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ABOUT AN INEQUALITY BY ELDENIZ HESENOV-III

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1) In ΔABC the following relationship holds:

$$\frac{AH^2}{b^2+c^2}+\frac{BH^2}{c^2+a^2}+\frac{CH^2}{b^2+c^2}\le\frac{R}{r}+\frac{2r}{R}-\frac{5}{2}$$

Proposed by Eldeniz Hesenov-Georgia

Solution

Lemma 1. 2) In ΔABC the following relationship holds:

$$AH^2=4R^2-a^2$$

Proof. Using $AH=2r|cosA|$, we get

$$AH^2=4R^2cos^2A=4R^2(1-sin^2A)=4R^2\left(1-\frac{a^2}{4R^2}\right)=4R^2-a^2$$

Lemma 2. 3) In ΔABC the following relationship holds:

$$\sum\frac{4R^2-a^2}{bc}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr}$$

Proof.

$$\sum\frac{4R^2-a^2}{bc}=\frac{\sum a(4R^2-a^2)}{abc}=\frac{4R^2\sum a-\sum a^3}{abc}=\frac{4R^2\cdot 2s-2s(s^2-3r^2-6Rr)}{4Rrs}=$$

$$=\frac{4R^2-(s^2-3r^2-6Rr)}{2Rr}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr};\ \sum a^3=2s(s^2-3r^2-6Rr)$$

Let's get back to the main problem.

Using AM-GM: $b^2+c^2\ge 2bc$, we get:

$$LHS=\sum\frac{AH^2}{b^2+c^2}\le\frac{1}{2}\sum\frac{AH^2}{bc}\overset{Lemma\ 1.}{=}\frac{1}{2}\sum\frac{4R^2-a^2}{bc}\overset{Lemma\ 2.}{=}$$

$$=\frac{1}{2}\frac{4R^2+6Rr+3r^2-s^2}{2Rr}=\frac{4R^2+6Rr+3r^2-s^2}{4Rr}\overset{Gerretsen}{\le}$$

$$\le\frac{4R^2+6Rr+3r^2-(16Rr-5r^2)}{4Rr}=\frac{4R^2-10Rr+8r^2}{4Rr}=$$

$$=\frac{2R^2-5Rr+4r^2}{2Rr}=\frac{R}{r}+\frac{2r}{R}-\frac{5}{2}=RHS.$$

4) In ΔABC the following relationship holds:

$$\sum\frac{AH^2}{bc}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr}$$

Proof. $\sum\frac{AH^2}{bc}\overset{Lemma\ 1.}{=}\sum\frac{4R^2-a^2}{bc}=\frac{\sum a(4R^2-a^2)}{abc}=\frac{4R^2\sum a-\sum a^3}{abc}=\frac{4R^2\cdot 2s-2s(s^2-3r^2-6Rr)}{4Rrs}=$

$$=\frac{4R^2-(s^2-3r^2-6Rr)}{2Rr}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr};\ \sum a^3=2s(s^2-3r^2-6Rr)$$

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$1\le\sum\frac{AH^2}{bc}\le\frac{2R}{r}+\frac{4r}{R}-5$$

Proposed by Marin Chirciu-Romania

Solution

Lemma1. 6) In ΔABC the following relationship holds:

$$AH^2=4R^2-a^2$$

Proof. Using $AH=2r|cosA|$, we get

$$AH^2=4R^2cos^2A=4R^2(1-sin^2A)=4R^2\left(1-\frac{a^2}{4R^2}\right)=4R^2-a^2$$

Lemma 2. 3) In ΔABC the following relationship holds:

$$\sum\frac{AH^2}{bc}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr}$$

Proof.

$$\sum\frac{AH^2}{bc}=\frac{\sum a(4R^2-a^2)}{abc}=\frac{4R^2\sum a-\sum a^3}{abc}=\frac{4R^2\cdot 2s-2s(s^2-3r^2-6Rr)}{4Rrs}=$$

$$=\frac{4R^2-(s^2-3r^2-6Rr)}{2Rr}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr};\ \sum a^3=2s(s^2-3r^2-6Rr)$$

Let's get back to the main problem. For RHS, we have:

$$E=\sum\frac{AH^2}{bc}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr}\overset{Gerretsen}{\leq}\frac{4R^2+6Rr+3r^2-(16Rr-5r^2)}{2Rr}=$$

$$=\frac{4R^2-10Rr+8r^2}{2Rr}=\frac{2R^2-5Rr+4r^2}{Rr}=\frac{2R}{r}+\frac{4r}{R}-5=RHS$$

Fro LHS, we have: $E=\sum\frac{AH^2}{bc}=\frac{4R^2+6Rr+3r^2-s^2}{2Rr}\overset{Gerretsen}{\geq}\frac{4R^2+6Rr+3r^2-(4R^2+4Rr+3r^2)}{2Rr}=\frac{2Rr}{2Rr}=$ $1=LHS$

Equality holds if and only if triangle is equilateral.

