ABOUT AN INEQUALITY BY NGUYEN VAN CANH-VI

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1) In ΔABC the following relationship holds:

$$\sum \frac{a}{b+c} + \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}} \geq 2$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Marin Chirciu-Romania

Using identities: $\sum \frac{a}{b+c} = \frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}, abc = 4Rrs, \prod(b+c) = 2s(s^2+r^2+2Rr)$,

the desired inequality becomes:

$$\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr} + \sqrt{\frac{2 \cdot 4Rrs}{2s(s^2+r^2+2Rr)}} \geq 2 \Leftrightarrow$$

$$\frac{s^2-r^2-Rr}{s^2+r^2+2Rr} + \sqrt{\frac{Rr}{s^2+r^2+2Rr}} \geq 1 \Leftrightarrow \sqrt{\frac{Rr}{s^2+r^2+2Rr}} \geq 1 - \frac{s^2-r^2-Rr}{s^2+r^2+2Rr}$$

$$\Leftrightarrow \sqrt{\frac{Rr}{s^2+r^2+2Rr}} \geq \frac{r(3R+2r)}{s^2+r^2+2Rr} \Leftrightarrow \frac{Rr}{s^2+r^2+2Rr} \geq \frac{r^2(3R+2r)^2}{(s^2+r^2+2Rr)^2} \Leftrightarrow$$

$R(s^2+r^2+2Rr) \geq r(3R+2r)^2$, which follows from $s^2 \geq 16Rr - 5r^2 (Gerretsen)$.

Remains to prove that:

$$R(16Rr - 5r^2 + r^2 + 2Rr) \geq r(3R+2r)^2 \Leftrightarrow 2R(9R-2r) \geq (3R+2r)^2$$

$\Leftrightarrow 9R^2 - 16Rr - 4r^2 \geq 0 \Leftrightarrow (R-2r)(9R+2r) \geq 0$, which is true from

$$R \geq 2r (Euler).$$

Equality holds if and only if triangle is equilateral.

Remark. Let's find an reverse inequality.

2) In ΔABC the following relationship holds:

$$\sum\frac{a}{b+c}+\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}\le\frac{R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using identities: $\sum\frac{a}{b+c}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}, abc=4Rrs, \prod(b+c)=2s(s^2+r^2+2Rr),$

the desired inequality becomes:

$$\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}+\sqrt{\frac{2\cdot 4Rrs}{2s(s^2+r^2+2Rr)}}\le\frac{R}{r}\Leftrightarrow$$

$$\frac{s^2-r^2-Rr}{s^2+r^2+2Rr}+\sqrt{\frac{Rr}{s^2+r^2+2Rr}}\le\frac{R}{2r}\Leftrightarrow\sqrt{\frac{Rr}{s^2+r^2+2Rr}}\le\frac{R}{2r}-\frac{s^2-r^2-Rr}{s^2+r^2+2Rr}$$

$$\Leftrightarrow\sqrt{\frac{Rr}{s^2+r^2+2Rr}}\le\frac{s^2(R-2r)+r(2R^2+3Rr+2r^2)}{2r(s^2+r^2+2Rr)}$$

$$\Leftrightarrow\frac{Rr}{s^2+r^2+2Rr}\le\frac{[s^2(R-2r)+r(2R^2+3Rr+2r^2)]^2}{4r^2(s^2+r^2+2Rr)^2}\Leftrightarrow$$

$$s^4(R-2r)^2+s^2r(4R^3-2R^2r-12Rr^2-8r^3)$$
$$+r^2(4R^4+12R^3r+9R^2r^2+8Rr^3+4r^4)\ge 0\Leftrightarrow$$

$$s^2[s^2(R-2r)^2+r(4R^3-2R^2r-12Rr^2-8r^3)]$$
$$+r^2(4R^4+12R^3r+9R^2r^2+8Rr^3+4r^4)\ge 0$$

We distinguish the cases:

Case 1) If $[s^2(R-2r)^2+r(4R^3-2R^2r-12Rr^2-8r^3)]\ge 0$ **inequality is obviously true.**

Case 2) If $[s^2(R-2r)^2+r(4R^3-2R^2r-12Rr^2-8r^3)]<0$ **inequality can be written:**

$r^2(4R^4+12R^3r+9R^2r^2+8Rr^2+4r^4)\ge s^2[-r(4R^3-2R^2r-12Rr^2-8r^3)-$
$-s^2(R-2r)^2]$ **which follows from**

$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen).$$

Remains to prove that:

$$r^2(4R^4+12R^3r+9R^2r^2+8Rr^3+4r^4)\ge$$
$$\ge(4R^2+4Rr+3r^2)[-r(4R^3-2R^2r-12Rr^2-8r^3)-(16Rr-5r^2)(R-2r)^2]\Leftrightarrow$$
$$r^2(4R^4+12R^3r+9R^2r^2+8Rr^3+4r^4)\ge$$
$$\ge(4R^2+4Rr+3r^2)(-20R^3+71R^2r-72Rr^2+28r^3)\Leftrightarrow$$
$$r^2(4R^4+12R^3r+9R^2r^2+8Rr^3+4r^4)\ge$$
$$\ge-80R^5+204R^4r-64R^3r^2+37R^2r^3-104Rr^4+84r^5\Leftrightarrow$$
$$20R^5-50R^4r+19R^3r^2-7R^2r^3+28Rr^4-20r^5\ge0\Leftrightarrow$$

$(R-2r)(20R^4-10R^3r-R^2r^2-9Rr^3+10r^4)\ge0$ **true from** $R\ge2r(Euler)$.

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$2\le\sum\frac{a}{b+c}+\sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}\le\frac{R}{r}$$

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Solution by proposer

See up these inequalities.

Equality holds if and only if triangle is equilateral.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro