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ABOUT AN INEQUALITY BY ERTAN YILDIRIM-V

By Marin Chirciu-Romania

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1) In ΔABC the following relationship holds:

$$\sum \frac{m_b^2+m_c^2}{h_a} \geq 9R$$

Proposed by Ertan Yildirim-Turkey

Solution by Marin Chirciu-Romania

Lemma. 2) In ΔABC the following relationship holds:

$$\sum \frac{m_b^2+m_c^2}{h_a} = \frac{5s^2-11r^2-26Rr}{4r}$$

Proof. Using identities $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$, $h_a = \frac{2F}{a}$, we get:

$$\sum \frac{m_b^2+m_c^2}{h_a} = \sum \frac{\frac{2a^2+2c^2-b^2}{4}+\frac{2a^2+2b^2-c^2}{4}}{\frac{2F}{a}} =$$

$$= \frac{1}{8F}\sum a(4a^2+b^2+c^2) = \frac{4\sum a^2 + \sum a(b^2+c^2)}{8rs} =$$

$$= \frac{4\cdot 2s(s^2-3r^2-6Rr)+2s(s^2+r^2-2Rr)}{8rs} = \frac{5s^2-11r^2-26Rr}{4r}$$

Let's get back to the main problem.

Using Lemma and $s^2 \geq 16Rr-5r^2 (Gerretsen)$, it follows that:

$$LHS = \sum \frac{m_b^2+m_c^2}{h_a} = \frac{5s^2-11r^2-26Rr}{4r} \geq \frac{5(16Rr-5r^2)-11r^2-26Rr}{4r} =$$

$$= \frac{80Rr-25r^2-11r^2-26Rr}{4r} = \frac{54Rr-36r^2}{4r} = \frac{2r(27R-18r)}{4r} =$$

$$= \frac{27R-18r}{2} \overset{Euler}{\geq} 9R = RHS$$

Equality holds if and only if triangle is equilateral.

Remark. Let's find an reverse inequality.

3) In ΔABC the following relationship holds:

$$\sum\frac{m_b^2+m_c^2}{h_a}\le\frac{9R^3}{4r^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. 4) In ΔABC the following relationship holds:

$$\sum\frac{m_b^2+m_c^2}{h_a}=\frac{5s^2-11r^2-26Rr}{4r}$$

Proof. Using identities $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, $h_a=\frac{2F}{a}$, we get:

$$\sum\frac{m_b^2+m_c^2}{h_a}=\sum\frac{\frac{2a^2+2c^2-b^2}{4}+\frac{2a^2+2b^2-c^2}{4}}{\frac{2F}{a}}=$$

$$=\frac{1}{8F}\sum a(4a^2+b^2+c^2)=\frac{4\sum a^2+\sum a(b^2+c^2)}{8rs}=$$

$$=\frac{4\cdot 2s(s^2-3r^2-6Rr)+2s(s^2+r^2-2Rr)}{8rs}=\frac{5s^2-11r^2-26Rr}{4r}$$

Let's get back to the main problem.

Using Lemma and $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$, it follows that:

$$LHS=\sum\frac{m_b^2+m_c^2}{h_a}=\frac{5s^2-11r^2-26Rr}{4r}\le\frac{5(4R^2+4Rr+3r^2)-11r^2-26Rr}{4r}=$$

$$=\frac{20R^2-6Rr+4r^2}{4r}=\frac{10R^2-3Rr+2r^2}{2r}\overset{(1)}{\le}\frac{9R^3}{4r^2}=RHS$$

$$(1)\Leftrightarrow\frac{10R^2-3Rr+2r^2}{2r}\le\frac{9R^3}{4r^2}\Leftrightarrow 9R^3-20R^2r+6Rr^2-4r^3\ge 0\Leftrightarrow$$

