ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT AN INEQUALITY BY ERTAN YILDIRIM-VIII

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1) In ΔABC the following relationship holds:

$$\sum \frac{cosA}{r_b + r_c} \le \frac{1}{4r}$$

Proposed by Ertan Yildirim-Ankara-Turkey

Solution. **Lemma 1. 2) In ΔABC the following relationship holds:**

$$\sum \frac{cosA}{r_b + r_c} = \frac{5s^2 - (4R + r)^2}{2s^2R}$$

Proof. Using Law of Cosines: $cosA = \frac{b^2+c^2-a^2}{2bc}$ and $r_a = \frac{F}{s-a}$, we get:

$$\sum \frac{cosA}{r_b + r_c} = \sum \frac{\frac{b^2 + c^2 - a^2}{2bc}}{\frac{F}{s-b} + \frac{F}{s-c}} = \frac{1}{2F}\sum \frac{(s-b)(s-c)(b^2 + c^2 - a^2)}{abc} =$$

$$= \frac{1}{2F}\frac{\sum(s-b)(s-c)(b^2 + c^2 - a^2)}{abc} = \frac{1}{2rs}\frac{2r^2(5s^2 - (4R + r)^2)}{4Rrs} = \frac{5s^2 - (4R + r)^2}{4s^2R}$$

Which follows from

$$\sum (s-b)(s-c)(b^2 + c^2 - a^2) = 2r^2(5s^2 - (4R + r)^2)$$

Let's get back to the main problem.

Using Lemma, inequality can be written as:

$$\frac{5s^2 - (4R + r)^2}{4s^2R} \le \frac{1}{4r} \Leftrightarrow 5s^2r - r(4R + r)^2 \le s^2r \Leftrightarrow s^2(R - 5r) + r(4R + r)^2 \ge 0$$

Distinguish the cases:

Case 1) If $(R - 5r) > 0$, inequality is obviously true.

Case 2) If $(R - 5r) < 0$, inequality can be written: $r(4R + r)^2 \ge s^2(5r - R)$, which follows from $s^2 \le \frac{R(4R+r)^2}{2(2R-r)} \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$.

Remains to prove that:

$$r(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}(5r-R) \Leftrightarrow 2r(2R-r) \geq R(5r-R) \Leftrightarrow$$

$R^2 - Rr - 2r^2 \geq 0 \Leftrightarrow (R-2r)(R+r) \geq 0$, true from $R \geq 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. Inequality it can be much stronger.

3) In ΔABC the following relationship holds:

$$\sum \frac{cosA}{r_b + r_c} \leq \frac{1}{2R}$$

Proposed by Marin Chirciu-Romania

Solution. Using Lemma, inequality can be written as:

$$\frac{5s^2-(4R+r)^2}{4s^2R} \leq \frac{1}{2R} \Leftrightarrow 5s^2-(4R+r)^2 \leq 2s^2 \Leftrightarrow 3s^2 \leq (4R+r)^2 (Doucet).$$

Equality holds if and only if triangle is equilateral.

Remark. Let's find an reverse inequality.

4) In ΔABC the following relationship holds:

$$\sum \frac{cosA}{r_b + r_c} \geq \frac{1}{R}\left(1 - \frac{R}{4r}\right)$$

Proposed by Marin Chirciu-Romania

Solution. Using Lemma, we get:

$$\sum \frac{cosA}{r_b + r_c} = \frac{5s^2-(4R+r)^2}{4s^2R} = \frac{1}{4R}\left(5 - \frac{(4R+r)^2}{s^2}\right) \overset{(1)}{\geq} \frac{1}{4R}\left(5 - \frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right) =$$

$$= \frac{1}{4R}\left(5 - \frac{R+r}{r}\right) = \frac{1}{4R}\left(4 - \frac{R}{r}\right) = \frac{1}{R}\left(1 - \frac{R}{4r}\right),$$

Where (1) it follows from $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r} (Gerretsen)$.

Equality holds if and only if triangle is equilateral.

Remark. Inequality it can be doubled.

5) In ΔABC the following relationship holds:

$$\frac{1}{R}\left(1-\frac{R}{4r}\right)\le\sum\frac{\cos A}{r_b+r_c}\le\frac{1}{2R}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma 6) In ΔABC the following relationship holds:**

$$\sum\frac{\cos A}{r_b+r_c}=\frac{5s^2-(4R+r)^2}{4s^2R}$$

Proof. . Using Law of Cosines: $\cos A=\frac{b^2+c^2-a^2}{2bc}$ and $r_a=\frac{F}{s-a}$, we get:

$$\sum\frac{\cos A}{r_b+r_c}=\sum\frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{F}{s-b}+\frac{F}{s-c}}=\frac{1}{2F}\sum\frac{(s-b)(s-c)(b^2+c^2-a^2)}{abc}=$$

$$=\frac{1}{2F}\frac{\sum(s-b)(s-c)(b^2+c^2-a^2)}{abc}=\frac{1}{2rs}\frac{2r^2(5s^2-(4R+r)^2)}{4Rrs}=\frac{5s^2-(4R+r)^2}{4s^2R}$$

Which follows from

$$\sum(s-b)(s-c)(b^2+c^2-a^2)=2r^2(5s^2-(4R+r)^2)$$

Let's get back to the main problem.

For RHS, using Lemma we get:

$$\frac{5s^2-(4R+r)^2}{4s^2R}\le\frac{1}{2R}\Leftrightarrow 5s^2-(4R+r)^2\le 2s^2\Leftrightarrow 3s^2\le(4R+r)^2(Doucet).$$

Equality holds if and only if triangle is equilateral.

