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ROMANIAN MATHEMATICAL MAGAZINE

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ABOUT AN INEQUALITY BY ERTAN YILDIRIM-IX

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1)In ΔABC the following relationship holds:

$$(1+m_a)(1+m_b)(1+m_c)\ge(1+3r)^3$$

Proposed by Ertan Yildirim-Turkey

Solution.

Using Huygens Inequality: $(1+x)(1+y)(1+z)\ge\left(1+\sqrt[3]{xyz}\right)^3$ **with**

$x=m_a, y=m_b, z=m_c$, **we get:**

$$(1+m_a)(1+m_b)(1+m_c)\ge\left(1+\sqrt[3]{m_am_bm_c}\right)^3\overset{(1)}{\ge}(1+3r)^3$$

$(1)\Leftrightarrow m_am_bm_c\ge 27r^3$, **which follows from** $m_a\ge\sqrt{s(s-a)}$ **and**

$s\ge 3\sqrt{3}r(Mitrinovic)$.

$$m_am_bm_c\ge\sqrt{s(s-a)}\sqrt{s(s-b)}\sqrt{s(s-c)}=sF=s\cdot rs=rs^2\ge r\cdot 27r^2=27r^3$$

Equality holds if and only if triangle is equilateral.

Remark. Inequality can be developed.

2) In ΔABC the following relationship holds:

$$(\lambda+m_a)(\lambda+m_b)(\lambda+m_c)\ge(\lambda+3r)^3, \lambda\ge 0$$

Proposed by Marin Chirciu-Romania

Solution. **Using Holder Inequality**

$$(a+x)(b+y)(c+z)\ge\left(\sqrt[3]{abc}+\sqrt[3]{xyz}\right)^3, \forall a,b,c,x,y,z\ge 0,$$

for $a=b=c=\lambda, x=m_a, y=m_b, z=m_c$, **we have:**

$$(\lambda+m_a)(\lambda+m_b)(\lambda+m_c)\ge\left(\sqrt[3]{\lambda^3}+\sqrt[3]{m_am_bm_c}\right)^3\overset{(1)}{\ge}(\lambda+3r)^3$$

$(1)\Leftrightarrow m_am_bm_c\ge 27r^3$, **which follows from** $m_a\ge\sqrt{s(s-a)}$ **and**

$s\ge 3\sqrt{3}r(Mitrinovic)$.

$$m_am_bm_c \geq \sqrt{s(s-a)}\sqrt{s(s-b)}\sqrt{s(s-c)} = sF = s\cdot rs = rs^2 \geq r\cdot 27r^2 = 27r^3$$

Equality holds if and only if triangle is equilateral.

Note.

For $\lambda = 0$, it follows well-known inequality: $m_am_bm_c \geq 27r^3$.

For $\lambda = 1$, it follows problem proposed by Ertan Yildirim in RMM 8/2019.

Remark. If replace m_a with r_a, we get:

3) In ΔABC the following relationship holds:

$$(\lambda + r_a)(\lambda + r_b)(\lambda + r_c) \geq (\lambda + 3r)^3, \lambda \geq 0$$

Proposed by Marin Chirciu-Romania

Solution. **Using Holder Inequality**

$$(a+x)(b+y)(c+z) \geq \left(\sqrt[3]{abc} + \sqrt[3]{xyz}\right)^3, \forall a,b,c,x,y,z \geq 0,$$

$$a = b = c = \lambda, x = r_a, y = r_b, z = r_c, \text{ we get:}$$

$$(\lambda + r_a)(\lambda + r_b)(\lambda + r_c) \geq \left(\sqrt[3]{\lambda^3} + \sqrt[3]{r_ar_br_c}\right)^3 \overset{(1)}{\geq} (\lambda + 3r)^3$$

$(1) \Leftrightarrow r_ar_br_c \geq 27r^3$, **which follows from** $r_a = \frac{F}{s-a}$ **and** $s \geq 3\sqrt{3}r(Mitrinovic)$.

$$r_ar_br_c = \frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c} = \frac{F^3}{(s-a)(s-b)(s-c)} = \frac{r^3s^3}{r^2s} = rs^2 \geq r\cdot 27r^2 = 27r^3$$

Equality holds if and only if triangle is equilateral.