8) In ΔABC the following relationship holds:

$$\frac{3}{2R(4R+r)}\leq\sum\left(\frac{AH}{bc}\right)^2\leq\frac{R}{24r^3}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma 1. 9) In ΔABC the following relationship holds:**

$$AH^2=4R^2-a^2$$

Proof. Using $AH=2r|cosA|$, we get

$$AH^2=4R^2cos^2A=4R^2(1-sin^2A)=4R^2\left(1-\frac{a^2}{4R^2}\right)=4R^2-a^2$$

Lemma 2. 10) In ΔABC the following relationship holds:

$$\sum\left(\frac{AH}{bc}\right)^2=\frac{s^2(4R^2+8Rr+6r^2-s^2)-r(4R+r)(2R+r)^2}{8R^2r^2s^2}$$

Proof.

$$\sum\left(\frac{AH}{bc}\right)^2=\sum\frac{4R^2-a^2}{b^2c^2}=\frac{\sum a^2(4R^2-a^2)}{a^2b^2c^2}=\frac{4R^2\sum a^2-\sum a^4}{(abc)^2}=$$

$$=\frac{4R^2\cdot 2(s^2-r^2-4Rr)-2[s^4-s^2(8Rr+6R^2)+r^2(4R+r)^2]}{(4Rrs)^2}=$$

$$=\frac{4R^2\cdot(s^2-r^2-4Rr)-s^4+s^2(8Rr+6r^2)-r^2(4R+r)^2}{8R^2r^2s^2}=$$

$$=\frac{s^2(4R^2+|8Rr+6r^2-s^2)-r(4R+r)(2R+r)^2}{8R^2r^2s^2}$$

$$\therefore \sum a^2 = 2(s^2 - r^2 - 4Rr; \sum a^4 = 2[s^4 - s^2(8Rr + 6r^2) + r^2(4R + r)^2].$$

Let's get back to the main problem. For RHS, we get:

$$E = \sum \left(\frac{AH}{bc}\right)^2 = \frac{s^2(4R^2 + 8Rr + 6r^2 - s^2) - r(4R + r)(2R + r)^2}{8R^2r^2s^2} =$$

$$= \frac{s^2(4R^2 + 8Rr + 6r^2 - s^2)}{8R^2r^2s^2} - \frac{r(4R + r)(2R + r)^2}{8R^2r^2s^2} \overset{Gerretsen}{\leq}$$

$$\leq \frac{4R^2 + 8Rr + 6r^2 - (16Rr - 5r^2)}{8R^2r^2} - \frac{r(4R + r)(2R + r)^2}{8R^2r^2\frac{R(4R + r)^2}{2(2R - r)}} =$$

$$= \frac{4R^2 - 8Rr + 11r^2}{8R^2r^2} - \frac{r(2R - r)(2R + r)^2}{4R^3r^2(4R + r)} =$$

$$= \frac{1}{4R^2r^2}\left(\frac{4R^2 - 8Rr + 11r^2}{2} - \frac{r(2R - r)(2R + r)^2}{R(4R + r)}\right) =$$

$$= \frac{1}{4R^2r^2}\left(\frac{R(4R + r)(4R^2 - 8Rr + 11r^2) - 2r(2R - r)(2R + r)^2}{2R(4R + r)}\right) =$$

$$= \frac{16R^4 - 44R^3r + 28R^2r^2 + 15Rr^3 + 2r^4}{8R^3r^3(4R + r)} \leq \frac{R}{24r^3} \Leftrightarrow$$

$$3r(16R^4 - 44R^3r + 28R^2r^2 + 15Rr^3 + 2r^4) \leq 2R^4(4R + r) \Leftrightarrow$$

$$4R^5 - 47R^4r + 132R^3r^2 - 84R^2r^3 - 45Rr^4 - 6r^5 \geq 0 \Leftrightarrow$$

$(R - 2r)(4R^4 - 39R^3r + 54R^2r^2 + 24Rr^3 + 3r^4) \geq 0$ **true from** $R \geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For LHS, we get: $E = \sum \left(\frac{AH}{bc}\right)^2 = \frac{s^2(4R^2+8Rr+6r^2-s^2)-r(4R+r)(2R+r)^2}{8R^2r^2s^2} =$