Fro LHS, using Lemma we get:

$$\sum\frac{\cos A}{r_b+r_c}=\frac{5s^2-(4R+r)^2}{4s^2R}=\frac{1}{4R}\left(5-\frac{(4R+r)^2}{s^2}\right)\overset{(1)}{\ge}\frac{1}{4R}\left(5-\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right)$$

Where (1)nit follows from $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$(*Gerretsen*.

Equality holds if and only if triangle is equilateral.

Remark. Let's replace r_a with h_a.

7) In ΔABC the following relationship holds:

$$\frac{1}{2R}\le\sum\frac{\cos A}{h_b+h_c}\le\frac{1}{4r}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 8) In ΔABC the following relationship holds:**

$$\sum\frac{\cos A}{h_b+h_c}=\frac{s^2(s^2+8r^2)-r^2(4R+r)^2}{4rs^2(s^2+r^2+2Rr)}$$

Proof. Using Law of Cosines: $\cos A=\frac{b^2+c^2-a^2}{2bc}$ and $h_a=\frac{2F}{a}$, we get:

$$\sum\frac{\cos A}{h_b+h_c}=\sum\frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{2F}{b}+\frac{2F}{c}}=\frac{1}{4F}\sum\frac{b^2+c^2-a^2}{b+c}=$$

$$=\frac{1}{4F}\cdot\frac{s^2(s^2+8r^2)-r^2(4R+r)^2}{s(s^2+r^2+2Rr)}=\frac{s^2(s^2+8r^2)-r^2(4R+r)^2}{4rs^2(s^2+r^2+2Rr)}$$

Which follows from:

$$\sum\frac{b^2+c^2-a^2}{b+c}=\frac{s^2(s^2+8r^2)-r^2(4R+r)^2}{s(s^2+r^2+2Rr)}$$

Let's get back to the main problem.

For RHS, using Lemma inequality can be written as:

$$\frac{s^2(s^2+8r^2)-r^2(4R+r)^2}{4rs^2(s^2+r^2+2Rr)}\le\frac{1}{4r}\Leftrightarrow s^2(s^2+8r^2)-r^2(4R+r)^2\le s^2(s^2+r^2+2Rr)$$

$$s^2(2R-7r)+r(4R+r)^2\ge 0$$

Distinguish the cases:

Case 1) If $(2R-7r)\ge 0$, inequality is obviously true.

Case 2) If $(2R-7r)<0$, inequality can be written as: $r(4R+r)^2\ge s^2(7r-2R)$,

Which follows from $s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2(Gerretsen)$.

Remains to prove that: $r(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}(7r-2R)\Leftrightarrow$

$$2r(2R-r)\ge R(7r-2R)\Leftrightarrow 2R^2-3Rr-2r^2\ge 0\Leftrightarrow(R-2r)(2R+r)\ge 0$$

Which is obviously true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Fro LHS, using Lemma, we have:

$$\sum\frac{cosA}{h_b+h_c}=\frac{s^2(s^2+8r^2)-r^2(4R+r)^2}{4rs^2(s^2+r^2+2Rr)}\overset{(1)}{\geq}\frac{1}{2R}$$

$$(1)\Leftrightarrow\frac{s^2(s^2+8r^2)-r^2(4R+r)^2}{4rs^2(s^2+r^2+2Rr)}\geq\frac{1}{2R}\Leftrightarrow$$

$$Rs^2(s^2+8r^2)-Rr^2(4R+r)^2\geq 2rs^2(s^2+r^2+2Rr)\Leftrightarrow$$

$$Rs^2(s^2+8r^2)-Rr^2(4R+r)^2\geq 2rs^2(s^2+r^2+2Rr)\Leftrightarrow$$

$$s^4(R-2r)+s^2r^2(4R-2r)\geq Rr^2(4R+r)^2\Leftrightarrow$$

$s^2[s^2(R-2r)+r^2(4R-2r)]\geq Rr^2(4R+r)^2$, **which follows from**

$$s^2\geq 16Rr-5r^2\geq\frac{r(4R+r)^2}{R+r}(Gerretsen).$$

Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(R-2r)+r^2(4R-2r)]\geq Rr^2(4R+r)^2\Leftrightarrow$$

$$(16R-5r)(R-2r)+r(4R-2r)\geq R(R+r)\Leftrightarrow$$

$$15R^2-34Rr+8R^2\geq 0\Leftrightarrow(R-2r)(15R-4r)\geq 0,$$

which is true from $R\geq 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

9) In ΔABC the following relationship holds:

$$\sum\frac{cosA}{r_b+r_c}\leq\sum\frac{cosA}{h_b+h_c}$$

Proposed by Marin Chirciu-Romania

Solution. **From up these Lemmas, we have:**

$$\frac{1}{R}\left(1-\frac{R}{4r}\right)\leq\sum\frac{cosA}{r_b+r_c}\leq\frac{1}{2R},\frac{1}{2R}\leq\sum\frac{cosA}{h_b+h_c}\leq\frac{1}{4r}$$

$$\Rightarrow\sum\frac{cosA}{r_b+r_c}\leq\frac{1}{2R}\leq\sum\frac{cosA}{h_b+h_c}$$

Equality if and only if triangle is equilateral.

10) In ΔABC the following relationship holds:

$$\frac{1}{R}\left(1-\frac{R}{4r}\right)\le\sum\frac{cosA}{r_b+r_c}\le\frac{1}{2R}\le\sum\frac{cosA}{h_b+h_c}\le\frac{1}{4r}$$

Proposed by Marin Chirciu-Romania

Solution. **See up these inequalities.**

Equality holds if and only if triangle is equilateral.

References:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro