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ABOUT AN INEQUALITY BY HAXVERDIYEV TAVERDI-III

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1) In ΔABC the following relationship holds:

$$\sqrt{\frac{a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}}{acosA+bcosB+ccosC}}\ge\sqrt{6}R$$

Proposed by Haxverdiyev Taverdi-Azerbaijan

Solution. Using well-known identity in triangle: $\sum a\,cosA=\frac{2rs}{R}$, inequality can be written:

$$\frac{a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}}{acosA+bcosB+ccosC}\ge 6R^2\Leftrightarrow a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}\ge 6R^2\cdot\frac{2rs}{R}\Leftrightarrow$$

$a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}\ge 12Rrs$, which follows from AM-GM inequality and $abc=4Rrs$.

Equality holds if and only if triangle is equilateral.

Remark. Let's find an reverse inequality.

2) In ΔABC the following relationship holds:

$$\sqrt{\frac{a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}}{acosA+bcosB+ccosC}}\le\sqrt{\frac{3}{2}\cdot\frac{R^2}{r}}$$

Proposed by Marin Chirciu-Romania

Solution. *Using well-known identity in triangle:* $\sum a\,cosA=\frac{2rs}{R}$, *inequality can be written:*

$$\frac{a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}}{acosA+bcosB+ccosC}\le\frac{3R^4}{2r^2}\Leftrightarrow a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}\le\frac{3R^4}{2r^2}\cdot\frac{2rs}{R}\Leftrightarrow$$

$a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}\le\frac{3R^3s}{r}$, *which follows from AM-GM inequality.*

We get: $\sum a^2\sqrt{bc}\le\sum a^2\frac{b+c}{2}=\frac{1}{2}\sum a^2(b+c)=\frac{1}{2}2s(s^2+r^2-2Rr)=$

$=s(s^2+r^2-2Rr)$, *which follows from* $\sum a^2(b+c)=2s(s^2+r^2-2Rr)$.

Next, we must show that:

$2s(s^2+r^2-2Rr)\le\frac{3R^2s}{r}\Leftrightarrow r(s^2+r^2-2Rr)\le 3R^3$, *which follows from*

$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$. *Remains to prove:*

$$r(4R^2+4Rr+3r^2+r^2-2Rr)\le 3R^3 \Leftrightarrow 3R^3-4R^2r-2Rr^2-4r^3\ge 0 \Leftrightarrow$$

$(R-2r)(3R^2+2Rr+2r^2)\ge 0$, *which is true from* $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sqrt{6}R\le\sqrt{\frac{a^2\sqrt{bc}+b^2\sqrt{ca}+c^2\sqrt{ab}}{acosA+bcosB+ccosC}}\le\sqrt{\frac{3}{2}\cdot\frac{R^2}{r}}$$

Proposed by Marin Chirciu-Romania

Solution. See inequalities 1) and 2).

Equality holds if and only if triangle is equilateral.

Reference:

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