$(R-2r)(9R^2-2Rr+2r^2)\ge 0$, true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$9R\le\sum\frac{m_b^2+m_c^2}{h_a}\le\frac{9R^3}{4r^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer **See inequalities 1) and 3).Equality holds if and only if triangle is equilateral. Let's replacing h_a with r_a.**

6) In ΔABC he following relationship holds:

$$9R \le \sum \frac{m_b^2 + m_c^2}{r_a} \le \frac{9R^2}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. 7) In ΔABC the following relationship holds:

$$\sum \frac{m_b^2 + m_c^2}{r_a} = \frac{s^2 + 5r^2 + 2Rr}{2r}$$

Proof. Using identities $m_a^2 = \frac{2b^2+2c^2-a^2}{4}, r_a = \frac{F}{s-a}$, it follows that:

$$\sum \frac{m_b^2 + m_c^2}{r_a} = \sum \frac{\frac{2a^2+2c^2-b^2}{4} - \frac{2a^2+2b^2-c^2}{4}}{\frac{F}{s-a}} = \frac{1}{4F}\sum (s-a)(4a^2+b^2+c^2)$$

$$= \frac{6s\sum a^2 - 4\sum a^3 - \sum a(b^2+c^2)}{4rs} =$$

$$= \frac{6s \cdot 2(s^2 - r^2 - 4Rr) - 4 \cdot 2s(s^2 - 3r^2 - 6Rr) - 2s(s^2 + r^2 - 2Rr)}{4rs} =$$

$$= \frac{s^2 + 5r^2 + 2Rr}{2r}$$

Le's get back to the main problem.

For RHD, using Lemma and $s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$, we get:

$$LHS = \sum \frac{m_b^2 + m_c^2}{r_a} = \frac{s^2 + 5r^2 + 2Rr}{2r} \le$$

$$\le \frac{(4R^2 + 4Rr + 3r^2) + 5r^2 + 2Rr}{2r} = \frac{4R^2 + 6Rr + 8r^2}{2r} =$$

$$= \frac{2R^2 + 3Rr + 4r^2}{r} \overset{(1)}{\le} \frac{9R^2}{2r} = RHS$$

$$(1) \Leftrightarrow \frac{2R^2+3Rr+4r^2}{r} \le \frac{9R^2}{2r} \Leftrightarrow 5R^2-6Rr-8r^2 \ge 0 \Leftrightarrow$$

$(R-2r)(5R+4r) \ge 0$, true from $R \ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and $s^2 \ge 16Rr-5r^2(Gerretsen)$, we get:

$$LHS = \sum \frac{m_b^2+m_c^2}{r_a} = \frac{s^2+5r^2+2Rr}{2r} \ge$$

$$\ge \frac{(16Rr-5r^2)+5r^2+2Rr}{2r} = \frac{18Rr}{2r} = 9R$$

Equality holds if and only if triangle is equilateral.

8) In ΔABC the following relationship holds:

$$\sum \frac{m_b^2+m_c^2}{r_a} \le \sum \frac{m_b^2+m_c^2}{h_a}$$

Proposed by Marin Chirciu-Romania

Solution by proposer Using up these Lemmas, we have:

$$\sum \frac{m_b^2+m_c^2}{h_a} = \frac{5s^2-11r^2-26Rr}{4r}; \sum \frac{m_b^2+m_c^2}{r_a} = \frac{s^2+5r^2+2Rr}{2r}$$

Inequality can be written:

$$\frac{s^2+5r^2+2Rr}{2r} \le \frac{5s^2-11r^2-26Rr}{4r} \Leftrightarrow 2(s^2+5r^2+2Rr) \le 5s^2-11r^2-26Rr \Leftrightarrow$$

$3s^2 \ge 30Rr+21r^2 \Leftrightarrow s^2 \ge 10Rr+7r^2$, which follows from $s^2 \ge 16Rr-5r^2(Gerretsen)$. Remains to prove that:

$16Rr-5r^2 \ge 10Rr+7r^2 \Leftrightarrow R \ge 2r(Euler)$. Remark. In same class of problems.

