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ABOUT AN INEQUALITY FROM SAUDI ARABIA MATHEMATICAL COMPETITION

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1) If $a,b,c>0$ then

$$a\sqrt{3a^2+6b^2}+b\sqrt{3b^2+6c^2}+c\sqrt{3c^2+6a^2}\geq(a+b+c)^2$$

Saudi Arabia Mathematical Competition

Solution.

$$a\sqrt{3a^2+6b^2}+b\sqrt{3b^2+6c^2}+c\sqrt{3c^2+6a^2}\geq(a+b+c)^2$$

$$\Leftrightarrow a\sqrt{3a^2+6b^2}+b\sqrt{3b^2+6c^2}+c\sqrt{3c^2+6a^2}\geq a^2+b^2+c^2+2ab+2bc+2ca$$

$$\left(a\sqrt{3a^2+6b^2}-a^2-2ab\right)+\left(b\sqrt{3b^2+6c^2}-b^2-2bc\right)+\left(c\sqrt{3c^2+6a^2}-c^2-2ca\right)\geq 0$$

$$a\left(\sqrt{3a^2+6b^2}-a-2b\right)+b\left(\sqrt{3b^2+6c^2}-b-2c\right)+c\left(\sqrt{3c^2+6a^2}-c-2a\right)\geq 0,(1)$$

which follows from: $\sqrt{3a^2+6b^2}-a-2b\geq 0\Leftrightarrow\sqrt{3a^2+6b^2}\geq a+2b\Leftrightarrow$

$3a^2+6b^2\geq(a+2b)^2\Leftrightarrow 2(a-b)^2\geq 0$. Equality holds for $a=b$.

Similarly, $\sqrt{3b^2+6c^2}-b-2c\geq 0$ and $\sqrt{3c^2+6a^2}-c-2a\geq 0$

Remark. The problem can be developed.

2) If $a,b,c\geq 0$ and $x\geq 1, y\geq 4, xy=4x+y$ then:

$$a\sqrt{xa^2+yb^2}+b\sqrt{xb^2+yc^2}+c\sqrt{xc^2+ya^2}\geq(a+b+c)^2$$

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Solution.

$$a\sqrt{xa^2+yb^2}+b\sqrt{xb^2+yc^2}+c\sqrt{xc^2+ya^2}\geq(a+b+c)^2\Leftrightarrow$$

$$a\sqrt{xa^2+yb^2}+b\sqrt{xb^2+yc^2}+c\sqrt{xc^2+ya^2}\geq a^2+b^2+c^2+2ab+2bc+2ca\Leftrightarrow$$

$$\left(a\sqrt{xa^2+yb^2}-a^2-2ab\right)+\left(b\sqrt{xb^2+yc^2}-b^2-2bc\right)+\left(c\sqrt{xc^2+ya^2}-c^2-2ca\right)\geq 0$$

$$a\left(\sqrt{xa^2+yb^2}-a-2b\right)+b\left(\sqrt{xb^2+yc^2}-b-2c\right)+c\left(\sqrt{xc^2+ya^2}-c-2a\right)\geq 0;(1)$$

Inequality (1) it follows from: $\sqrt{xa^2+yb^2}-a-2b\geq 0\Leftrightarrow\sqrt{xa^2+yb^2}\geq a+2b\Leftrightarrow$

$$xa^2+yb^2\ge(a+2b)^2\Leftrightarrow xa^2+yb^2\ge a^2+4ab+4b^2\Leftrightarrow$$

$(x-1)a^2-4ab+(y-4)b^2\ge 0\Leftrightarrow\left(a\sqrt{x-1}-b\sqrt{y-4}\right)^2\ge 0$, which is true from conditions $x\ge 1, y\ge 4, xy=4x+y\Leftrightarrow(x-1)(y-4)=4$.

Equality holds for $a\sqrt{x-1}=b\sqrt{y-4}$.

Similarly, $\sqrt{xb^2+yc^2}-b-2c\ge 0$ and $\sqrt{xc^2+ya^2}-c-2a\ge 0\Rightarrow(1)$ −true.

Note.

For $(x,y)=(3,6)$, we get problem Saudi Arabia Mathematical Competition, 2020.

3) If $a,b,c\ge 0$ then:

$$a\sqrt[3]{9(a^3+2b^3)}+b\sqrt[3]{9(b^3+2c^3)}+c\sqrt[3]{9(c^3+2a^3)}\ge(a+b+c)^2$$

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Solution.

$$a\sqrt[3]{9(a^3+2b^3)}+b\sqrt[3]{9(b^3+2c^3)}+c\sqrt[3]{9(c^3+2a^3)}\ge(a+b+c)^2\Leftrightarrow$$

$$a\sqrt[3]{9(a^3+2b^3)}+b\sqrt[3]{9(b^3+2c^3)}+c\sqrt[3]{9(c^3+2a^3)}\ge a^2+b^2+c^2+2ab+2bc+2ca\Leftrightarrow$$

$$\left(a\sqrt[3]{9(a^3+2b^3)}-2a^2-2ab\right)+\left(b\sqrt[3]{9(b^3+2c^3)}-b^2-2bc\right)$$
$$+\left(c\sqrt[3]{9(c^3+2a^3)}-2c^2-2ca\right)\ge 0$$

$$a\left(\sqrt[3]{9(a^3+2b^3)}-a-2b\right)+b\left(\sqrt[3]{9(b^3+2c^3)}-b-2c\right)+c\left(\sqrt[3]{9(c^3+2a^3)}-c-2a\right)\ge 0;\ (1)$$, which follows from: $\sqrt[3]{9(a^3+2b^3)}-a-2b\ge 0\Leftrightarrow$

$$\sqrt[3]{9(a^3+2b^3)}\ge a+2b\Leftrightarrow 9(a^3+2b^3)\ge(a+2b)^3\Leftrightarrow$$

$4a^3-3a^2b-6ab^2+5b^2\ge 0\Leftrightarrow(a-b)^2(4a+5b)\ge 0$. Equality holds for $a=b$.

