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ABOUT AN INEQUALITY BY MUSTAFA TAREK-II

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\sqrt{\tan\frac{A}{2}}+\sqrt{\tan\frac{B}{2}}+\sqrt{\tan\frac{C}{2}}\le\sqrt{\frac{s}{r}}$$

Proposed by Mustafa Tarek – Egypt

Solution *Using the identity* $\tan\frac{A}{2}=\frac{r}{s-a}$ *and the CBS inequality, we obtain:*

$$M_s=\sum\sqrt{\tan\frac{A}{2}}=\sum\sqrt{\frac{r}{s-a}}\le\sum\sqrt{r}\cdot\frac{1}{\sqrt{s-a}}\le\sqrt{\sum r\cdot\sum\frac{1}{s-a}}=\sqrt{3r\cdot\frac{4R+r}{rs}}=$$

$$=\sqrt{\frac{3(4R+r)}{s}}\le\sqrt{\frac{s}{r}}=M_d,$$ *where the last inequality is equivalent with:*

$\frac{3(4R+r)}{s}\le\frac{s}{r}\Leftrightarrow s^2\ge 3r(4R+r)$, *which follows from Gerretsen's inequality*

$s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2\ge 3r(4R+r)\Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark.

Let's find an inequality having an opposite sense:

2) In ΔABC the following inequality holds:

$$\sqrt{\tan\frac{A}{2}}+\sqrt{\tan\frac{B}{2}}+\sqrt{\tan\frac{C}{2}}\ge 3\sqrt{\frac{3r}{s}}$$

Proposed by Marin Chirciu – Romania

Solution *Using the identity* $\prod\frac{A}{2}=\frac{r}{s}$ *and the means inequality, we obtain:*

$$M_s=\sum\sqrt{\tan\frac{A}{2}}\ge 3\sqrt[3]{\prod\sqrt{\frac{A}{2}}}=3\sqrt[6]{\frac{r}{s}}\ge 3\sqrt{\frac{3r}{s}}=M_d$$

Wherefrom the last inequality is equivalent with:

$$\sqrt[6]{\frac{r}{s}} \geq \sqrt{\frac{3r}{s}} \Leftrightarrow s^2 \geq 27r^2,\ \textit{which follows from Mitrinovic's inequality } s \geq 3r\sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

Remark.

We can write the double inequality:

3) In ΔABC the following inequality holds:

$$3\sqrt{\frac{3r}{s}} \leq \sqrt{\tan\frac{A}{2}} + \sqrt{\tan\frac{B}{2}} + \sqrt{\tan\frac{C}{2}} \leq \sqrt{\frac{s}{r}}$$

Solution*See inequalities 1) and 2).Equality holds if and only if the triangle is equilateral.*

Remark.*If we replace* $\tan\frac{A}{2}$ *with* $\cot\frac{A}{2}$ *we propose:*

4) In ΔABC the following relationship holds:

$$9\sqrt{\frac{r}{s}} \leq \sqrt{\cot\frac{A}{2}} + \sqrt{\cot\frac{B}{2}} + \sqrt{\cot\frac{C}{2}} \leq \sqrt{\frac{3s}{r}}$$

Proposed by Marin Chirciu – Romania

Solution *LHS inequality:*

Using the identity $\cot\frac{A}{2} = \frac{s-a}{r}$ *and CBS inequality, we obtain:*

$$M_s = \sum\sqrt{\cot\frac{A}{2}} = \sum\sqrt{\frac{s-a}{r}} \leq \sum\frac{1}{\sqrt{r}}\cdot\sqrt{s-a} \leq \sqrt{\sum\frac{1}{r}\cdot\sum(s-a)} = \sqrt{\frac{3}{r}\cdot s} = \sqrt{\frac{3s}{r}}$$

Equality holds if and only if the triangle is equilateral.RHS inequality:

Using the identity $\prod\cot\frac{A}{2} = \frac{s}{r}$ *and the means inequality we obtain:*

$$M_s = \sum\sqrt{\cot\frac{A}{2}} \geq 3\sqrt[3]{\prod\sqrt{\cot\frac{A}{2}}} = 3\sqrt[6]{\frac{s}{r}} \geq 9\sqrt{\frac{r}{s}} = M_d$$

where the last inequality is equivalent with:

$\sqrt[6]{\frac{s}{r}} \geq 3\sqrt{\frac{r}{s}} \Leftrightarrow s^2 \geq 27r^2$, *which follows from Mitrinoic's inequality* $s \geq 3r\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

Remark. *Between the sums* $\sum \tan\frac{A}{2}$ *and* $\sum \cot\frac{A}{2}$ *the following relationship holds:*

5) In ΔABC the following relationship holds:

$$\sum \tan\frac{A}{2} \leq \frac{1}{\sqrt{3}} \cdot \frac{R}{2r} \sum \cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

Solution *See inequalities 1), 4) and Mitrinovic's inequality* $s \leq \frac{3R\sqrt{3}}{2}$.

Indeed, $\sum \tan\frac{A}{2} \overset{1)}{\leq} \sqrt{\frac{s}{r}} \overset{(*)}{\leq} \frac{1}{\sqrt{3}} \cdot \frac{R}{2r} \cdot 9\sqrt{\frac{r}{s}} \overset{4)}{\leq} \frac{1}{\sqrt{3}} \cdot \frac{R}{2r} \cdot \sum \cot\frac{A}{2}$, *where the inequality (*) is*

equivalent with: $\sqrt{\frac{s}{r}} \overset{(*)}{\leq} \frac{1}{\sqrt{3}} \cdot \frac{R}{2r} \cdot 9\sqrt{\frac{r}{s}} \Leftrightarrow \frac{s}{r} \leq \frac{1}{3} \cdot \frac{R^2}{4r^2} \cdot 81 \cdot \frac{r}{s} \Leftrightarrow s^2 \leq \frac{27R^2}{4}$, *which follows from*

Mitrinovic's inequality $s \leq \frac{3R\sqrt{3}}{2}$

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the sequences of inequalities:*

6) In ΔABC the following relationship holds:

$$3\sqrt{\frac{3r}{s}} \leq \sum \tan\frac{A}{2} \leq \frac{1}{\sqrt{3}} \cdot \frac{R}{2r} \sum \cot\frac{A}{2} \leq \frac{R}{2r}\sqrt{\frac{s}{r}}$$

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Solution:*See inequalities 1) and 4).Equality holds if and only if the triangle is equilateral.*

Reference:

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