9) In ΔABC the following relationship holds:

$$9r \le \sum \frac{m_b^2+m_c^2}{r_b+r_c} \le \frac{9R}{2}$$

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Solution by proposer

Lemma. 10) In ΔABC the following relationship holds:

$$\sum\frac{m_b^2+m_c^2}{r_b+r_c}=\frac{s^2r^2(s^2+2r^2-12Rr+64R^2)+r^3(4R+r)^3}{16s^2r^2R^2}$$

Proof. Using identities $m_a^2=\frac{2b^2+2c^2-a^2}{4}, r_a=\frac{F}{s-a}$, it follows that:

$$\sum\frac{m_b^2+m_c^2}{r_b+r_c}=\sum\frac{\frac{2a^2+2c^2-b^2}{4}-\frac{2a^2+2b^2-c^2}{4}}{\frac{F}{s-b}+\frac{F}{s-c}}=$$

$$=\frac{1}{4F}\sum\frac{(s-b)(s-c)(4a^2+b^2+c^2)}{a}=\frac{1}{4rs}\frac{\sum bc(s-b)(s-c)(4a^2+b^2+c^2)}{abc}=$$

$$=\frac{s^2r^2(s^2+2r^2-12Rr+64R^2)+r^3(4R+r)^3}{16s^2r^2R^2}$$

Let's get back to the main problem.

For RHS, using Lemma we get:

$$\sum\frac{m_b^2+m_c^2}{r_b+r_c}=\frac{s^2r^2(s^2+2r^2-12Rr+64R^2)+r^3(4R+r)^3}{16s^2r^2R^2}=$$

$$\frac{1}{16R}\left[s^2+2r^2-12Rr+64R^2+\frac{r(4R+r)^3}{s^2}\right]\le$$

$$\le\frac{1}{6R}\left[4R^2+4Rr+3r^2+2r^2-12Rr+64R^2+\frac{r(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{72R^2-3Rr+6r^2}{16R}\overset{Euler}{\le}\frac{9R}{2}$$

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma we get:

$$\sum\frac{m_b^2+m_c^2}{r_b+r_c}=\frac{s^2r^2(s^2+2r^2-12Rr+64R^2)+r^3(4R+r)^3}{16s^2r^2R^2}=$$

$$=\frac{1}{16R}\left[s^2+2r^2-12Rr+64R^2+\frac{r(4R+r)^3}{s^2}\right]\ge$$

$$\geq \frac{1}{16R}\left[16Rr-5r^2+2r^2-12Rr+64R^2+\frac{r(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{64R^3+20R^2r-7Rr^2-2r^3}{16R^2}\overset{(1)}{\geq} 9r$$

$$(1)\Leftrightarrow \frac{64R^3+20R^2r-7Rr^2-2r^3}{16R^2}\geq 9r \Leftrightarrow 64R^3-124R^2r-7Rr^2-3r^3\geq 0 \Leftrightarrow$$

$(R-2r)(64R^2+4Rr+r^2)\geq 0$, true from $R\geq 2r(Euler)$.

11) In ΔABC the following relationship holds:

$$9r\leq \sum \frac{m_b^2+m_c^2}{h_b+h_c}\leq \frac{9R^2}{4r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. 12) In ΔABC the following relationship holds:

$$\sum \frac{m_b^2+m_c^2}{h_b+h_c}=\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}$$

Proof. Using identities $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, $h_a=\frac{2F}{a}$ it follows that:

$$\sum \frac{m_b^2+m_c^2}{h_b+h_c}=\sum \frac{\frac{2a^2+2c^2-b^2}{4}+\frac{2a^2+2b^2-c^2}{4}}{\frac{2F}{b}+\frac{2F}{c}}=$$

$$=\frac{1}{8F}\sum \frac{bc(4a^2+b^2+c^2)}{b+c}=\frac{1}{8rs}\frac{\sum bc(a+b)(a+c)(4a^2+b^2+c^2)}{\prod(b+c)}=$$

$$=\frac{1}{8rs}\frac{2[s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{2s(s^2+r^2+2Rr)}=$$

$$=\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}$$

Let's get back to the main problem.

