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ABOUT AN INEQUALITY BY NGUYEN VAN CANH-V

By Marin Chirciu-Romania

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1) In ΔABC the following relationship holds:

$$\sum \sqrt{r_a h_a m_a} \le (4R+r)\sqrt{\frac{3R}{2}}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Marin Chirciu-Romania

Using inequality $h_a \le m_a$, we get:

$$LHS = \sum \sqrt{r_a h_a m_a} \le \sum \sqrt{r_a m_a m_a} = \sum m_a \sqrt{r_a} \overset{CBS}{\le} \sqrt{\sum m_a^2 \cdot \sum r_a} =$$

$$= \sqrt{\frac{3}{4}\sum a^2 \cdot (4R+r)} \overset{(1)}{\le} (4R+r)\sqrt{\frac{3R}{2}} = RHD$$

Where $(1) \Leftrightarrow \sqrt{\frac{3}{4}\sum a^2 \cdot (4R+r)} \le (4R+r)\sqrt{\frac{3R}{2}} \Leftrightarrow$

$$\frac{3}{4}\sum a^2 \cdot (4R+r) \le (4R+r)^2 \frac{3R}{2} \Leftrightarrow \sum a^2 \le 2R(4R+r) \Leftrightarrow$$

$2(s^2 - r^2 - 4Rr) \le 2R(4R+r) \Leftrightarrow s^2 \le 4R^2 + 5Rr + r^2$, which follows from

$$s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen).$$

It remains to prove that:

$$4R^2 + 4Rr + 3r^2 \le 4R^2 + 5Rr + r^2 \Leftrightarrow R \ge 2r (Euler).$$

Equality holds if and only if triangle ids equilateral.

Remark. In same class of the problem.

2) In ΔABC the following relationship holds:

$$\sum \sqrt{r_a w_a m_a} \le (4R+r)\sqrt{\frac{3R}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using inequality $w_a \leq m_a$, we get:

$$LHS = \sum \sqrt{r_a w_a m_a} \leq \sum \sqrt{r_a m_a m_a} = \sum m_a \sqrt{r_a} \overset{CBS}{\leq} \sqrt{\sum m_a^2 \cdot \sum r_a} =$$

$$= \sqrt{\frac{3}{4}\sum a^2 \cdot (4R+r)} \overset{(1)}{\leq} (4R+r)\sqrt{\frac{3R}{2}} = RHD$$

Where $(1) \Leftrightarrow \sqrt{\frac{3}{4}\sum a^2 \cdot (4R+r)} \leq (4R+r)\sqrt{\frac{3R}{2}} \Leftrightarrow$

$$\frac{3}{4}\sum a^2 \cdot (4R+r) \leq (4R+r)^2 \frac{3R}{2} \Leftrightarrow \sum a^2 \leq 2R(4R+r) \Leftrightarrow$$

$2(s^2 - r^2 - 4Rr) \leq 2R(4R+r) \Leftrightarrow s^2 \leq 4R^2 + 5Rr + r^2$, which follows from

$$s^2 \leq 4R^2 + 4Rr + 3r^2 (Gerretsen).$$

It remains to prove that:

$$4R^2 + 4Rr + 3r^2 \leq 4R^2 + 5Rr + r^2 \Leftrightarrow R \geq 2r (Euler).$$

Equality holds if and only if triangle ids equilateral.

3) In ΔABC the following relationship holds:

$$\sum \sqrt{r_a s_a m_a} \leq (4R+r)\sqrt{\frac{3R}{2}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Marin Chirciu-Romania

Using inequality $s_a = \frac{2bc}{b^2+c^2} m_a \leq m_a$, we get:

$$LHS = \sum \sqrt{r_a s_a m_a} \leq \sum \sqrt{r_a m_a m_a} = \sum m_a \sqrt{r_a} \overset{CBS}{\leq} \sqrt{\sum m_a^2 \cdot \sum r_a} =$$

$$= \sqrt{\frac{3}{4}\sum a^2 \cdot (4R+r)} \overset{(1)}{\leq} (4R+r)\sqrt{\frac{3R}{2}} = RHD$$

Where $(1) \Leftrightarrow \sqrt{\frac{3}{4}\sum a^2 \cdot (4R+r)} \leq (4R+r)\sqrt{\frac{3R}{2}} \Leftrightarrow$

$$\frac{3}{4}\sum a^2\cdot(4R+r)\le(4R+r)^2\frac{3R}{2}\Leftrightarrow\sum a^2\le 2R(4R+r)\Leftrightarrow$$

$2(s^2-r^2-4Rr)\le 2R(4R+r)\Leftrightarrow s^2\le 4R^2+5Rr+r^2$, which follows from

$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$.

It remains to prove that:

$$4R^2+4Rr+3r^2\le 4R^2+5Rr+r^2\Leftrightarrow R\ge 2r(Euler).$$

Equality holds if and only if triangle ids equilateral.

4) In ΔABC the following relationship holds:

$$\sum m_a\sqrt{r_a}\le(4R+r)\sqrt{\frac{3R}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using CBS inequality, we have:

$$LHS=\sum m_a\sqrt{r_a}\overset{CBS}{\le}\sqrt{\sum m_a^2\cdot\sum r_a}=\sqrt{\frac{3}{4}\sum a^2\cdot(4R+r)}\overset{(1)}{\le}$$

$$\overset{(1)}{\le}(4R+r)\sqrt{\frac{3R}{2}}=RHD$$

$$(1)\Leftrightarrow\sqrt{\frac{3}{4}\sum a^2\cdot(4R+r)}\le(4R+r)\sqrt{\frac{3R}{2}}\Leftrightarrow$$

$$\frac{3}{4}\sum a^2\cdot(4R+r)\le(4R+r)^2\frac{3R}{2}\Leftrightarrow\sum a^2\le 2R(4R+r)\Leftrightarrow$$

$2(s^2-r^2-4Rr)\le 2R(4R+r)\Leftrightarrow s^2\le 4R^2+5Rr+r^2$, which follows from $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$.

Remains to prove that:

$$4R^2+4Rr+3r^2\le 4R^2+5Rr+r^2\Leftrightarrow R\ge 2r(Euler).$$

Equality if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum a\sqrt{r_a} \le (4R+r)\sqrt{2R}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using CBS inequality, we have:

$$LHS = \sum a\sqrt{r_a} \overset{CBS}{\le} \sqrt{\sum a^2 \cdot \sum r_a} = \sqrt{\sum a^2 \cdot (4R+r)} \overset{(1)}{\le}$$

$$\overset{(1)}{\le} (4R+r)\sqrt{2R} = RHD$$

$$(1) \Leftrightarrow \sqrt{\sum a^2 \cdot (4R+r)} \le (4R+r)\sqrt{2R} \Leftrightarrow$$

$$\sum a^2 \cdot (4R+r) \le (4R+r)^2 2R \Leftrightarrow \sum a^2 \le 2R(4R+r) \Leftrightarrow$$

$2(s^2-r^2-4Rr) \le 2R(4R+r) \Leftrightarrow s^2 \le 4R^2+5Rr+r^2$, **which follows from** $s^2 \le 4R^2+4Rr+3r^2(Gerretsen)$.

Remains to prove that:

$$4R^2+4Rr+3r^2 \le 4R^2+5Rr+r^2 \Leftrightarrow R \ge 2r(Euler).$$

Equality if and only if triangle is equilateral.

Reference:

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