\frac{4cos^2\frac{A}{2}+3+3}{3}=\frac{2}{3\sqrt[3]{9}}\sum\left(2cos^2\frac{A}{2}+3\right)=$$

$$=\frac{2}{3\sqrt[3]{9}}\left(2\sum cos^2\frac{A}{2}+9\right)=\frac{2}{3\sqrt[3]{9}}\left(2\frac{4R+r}{2R}+9\right)=\frac{2}{3\sqrt[3]{9}}\left(13+\frac{r}{R}\right)$$

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and AM-GM, we have:

$$\sum\sqrt[3]{\left(\frac{a}{R_a}\right)^2}=\sum\sqrt[3]{4cos^2\frac{A}{2}}\ge 3\sqrt[3]{64\prod cos^2\frac{A}{2}}=12\sqrt[3]{\prod cos^2\frac{A}{2}}=$$

$$=12\sqrt[3]{\left(\frac{s}{4R}\right)^2}\overset{\substack{Mitrinovic\\ s^2\ge 27r^2}}{\ge}12\sqrt[3]{\frac{27r^2}{16R^2}}=9\sqrt[3]{\left(\frac{2r}{R}\right)^2}.$$

Equality holds if and only if triangle is equilateral.

Remark. In same class of problems.

5) In $\Delta ABC, I$ −incenter,R_a, R_b, R_c −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$3\sqrt[3]{\frac{s}{4R}} \le \frac{a}{R_a} + \frac{b}{R_b} + \frac{c}{R_c} \le \frac{3\sqrt{3}}{2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma 1. 6) In $\Delta ABC, I$ —incenter, R_a, R_b, R_c —circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$R_a = 2r\sin\frac{A}{2}$$

Proof. Using identity $F = \frac{abc}{4R}$ in ΔIBC, we have:

$$R_a = \frac{IB \cdot IC \cdot BC}{4F_{\Delta IBC}} = \frac{IB \cdot IC \cdot a}{4\frac{IB \cdot IC \cdot \sin(BIC)}{2}} = \frac{a}{2\cos\frac{A}{2}} = \frac{2R \cdot \sin A}{2\cos\frac{A}{2}} =$$

$$= \frac{2R \cdot 2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}} = 2R \cdot \sin\frac{A}{2}$$

Lemma2. 7) In $\Delta ABC, I$ —incenter, R_a, R_b, R_c —circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$\frac{a}{R_a} + \frac{b}{R_b} + \frac{c}{R_c} = 2\sum \cos\frac{A}{2}$$

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Proof. Using $a = 2R\sin A, R_a = 2r\sin\frac{A}{2}$, we have:

$$\sum \frac{a}{R_a} = \sum \frac{2R\sin A}{2R\sin\frac{A}{2}} = \sum \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}} = 2\sum \cos\frac{A}{2}.$$

Let's get back to the main problem.

For RHS, using Lemma and Jensen inequality, we have:

$$\sum \cos\frac{A}{2} \le \frac{3\sqrt{3}}{2} \Rightarrow \sum \frac{a}{R_a} = \sum \cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$$

For LHS, using Lemma and Jensen inequality, we have:

$$\sum\frac{a}{R_a}=\sum cos\frac{A}{2}\geq 3\sqrt[3]{\prod cos\frac{A}{2}}=3\sqrt[3]{\frac{s}{4R}}$$

Equality holds if and only if triangle is equilateral.

8) In $\Delta ABC, I$ –incenter,R_a, R_b, R_c –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$\frac{18r}{R}\leq\left(\frac{a}{R_a}\right)^2+\left(\frac{b}{R_b}\right)^2+\left(\frac{c}{R_c}\right)^2\leq 9$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma 1. 9) In $\Delta ABC, I$ –incenter,R_a, R_b, R_c –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$R_a=2rsin\frac{A}{2}$$

Proof. Using identity $F=\frac{abc}{4R}$ in ΔIBC, we have:

$$R_a=\frac{IB\cdot IC\cdot BC}{4F_{\Delta IBC}}=\frac{IB\cdot IC\cdot a}{4\frac{IB\cdot IC\cdot sin(BIC)}{2}}=\frac{a}{2cos\frac{A}{2}}=\frac{2R\cdot sinA}{2cos\frac{A}{2}}=$$

$$=\frac{2R\cdot 2sin\frac{A}{2}cos\frac{A}{2}}{2cos\frac{A}{2}}=2R\cdot sin\frac{A}{2}$$

Lemma 2. 10)) In $\Delta ABC, I$ –incenter,R_a, R_b, R_c –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$ the following relationship holds:

$$\left(\frac{a}{R_a}\right)^2+\left(\frac{b}{R_b}\right)^2+\left(\frac{c}{R_c}\right)^2=8+\frac{2r}{R}$$

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Proof. Using $a=2rsinA, R_a=2Rsin\frac{A}{2}$, we get:

$$\sum\left(\frac{a}{R_a}\right)^2=\sum\left(\frac{2RsinA}{2Rsin\frac{A}{2}}\right)^2=\sum\left(\frac{sinA}{sin\frac{A}{2}}\right)^2=\sum\left(\frac{2sin\frac{A}{2}cos\frac{A}{2}}{sin\frac{A}{2}}\right)^2=$$

$$=4\sum cos^2\frac{A}{2}=4\left(2+\frac{r}{2R}\right)=4\left(2+\frac{r}{2R}\right)=8+\frac{2r}{R}.$$

Let's get back to the main problem.

For RHS, using Lemma and Euler inequality, we have:

$$\sum\left(\frac{a}{R_a}\right)^2=8+\frac{2r}{R}\overset{Euler}{\leq}8+1=9$$

For LHS, using Lemma and Euler inequality, we get:

$$\sum\left(\frac{a}{R_a}\right)^2=\frac{2(4R+r)}{R}\overset{Euler}{\geq}\frac{2(8r+r)}{R}=\frac{18r}{R}$$

Equality holds if and only if triangle is equilateral.

REFERENCE:

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