For RHS, using Lemma we get:

$$\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}\leq \frac{9R^2}{4r}\Leftrightarrow$$

$$s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3$$
$$\le 18R^2s^2(s^2+r^2+2Rr)\Leftrightarrow$$
$$s^4(18R^2-36Rr-r^2-s^2)+s^2r(36R^3+90R^2r+40Rr^2+r^3)+r^3(4R+r)^3\ge 0$$

Distinguish the cases:

Case 1) If $(18R^2-36Rr-r^2-s^2)\ge 0$, inequality is obviously true.

Case 2) If $(18R^2-36Rr-r^2-s^2)<0$, inequality can be written:

$$s^2r(36R^3+90R^2r+40Rr^2+r^3)+r^3(4R+r)^3\ge s^4(s^2+r^2+36Rr-18R^2),$$

which follows from Blundon-Gerretsen inequality:

$$\frac{r(4R+r)^2}{R+r}\le 16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$$

Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}r(36R^3+90R^2r+40Rr^2+r^3)+r^3(4R+r)^2\ge$$
$$\ge\frac{R(4R+r)^2}{2(2R-r)}(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+r^2+36Rr-18R^2)\Leftrightarrow$$

$$2r^2(2R-r)(36R^3+90R^2r+40Rr^2+r^3)+2r^3(2R-r)(4R+r)(R+r)\ge$$
$$\ge R(4R^2+4Rr+3r^2)(-14R^2+40Rr+4r^2)(R+r)\Leftrightarrow$$
$$2r^2(72R^4+180R^3r+80R^2r^2+2Rr^3-36R^3r-90R^2r^2-40Rr^3-r^4)+$$
$$+2r^3(8R^3+10R^2r+2Rr^2-4R^2r-5Rr^2-r^3)\ge$$
$$\ge R(-56R^4+160R^3r+16R^2r^2-56R^3r+160R^2r^2+16Rr^3-42R^2r^2+120Rr^3$$
$$+12R^4)(R+r)\Leftrightarrow$$
$$2r^2(72R^4+144R^3r-10R^2r^2-38Rr^3-r^4)+2r^3(8R^3+6R^2r-3Rr^2-r^3)\ge$$
$$\ge R(-56R^4+104R^3r+134R^2r^2+136Rr^3+12r^4)(R+r)\Leftrightarrow$$
$$144R^4r^2+288R^3r^3-20R^2r^4-76Rr^5-2r^6+16R^3r^3+12R^2r^4-6Rr^5-2r^6\ge$$
$$\ge R\big(-56R^5+104R^4r+134R^3r^2+136R^2r^3+12Rr^4-56R^4r+104R^3r^2$$
$$+134R^2r^3+136Rr^4+12r^5\big)\Leftrightarrow$$
$$144R^4r^2+304R^3r^3-8R^2r^4-82Rr^5-4r^6\ge$$
$$\ge R\big(-56R^5+8R^4r+238R^3r^2+270R^2r^3+148Rr^4+12r^5\big)\Leftrightarrow$$

$$144R^4r^2+304R^3r^3-8R^2r^4-82Rr^5-4r^6$$
$$\geq -56R^6+48R^5r+238R^4r^2+270R^3r^3+148R^2r^4+12Rr^5 \Leftrightarrow$$
$$56R^6-48R^5r-94R^4r^2+34R^3r^3-156R^2r^4-94Rr^5-4r^6\geq 0 \Leftrightarrow$$
$$28R^6-24R^5r-47R^4r^2+17R^3r^3-78R^2r^4-47Rr^5-2r^6\geq 0 \Leftrightarrow$$

