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ABOUT AN INEQUALITY BY KOSTAS GERONIKOLAS-II

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1) In ΔABC the following relationship holds:

$$\sum \frac{m_a^4}{h_a^2} \geq \left(3\sqrt{3}r\right)^2$$

Proposed by Kostas Geronikolas-Greece

Solution.

Lemma. 2) In ΔABC the following relationship holds:

$$\sum \frac{m_a^4}{h_a^2} \geq \frac{(4R+r)^2 - s^2}{2}$$

Proof. Using inequality $m_a \geq \sqrt{s(s-a)}$ and $h_a = \frac{2F}{a}$, it follows that:

$$\sum \frac{m_a^4}{h_a^2} \geq \sum \frac{s^2(s-a)^2}{\left(\frac{2F}{a}\right)^2} = \frac{s^2}{4F^2} \sum a^2(s-a)^2 = \frac{s^2}{4s^2r^2} 2r^2[(4R+r)^2 - s^2] =$$

$= \frac{(4R+r)^2 - s^2}{2}$, which follows from $\sum a^2(s-a)^2 = 2r^2[(4R+r)^2 - s^2]$.

Le's get back to the main problem.

Using Lemma, we get:

$$LHS = \sum \frac{m_a^4}{h_a^2} \geq \frac{(4R+r)^2 - s^2}{2} \overset{(1)}{\geq} 27r^2 = RHS, (1) \Leftrightarrow \frac{(4R+r)^2 - s^2}{2} \geq 27r^2 \Leftrightarrow$$

$(4R+r)^2 - s^2 \geq 54r^2 \Leftrightarrow 16R^2 + 8Rr - 53r^2 \geq s^2$, which follows from

$$s^2 \leq 4R^2 + 4Rr + 3r^2 (Gerretsen).$$

Remains to prove that: $16R^2 + 8Rr - 53r^2 \geq 4R^2 + 4Rr + 3r^2 \Leftrightarrow$

$3R^2 + Rr - 14r^2 \geq 0 \Leftrightarrow (R-2r)(3R+r) \geq 0$, true from $R \geq 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. Inequality can be much stronger.

3) In ΔABC the following relationship holds:

$$\sum \frac{m_a^4}{h_a^2} \ge \frac{(4R+r)^2}{3}$$

Proposed by Marin Chirciu-Romania

Solution.

Lemma. 4) In ΔABC the following relationship holds:

$$\sum \frac{m_a^4}{h_a^2} \ge \frac{(4R+r)^2 - s^2}{2}$$

Using inequality $m_a \ge \sqrt{s(s-a)}$ and $h_a = \frac{2F}{a}$, it follows that:

$$\sum \frac{m_a^4}{h_a^2} \ge \sum \frac{s^2(s-a)^2}{\left(\frac{2F}{a}\right)^2} = \frac{s^2}{4F^2}\sum a^2(s-a)^2 = \frac{s^2}{4s^2r^2}2r^2[(4R+r)^2 - s^2] =$$

$= \frac{(4R+r)^2-s^2}{2}$, **which follows from** $\sum a^2(s-a)^2 = 2r^2[(4R+r)^2 - s^2]$.

Le's get back to the main problem.

Using Lemma, we get:

$$LHS = \sum \frac{m_a^4}{h_a^2} \ge \frac{(4R+r)^2 - s^2}{2} \overset{(1)}{\ge} \frac{(4R+r)^2}{3} = RHS,$$

$$(1) \Leftrightarrow \frac{(4R+r)^2 - s^2}{2} \ge \frac{(4R+r)^2}{3} \Leftrightarrow 3(4R+r)^2 - 3s^2 \ge 2(4R+r)^2 \Leftrightarrow$$

$(4R+r)^2 \ge 3s^2 (Doucet)$, **which follows from** $s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$.

Remains to prove that: $(4R+r)^2 \ge 3(4R^2 + 4Rr + 3r^2) \Leftrightarrow 4R^2 - 4Rr - 8r^2 \ge 0 \Leftrightarrow$

$R^2 - R - 2r^2 \ge 0 \Leftrightarrow (R-2r)(R+r) \ge 0$, **true from** $R \ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum \frac{m_a^4}{h_a^2} \ge \frac{(4R+r)^2}{3} \ge \left(3\sqrt{3}r\right)^2$$

Proposed by Marin Chirciu-Romania

Solution. See inequality 3) and $\frac{(4R+r)^2}{3} \geq \left(3\sqrt{3}r\right)^2 \Leftrightarrow (4R+r)^2 \geq 81r^2 \Leftrightarrow$

$$4R + r \geq 9r \Leftrightarrow R \geq 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

Remark. Let's replace h_a with r_a.

6) In ΔABC the following relationship holds:

$$\sum \frac{m_a^4}{r_a^2} \geq s^2\left(3 - \frac{4r}{R}\right)$$

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Solution.

