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ABOUT AN INEQUALITY BY ERTAN YILDIRIM-VII

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1) In ΔABC the following relationship holds:

$$\frac{AI_a}{\cos\frac{A}{2}}+\frac{BI_b}{\cos\frac{B}{2}}+\frac{CI_c}{\cos\frac{C}{2}}\ge 2(a+b+c)$$

Proposed by Ertan Yildirim-Turkey

Solution by Marin Chirciu-Romania

Lemma. 2) In ΔABC the following relationship holds:

$$\frac{AI_a}{\cos\frac{A}{2}}+\frac{BI_b}{\cos\frac{B}{2}}+\frac{CI_c}{\cos\frac{C}{2}}=s\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Proof. Using identities: $AI_a=\frac{s}{\cos\frac{A}{2}}$; $\sum\frac{1}{\cos^2\frac{A}{2}}=1+\left(\frac{4R+r}{s}\right)^2$, we have

$$\sum\frac{AI_a}{\cos\frac{A}{2}}=\sum\frac{\frac{s}{\cos\frac{A}{2}}}{\cos\frac{A}{2}}=s\sum\frac{1}{\cos^2\frac{A}{2}}\ge s\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Let's get back to the main problem. Using Lemma, we have:

$$\sum\frac{AI_a}{\cos\frac{A}{2}}=s\left[1+\left(\frac{4R+r}{s}\right)^2\right]\overset{Doucet}{\ge}4s;\ s\sqrt{3}\le 4R+r(Doucet)$$

Equality holds if and only if triangle is equilateral.

Remark. Let's determine the reverse inequality.

3) In ΔABC the following relationship holds:

$$\frac{AI_a}{\cos\frac{A}{2}}+\frac{BI_b}{\cos\frac{B}{2}}+\frac{CI_c}{\cos\frac{C}{2}}\le s\left(5-\frac{2r}{R}\right)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Lemma, we get:

$$\sum \frac{AI_a}{\cos\frac{A}{2}} = s\left[1+\left(\frac{4R+r}{s}\right)^2\right] \overset{Gerretsen}{\leq} s\left[1+\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$= s\left(1+\frac{2(2R-r)}{R}\right) = s\left(5-\frac{2r}{R}\right)$$

Where $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2 (Blundon-Gerretsen)$

4) In ΔABC the following relationship holds:

$$4s \leq \frac{AI_a}{\cos\frac{A}{2}} + \frac{BI_b}{\cos\frac{B}{2}} + \frac{CI_c}{\cos\frac{C}{2}} \leq s\left(5-\frac{2r}{R}\right)$$

Proposed by Marin Chirciu-Romania

Solution by proposer

See up these inequalities.

Remark. In same class of problems.

5) In ΔABC the following relationship holds:

$$\frac{AI_a}{\cos\frac{B}{2}+\cos\frac{C}{2}} + \frac{BI_b}{\cos\frac{C}{2}+\cos\frac{A}{2}} + \frac{CI_c}{\cos\frac{A}{2}+\cos\frac{B}{2}} \geq 2s$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using identity: $AI_a = \frac{s}{\cos\frac{A}{2}}$, we get:

$$LHS = \sum \frac{AI_a}{\cos\frac{B}{2}+\cos\frac{C}{2}} = \sum \frac{\frac{s}{\cos\frac{A}{2}}}{\cos\frac{B}{2}+\cos\frac{C}{2}} = s\sum \frac{1}{\cos\frac{A}{2}\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)} \overset{BCS}{\geq}$$

$$\geq s\frac{9}{\sum\cos\frac{A}{2}\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)} = \frac{9s}{2\sum\cos\frac{B}{2}\cos\frac{C}{2}} \overset{(1)}{\geq} 2s = RHD,$$

$(\mathbf{1}) \Leftrightarrow \sum \boldsymbol{cos}\frac{\boldsymbol{B}}{\mathbf{2}}\boldsymbol{cos}\frac{\boldsymbol{C}}{\mathbf{2}} \leq \frac{\mathbf{9}}{\mathbf{4}}$, **true from** $\sum \boldsymbol{xy} \leq \sum \boldsymbol{x}^{\mathbf{2}}$ **for** $\boldsymbol{x} = \boldsymbol{cos}\frac{\boldsymbol{A}}{\mathbf{2}}, \boldsymbol{y} = \boldsymbol{cos}\frac{\boldsymbol{B}}{\mathbf{2}}$,

$$\boldsymbol{z} = \boldsymbol{cos}\frac{\boldsymbol{C}}{\mathbf{2}}, \sum \boldsymbol{cos}^2\frac{\boldsymbol{A}}{\mathbf{2}} = \mathbf{2} + \frac{\boldsymbol{r}}{\mathbf{2}\boldsymbol{R}} \leq \frac{\mathbf{9}}{\mathbf{4}}.$$

Equality holds if and only if triangle is equilateral.

REFERENCES:

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