$$\frac{s^2(4R^2 + 8Rr + 6r^2 - s^2)}{8R^2r^2s^2} - \frac{r(4R + r)(2R + r)^2}{8R^2r^2s^2} \overset{Gerretsen}{\geq}$$

$$\geq \frac{4R^2 + 8Rr + 6r^2 - (4R^2 + 4Rr + 3r^2)}{8R^2r^2} - \frac{r(4R + r)(2R + r)^2}{8R^2r^2\frac{r(4R + r)^2}{R + r}} =$$

$$= \frac{4Rr + 3r^2}{8R^2r^2} - \frac{(R + r)(2R + r)^2}{8R^2r^2} = \frac{(4R + r)(4Rr + 3r^2) - (R + r)(2R + r)^2}{8R^2r^2(4R + r)} =$$

$$= \frac{04R^3 + 8R^2r + 11Rr^2 + 2r^3}{8R^2r^2(4R + r)} \overset{Euler}{\leq} \frac{11Rr^2 + 2r^3}{8R^2r^2(4R + r)} = \frac{11R + 2r}{8R^2(4R + r)} \overset{Euler}{\leq}$$

$$\le\frac{11R+R}{8R^2(4R+r)}=\frac{12R}{8R^2(4R+r)}=\frac{3}{2R(4R+r)}$$

Equality holds if and only if triangle is equilateral.

11) In ΔABC the following relationship holds:

$$1\le\sum\left(\frac{AH}{a}\right)^2\le\frac{R^2}{r^2}-\frac{3R}{4r}-\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution

Lemma 1. 12) In ΔABC the following relationship holds:

$$AH^2=4R^2-a^2$$

Proof. Using $AH=2r|cosA|$, we get

$$AH^2=4R^2cos^2A=4R^2(1-sin^2A)=4R^2\left(1-\frac{a^2}{4R^2}\right)=4R^2-a^2$$

Lemma 2. 13) In ΔABC the following relationship holds:

$$\sum\left(\frac{AH}{a}\right)^2=\frac{s^2(s^2-10r^2-8Rr)+r^2(4R+r)^2}{4r^2s^2}$$

Proof. $\sum\left(\frac{AH}{a}\right)^2=\sum\frac{4R^2-a^2}{a^2}=\frac{\sum b^2c^2(4R^2-a^2)}{a^2b^2c^2}=\frac{4R^2\sum b^2c^2-3a^2b^2c^2}{(abc)^2}=$

$$=\frac{4R^2[s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2]-3\cdot16R^2r^2s^2}{(4Rrs)^2}=$$

$$=\frac{s^2(s^2-10r^2-8Rr)+r^2(4R+r)^2}{4r^2s^2};\ \sum a^3=2s(s^2-3r^2-6Rr).$$

Let's get back to the main problem. For RHS, we have:

$$E=\sum\left(\frac{AH}{a}\right)^2=\frac{s^2(s^2-10r^2-8Rr)+r^2(4R+r)^2}{4r^2s^2}=$$

$$=\frac{s^2-10r^2-8Rr}{4r^2}=\frac{(4R+r)^2}{4s^2}\overset{Gerretsen}{\le}$$

$$\le\frac{4R^2+4Rr+3r^2-10r^2-8Rr}{4r^2}+\frac{(4R+r)^2}{4\frac{r(4R+r)^2}{R+r}}=$$

$$=\frac{4R^2-4Rr-7r^2}{4r^2}+\frac{R+r}{4r}=\frac{4R^2-3Rr-6r^2}{4r^2}=\frac{R^2}{r^2}-\frac{3R}{4r}-\frac{3}{2}=RHS.$$

Fro LHS, we have: $E=\sum\left(\frac{AH}{a}\right)^2=\frac{s^2(s^2-10r^2-8Rr)+r^2(4R+r)^2}{4r^2s^2}=\frac{s^2-10r^2-8Rr}{4r^2}=\frac{(4R+r)^2}{4s^2}\geq$

$$\overset{Gerretsen}{\geq}\frac{16Rr-5r^2-10r^2-8Rr}{4r^2}+\frac{(4R+r)^2}{4\frac{R(4R+r)^2}{2(2R-r)}}=$$

$$=\frac{8Rr-15r^2}{4r^2}+\frac{2R-r}{2R}=\frac{8R-15r}{4r}+\frac{2R-r}{2R}=\frac{R(8R-15r)+2(2R-r)}{4Rr}=$$

$$=\frac{8R^2-11Rr-2r^2}{4Rr}\geq 1=RHS.$$

Equality holds if and only if triangle is equilateral.

References:

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