$(R-2r)\left(28R^5+32R^4r+17R^3r^2+51R^2r^3+24Rr^4+r^5\right)\geq 0$, which is true from

$R\geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma we get:

$$\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}\geq 9r \Leftrightarrow$$
$$s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3$$
$$\geq 72r^2s^2(s^2+r^2+2Rr) \Leftrightarrow$$
$$s^6+s^4(36Rr-71r^2)-s^2r^2(72(R^2+184Rr+73r^2)\geq r^3(4R+r)^3 \Leftrightarrow$$

$s^2[s^2(s^2-71r^2+36Rr)-r^2(72R^2+184Rr+73r^2)]\geq r^3(4R+r)^3$, which follows

from $s^2\geq 16Rr-5r^2\geq \frac{r(4R+r)^2}{R+r}(Gerretsen)$.

Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(16Rr-5r^2-71r^2+36)-r^2(72R^2+184Rr+73r^2)]$$
$$\geq r^3(4R+r)^3 \Leftrightarrow$$
$$(16R-5r)(52R-76r)-(72R^2+184Rr+73r^2)\geq (4R+r)(R+r) \Leftrightarrow$$

$756R^2-1665Rr+306r^2\geq 0 \Leftrightarrow (R-2r)(756R-153r)\geq 0$, which is rue from

$R\geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

13) In ΔABC the following relationship holds:

$$\sum \frac{m_b^2+m_c^2}{r_b+r_c}\leq \sum \frac{m_b^2+m_c^2}{h_b+h_c}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using up these Lemmas, we have:

$$\sum \frac{m_b^2+m_c^2}{h_b+h_c} = \frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}$$

$$\sum \frac{m_b^2+m_c^2}{r_b+r_c} = \frac{s^2r^2(s^2+2r^2-12Rr+64R^2)+r^3(4R+r)^3}{16s^2r^2R^2}$$

Inequality can be written:

$$\frac{s^2r^2(s^2+2r^2-12Rr+64R^2)+r^3(4R+r)^3}{16s^2r^2R^2} \le \frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)} \Leftrightarrow$$

$$2Rr[s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3] \ge$$

$$\ge s^2r^2(s^2+r^2-12Rr+64r^2)+r^3(4R+r)^3 \Leftrightarrow$$

$$s^6(2Rr-r^2)+s^4(8R^2r+12Rr^2-3r^3)-s^2r^2(336R^3r+168R^2r^2+6Rr^3+3r^4) \ge r^4(4R+r)^4 \Leftrightarrow$$

$s^2\left[[s^2(2Rr-r^2)+(8R^2r+12Rr^2-3r^3)]-r^2(336R^3r+168R^2r^2+6Rr^3+3r^4)\right] \ge r^4(4R+r)^4$, which follows from $s^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r} (Gerretsen)$.

Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}s^2\left[[s^2(2Rr-r^2)+(8R^2r+12Rr^2-3r^3)]-r^2(336R^3r+168R^2r^2+6Rr^3+3r^4)\right] \ge r^4(4R+r)^4 \Leftrightarrow$$

$$s^2\left[[s^2(2Rr-r^2)+(8R^2r+12Rr^2-3r^3)]-r^2(336R^3r+168R^2r^2+6Rr^3+3r^4)\right] \ge r^3(4R+r)^2(R+r) \Leftrightarrow$$

$$(16Rr-5r^2)[(16Rr-5r^2)(2Rr-r^2)+(8R^2r+12Rr^2-3r^3)]-$$

$$-r^2(336R^3r+168R^2r^2+6Rr^3+3r^4) \ge r^3(4R+r)^2(R+r) \Leftrightarrow$$

$$288R^3-616R^2r+87Rr^2-14r^3 \ge 0 \Leftrightarrow$$

$(R-2r)(288R^2-49Rr+7r^2) \ge 0$, true from $R \ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

REFERENCES:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro