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ABOUT AN INEQUALITY BY MUSTAFA TAREK-I

By Marin Chirciu – Romania

1) Prove that in any ΔABC the following relationship holds:

$$\frac{AI\cdot h_a}{w_a(r_a+h_a)}+\frac{BI\cdot h_b}{w_b(r_b+h_b)}+\frac{CI\cdot h_c}{w_c(r_c+h_c)}=1$$

Proposed by Mustafa Tarek – Egypt

Solution *Using the relationships in triangle* $AI=\frac{r}{\sin\frac{A}{2}}, h_a=\frac{2S}{a}, w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$ *and*

$r_a+h_a=\frac{S(b+c)}{a(s-a)}$, *we obtain* $\sum\frac{AI\cdot h_a}{w_a(r_a+h_a)}=\sum\frac{\frac{r}{\sin\frac{A}{2}}\cdot\frac{2S}{a}}{\frac{2bc}{b+c}\cos\frac{A}{2}\cdot\frac{S(b+c)}{a(s-a)}}=\frac{1}{S}\cdot\frac{r}{2R}\sum\frac{a(s-a)}{\sin A}=\frac{1}{rs}\cdot\frac{r}{2R}\cdot$

$2R\sum(s-a)=\frac{1}{s}\cdot s=1$

2) Prove that in any ΔABC the following relationship holds:

$$\frac{w_a(r_a+h_a)}{AI\cdot h_a}+\frac{w_b(r_b+h_b)}{BI\cdot h_b}+\frac{w_c(r_c+h_c)}{CI\cdot h_c}\geq 9$$

Proposed by Mustafa Tarek – Egypt

Solution *Using the inequality* $x+y+z\geq\frac{9}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$, *with* $x=\frac{w_a(r_a+h_a)}{AI\cdot h_a}, y=\frac{w_b(r_b+h_b)}{BI\cdot h_b}$,

$z=\frac{w_c(r_c+h_c)}{CI\cdot h_c}$ *and the above identity* $\frac{AI\cdot h_a}{w_a(r_a+h_a)}+\frac{BI\cdot h_b}{w_b(r_b+h_b)}+\frac{CI\cdot h_c}{w_c(r_c+h_c)}=1$ *we obtain the conclusion. Equality holds if and only if the triangle is equilateral.*

Remark. *Let's find an inequality having an opposite sense:*

3) Prove that in any ΔABC the following relationship holds:

$$\frac{w_a(r_a+h_a)}{AI\cdot h_a}+\frac{w_b(r_b+h_b)}{BI\cdot h_b}+\frac{w_c(r_c+h_c)}{CI\cdot h_c}\leq\frac{9R}{2r}$$

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Solution *We prove the identity:*

Lemma.

4) Prove that in any ΔABC the following relationship holds:

$$\frac{w_a(r_a+h_a)}{AI\cdot h_a}+\frac{w_b(r_b+h_b)}{BI\cdot h_b}+\frac{w_c(r_c+h_c)}{CI\cdot h_c}=1+\frac{4R}{r}$$

Proof.

Using the known inequalities in triangle $AI=\frac{r}{\sin\frac{A}{2}}, h_a=\frac{2S}{a}, w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$ *and*

$r_a+h_a=\frac{S(b+c)}{a(s-a)}$, *we obtain:* $\sum\frac{w_a(r_a+h_a)}{AI\cdot h_a}=\sum\frac{\frac{2bc}{b+c}\cos\frac{A}{2}\cdot\frac{S(b+c)}{a(s-a)}}{\frac{r}{\sin\frac{A}{2}}}=S\cdot\frac{2R}{r}\sum\frac{\sin A}{a(s-a)}=$

$$=rs\cdot\frac{2R}{r}\cdot\frac{1}{2R}\sum\frac{1}{s-a}=s\cdot\sum\frac{1}{s-a}=s\cdot\frac{4R+r}{rs}=\frac{4R+r}{r}$$

Let's get back to the main problem.

Using the Lemma the inequality can be written: $1+\frac{4R}{r}\le\frac{9R}{2r}\Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark.

We can write the double inequality:

5) Prove that in any ΔABC the following relationship holds:

$$9\le\frac{w_a(r_a+h_a)}{AI\cdot h_a}+\frac{w_b(w_b+h_b)}{BI\cdot h_b}+\frac{w_c(r_c+h_c)}{CI\cdot h_c}\le\frac{9R}{2r}$$

Solution.

See inequalities 2) and 4)

Equality holds if and only if the triangle is equilateral.

Reference:

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