Note.

For $\lambda = 0$, it follows well-known inequality $r_ar_br_c \geq 27r^3$.

For $\lambda = 1$, it follows $(1 + r_a)(1 + r_b)(1 + r_c) \geq (1 + 3r)^3$.

4) In ΔABC the following relationship holds:

$$(\lambda + h_a)(\lambda + h_b)(\lambda + h_c) \geq \left(\lambda + 3r\cdot\sqrt[3]{\frac{2r}{R}}\right)^3, \lambda \geq 0$$

Proposed by Marin Chirciu-Romania

Solution. **Using Holder Inequality**

$$(a+x)(b+y)(c+z) \ge \left(\sqrt[3]{abc}+\sqrt[3]{xyz}\right)^3, \forall a,b,c,x,y,z \ge 0,$$

$a=b=c=\lambda, x=h_a, y=h_b, z=h_c$, we get:

$$(\lambda+h_a)(\lambda+h_b)(\lambda+h_c) \ge \left(\sqrt[3]{\lambda^3}+\sqrt[3]{h_ah_bh_c}\right)^3 \overset{(1)}{\ge} \left(\lambda+3r\cdot\sqrt[3]{\frac{2r}{R}}\right)^3$$

$(1) \Leftrightarrow h_ah_bh_c \ge 27r^3\cdot\frac{2r}{R}$, which follows from $h_a=\frac{2F}{a}$ and $s \ge 3\sqrt{3}r(Mitrinovic)$.

$$h_ah_bh_c = \frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c} = \frac{8F^3}{abc} = \frac{8r^3s^3}{4Rrs} = \frac{2r^2s^2}{4Rrs} = \frac{2r^2s^2}{R} \ge \frac{2r^2\cdot 27r^2}{R} = \frac{2r\cdot 27r^3}{R}$$

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$(\lambda+w_a)(\lambda+w_b)(\lambda+w_c) \ge \left(\lambda+3r\cdot\sqrt[3]{\frac{2r}{R}}\right)^3, \lambda \ge 0$$

Proposed by Marin Chirciu-Romania

Solution. Using Holder Inequality

$$(a+x)(b+y)(c+z) \ge \left(\sqrt[3]{abc}+\sqrt[3]{xyz}\right)^3, \forall a,b,c,x,y,z \ge 0,$$

$a=b=c=\lambda, x=w_a, y=w_b, z=w_c$, we get:

$$(\lambda+w_a)(\lambda+w_b)(\lambda+w_c) \ge \left(\sqrt[3]{\lambda^3}+\sqrt[3]{w_aw_bw_c}\right)^3 \overset{(1)}{\ge} \left(\lambda+3r\cdot\sqrt[3]{\frac{2r}{R}}\right)^3$$

$(1) \Leftrightarrow w_aw_bw_c \ge 27r^3\cdot\frac{2r}{R}$, which follows from $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$ and

$s \ge 3\sqrt{3}r(Mitrinovic)$.

$$w_aw_bw_c = \frac{2bc}{b+c}\cos\frac{A}{2}\cdot\frac{2ca}{c+a}\cos\frac{B}{2}\cdot\frac{2ab}{a+b}\cos\frac{C}{2} = \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}\prod_{cyc}\cos\frac{A}{2} =$$

$$= \frac{8\cdot 16R^2r^2s^2}{2s(s^2+r^2+2Rr)}\cdot\frac{s}{4R} = \frac{16Rr^2s^2}{s^2+r^2+2Rr} \ge \frac{16Rr^2\cdot 27r^2}{s^2+r^2+2Rr} \overset{(1)}{\ge} 27r^3\cdot\frac{2r}{R},$$

$(1) \Leftrightarrow \frac{16Rr}{s^2+r^2+2Rr} \ge \frac{2r}{R} \Leftrightarrow s^2 \le 8R^2-2Rr-r^2$, which follows from

$s^2 \le 4R^2+4Rr+3r^2(Gerretsen)$.

Remains to prove that: $4R^2+4Rr+3r^2\le 8R^2-2Rr-r^2 \Leftrightarrow$

$2R^2-3Rr-2r^2\ge 0 \Leftrightarrow (R-2r)(2R+r)\ge 0$**, which is obviously true from** $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

References:

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