Lemma. 7) In ΔABC the following relationship holds:

$$\sum \frac{(s-a)^2}{r_a^2} = \frac{s^2(s^2 - 16Rr) + 2r^2(4R+r)^2}{s^2r^2}$$

Proof. Using $r_a = \frac{F}{s-a}$, we get:

$$\sum \frac{(s-a)^2}{r_a^2} = \sum \frac{(s-a)^2}{\frac{F^2}{(s-a)^2}} = \frac{1}{F^2}\sum (s-a)^4 = \frac{1}{s^2r^2}[s^2(s^2 - 16Rr) + 2r^2(4R+r)^2]$$

$= \frac{s^2(s^2-16Rr)+2r^2(4R+r)^2}{s^2r^2}$, which follows from $\sum (s-a)^4 = s^2(s^2 - 16Rr) + 2r^2(4R+r)^2$

Let's get back to the main problem.

Using Lemma and $m_a \geq \sqrt{s(s-a)}$ it follows that:

$$LHS = \sum \frac{m_a^4}{r_a^2} \geq \sum \frac{s^2(s-a)^2}{r_a^2} = s^2 \sum \frac{(s-a)^2}{r_a^2} = s^2 \frac{s^2(s^2 - 16Rr) + 2r^2(4R+r)^2}{s^2r^2}$$

$$= \frac{s^2}{r^2}\left[(s^2 - 16Rr) + \frac{2r^2(4R+r)^2}{s^2}\right] \geq \frac{s^2}{r^2}\left[(16Rr - 5r^2 - 16Rr) + \frac{2r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$= \frac{s^2}{r^2}\left[-5r^2 + \frac{4r^2(4R+r)^2}{R}\right] = \frac{s^2}{r^2}\left[\frac{-5Rr^2 + 4r^2(2R-r)}{R}\right] =$$

$$= \frac{s^2}{r^2}\frac{3Rr^2 - 4r^3}{R} = s^2\left(3 - \frac{4r}{R}\right) = RHS.$$

$$16Rr-5r^2\le s^2\le \frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2(Blundon-Gerretsen)$$

8) In ΔABC the following relationship holds:

$$\sum \frac{(s-a)^2}{h_a^2}\le \sum \frac{(s-a)^2}{r_a^2}$$

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Solution.

Lemma 1. 9) In ΔABC the following relationship holds:

$$\sum \frac{(s-a)^2}{h_a^2}=\frac{(4R+r)^2-s^2}{2s^2}$$

Proof. Using identity $h_a=\frac{2F}{a}$, we get:

$$\sum \frac{(s-a)^2}{h_a^2}=\sum \frac{(s-a)^2}{\frac{F^2}{a^2}}=\frac{1}{F^2}\sum a^2(s-a)^2=\frac{1}{4s^2r^2}2r^2[(4R+r)^2-s^2]=$$

$=\frac{(4R+r)^2-s^2}{2s^2}$, **which follows from** $\sum a^2(s-a)^2=2r^2[(4R+r)^2-s^2]$

Lemma 2. 10) In ΔABC the following relationship holds:

$$\sum \frac{(s-a)^2}{r_a^2}=\frac{s^2(s^2-16Rr)+2r^2(4R+r)^2}{s^2r^2}$$

Proof. Using $r_a=\frac{F}{s-a}$, we get:

$$\sum \frac{(s-a)^2}{r_a^2}=\sum \frac{(s-a)^2}{\frac{F^2}{(s-a)^2}}=\frac{1}{F^2}\sum (s-a)^4=\frac{1}{s^2r^2}[s^2(s^2-16Rr)+2r^2(4R+r)^2]$$

$=\frac{s^2(s^2-16Rr)+2r^2(4R+r)^2}{s^2r^2}$, **which follows from** $\sum (s-a)^4=s^2(s^2-16Rr)+2r^2(4R+r)^2$

Let's get back to the main problem. Using Lemmas, we have:

$$\sum \frac{(s-a)^2}{r_a^2}=\frac{s^2(s^2-16Rr)+2r^2(4R+r)^2}{s^2r^2},\sum \frac{(s-a)^2}{h_a^2}=\frac{(4R+r)^2-s^2}{2s^2}$$

$$\sum \frac{(s-a)^2}{h_a^2}\le \sum \frac{(s-a)^2}{r_a^2}\Leftrightarrow$$

$$\frac{(4R+r)^2-s^2}{2s^2}\le\frac{s^2(s^2-16Rr)+2r^2(4R+r)^2}{s^2r^2}\Leftrightarrow$$

$$r^2(4R+r)^2-r^2s^2\le 2s^2(s^2-16Rr)+4r^2(4R+r)^2\Leftrightarrow$$

$$s^2(2s^2+r^2-32Rr)+3r^2(4R+r)^2\ge 0$$

Distinguish the cases:

Case 1) If $(2s^2+r^2-32Rr)\ge 0$ is obviously true.

Case 2) If $(2s^2+r^2-32Rr)<0$, inequality can be written:

$3r^2(4R+r)^2\ge s^2(32Rr-r^2-2s^2)$, true from

$$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2(Blundon-Gerretsen)$$

Remains to prove that:

$$3r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}\left(32Rr-r^2-2(16Rr-5r^2)\right)\Leftrightarrow$$

$$6r^2(2R-r)\ge R(Rr-r^2-32Rr+10r^2)\Leftrightarrow 6r^2(2R-r)\ge 9Rr^2\Leftrightarrow$$

$$2(2R-r)\ge 3R\Leftrightarrow R\ge 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

Reference:

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