Similarly, $\sqrt[3]{9(b^3+2c^3)}-b-2c\ge 0$ and $\sqrt[3]{9(c^3+2a^3)}-c-2a\ge 0$.

Equality hold if and only if $a=b=c$.

4) If $a,b,c\ge 0$ then:

$$a\sqrt[4]{27(a^4+2b^4)}+b\sqrt[4]{27(b^4+2c^4)}+c\sqrt[4]{27(c^4+2a^4)}\ge(a+b+c)^2$$

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Solution.

$$a\sqrt[4]{27(a^4+2b^4)}+b\sqrt[4]{27(b^4+2c^4)}+c\sqrt[4]{27(c^4+2a^4)}\geq(a+b+c)^2\Leftrightarrow$$

$$a\sqrt[4]{27(a^4+2b^4)}+b\sqrt[4]{27(b^4+2c^4)}+c\sqrt[4]{27(c^4+2a^4)}\geq a^2+b^2+c^2+2ab+2bc+2ca$$

$$\Leftrightarrow\left(a\sqrt[4]{27(a^4+2b^4)}-a^2-2ab\right)+\left(b\sqrt[4]{27(b^4+2c^4)}-2bc\right)$$

$$+\left(c\sqrt[4]{27(c^4+2a^4)}-2ca\right)\geq 0\Leftrightarrow$$

$a\left(\sqrt[4]{27(a^4+2b^4)}-a-2b\right)+b\left(\sqrt[4]{27(b^4+2c^4)}-b-2c\right)+c\left(\sqrt[4]{27(c^4+2a^4)}-c-2a\right)\geq 0$, which is true from $\sqrt[4]{27(a^4+2b^4)}-a-2b\geq 0\Leftrightarrow$

$$\sqrt[4]{27(a^4+2b^4)}\geq a+2b\Leftrightarrow 27(a^4+2b^4)\geq(a+2b)^4\Leftrightarrow$$

$$13a^4-4a^3b-12a^2b^2-16ab^3+19b^4\geq 0\Leftrightarrow(a-b)^2(13a^2+22ab+19b^2)\geq 0$$

Equality holds for $a=b$

Similarly, $\sqrt[4]{27(b^4+2c^4)}-b-2c\geq 0$ and $\sqrt[4]{27(c^4+2a^4)}-c-2a\geq 0$.

Equality holds if and only if $a=b=c$.

5) If $a,b,c\geq 0$ and $n\in\mathbb{N},n\geq 2$ then:

$$a\sqrt[n]{3^{n-1}(a^n+2b^n)}+b\sqrt[n]{3^{n-1}(b^n+2c^n)}+c\sqrt[n]{3^{n-1}(c^n+2b^n)}\geq(a+b+c)^2$$

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Solution.

$$a\sqrt[n]{3^{n-1}(a^n+2b^n)}+b\sqrt[n]{3^{n-1}(b^n+2c^n)}+c\sqrt[n]{3^{n-1}(c^n+2b^n)}\geq(a+b+c)^2\Leftrightarrow$$

$$a\sqrt[n]{3^{n-1}(a^n+2b^n)}+b\sqrt[n]{3^{n-1}(b^n+2c^n)}+c\sqrt[n]{3^{n-1}(c^n+2b^n)}$$

$$\geq a^2+b^2+c^2+2ab+2bc+2ca\Leftrightarrow$$

$$\left(a\sqrt[n]{3^{n-1}(a^n+2b^n)}-a^2-2ab\right)+\left(b\sqrt[n]{3^{n-1}(b^n+2c^n)}-2b^2-2bc\right)+$$

$\left(c\sqrt[n]{3^{n-1}(c^n+2b^n)}-c^2-2ca\right)\geq 0,(1)$ which follows from

$$a\sqrt[n]{3^{n-1}(a^n+2b^n)}-a^2-2ab\geq 0\Leftrightarrow a\sqrt[n]{3^{n-1}(a^n+2b^n)}\geq a^2+2ab\Leftrightarrow$$

$$3^{n-1}(a^n+2b^n)\geq(a+2b)^n\Leftrightarrow\frac{a^n+2b^n}{3}\geq\left(\frac{a+2b}{3}\right)^n\ (Jensen).$$

Equality holds for $a = b$.

Similarly, $b\sqrt[n]{3^{n-1}(b^n + 2c^n)} - 2b^2 - 2bc \geq 0$ and $c\sqrt[n]{3^{n-1}(c^n + 2b^n)} - c^2 - 2ca \geq 0$.

Equality holds if $a = b = c$.

Note.

For $n = 2$, we get Problem 2, Saudi Arabia Mathematical Competition, 2020.