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AMAZING IDENTITIES AND INEQUALITIES WITH MEDIANS

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In ΔABC the following relationship holds:

$$cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{r_br_c}{bc}}$$

$$\begin{cases}(b+c)^2=b^2+c^2+2bc\\(b-c)^2=b^2+c^2-2bc\end{cases}\Rightarrow(b+c)^2+(b-c)^2=4bc$$

$$((b+c)^2+(b-c)^2)cos^2\frac{A}{2}=4bc\cdot\frac{r_br_c}{bc}=4r_br_c;(1)$$

$$m_a^2=\frac{2(b^2+c^2)-a^2}{4}\Rightarrow4m_a^2=2(b^2+c^2)-a^2$$

$$r_br_c=s(s-a)=\frac{(a+b+c)(b+c-a)}{4}\Rightarrow4r_br_c=(b+c)^2-a^2$$

$$4r_br_c+(b-c)^2=2(b^2+c^2)-a^2\Rightarrow4m_a^2=4r_br_c+(b-c)^2;(2)$$

From (1),(2) it follows that:

$$4m_a^2=(b+c)cos^2\frac{A}{2}+(b-c)^2-(b-c)^2cos^2\frac{A}{2}$$

Therefore, we get a new inequality:

$$4m_a^2=(b+c)^2cos^2\frac{A}{2}+(b-c)^2sin^2\frac{A}{2}$$

Next, $4m_a^2\geq(b+c)^2cos^2\frac{A}{2}\Rightarrow m_a\geq\frac{b+c}{2}cos\frac{A}{2}$

$$bc=r_br_c+rr_a(and\ analogs),m_ag_a\geq m_aw_a\geq r_br_c$$

$$(b-c)^2=b^2+c^2-2bc=n_a^2+g_a^2-2r_br_c,(n_a-g_a)^2=n_a^2+g_a^2-2n_ag_a\Rightarrow$$

$|b-c|\geq n_a-g_a$. **So, we get:**

$$4m_a^2=(b+c)^2cos^2\frac{A}{2}+(n_a-g_a)^2sin^2\frac{A}{2}$$

But $\begin{cases}4m_a^2=n_a^2+g_a^2+2r_br_c\\(n_a+g_a)^2=n_a^2+g_a^2+2n_ag_a\end{cases}\Rightarrow n_a+g_a\geq2m_a(and\ analogs).$

Hence, $n_a-g_a\geq2(m_a-g_a),(and\ analogs)$**.**

Therefore, we get a new inequality:

$$4m_a^2 \geq (b+c)^2 cos^2\frac{A}{2} + 4(m_a - g_a)^2 sin^2\frac{A}{2}$$

Next,

$$g_a \leq AI + r, w_a = AI + \frac{w_a r}{h_a} \Rightarrow g_a \leq w_a \Rightarrow n_a - g_a \geq 2(m_a - w_a), (and\ analogs).$$

$$4m_a^2 \geq (b+c)^2 cos^2\frac{A}{2} + 4(m_a - w_a)^2 sin^2\frac{A}{2}$$

$$4m_a^2 \geq (b+c)^2 cos^2\frac{A}{2} + (n_a - w_a)^2 sin^2\frac{A}{2}$$

We know that: $4m_a^2 = (b+c)^2 cos^2\frac{A}{2} + (b-c)^2 sin^2\frac{A}{2}$,

$$\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} \Rightarrow \sqrt{x^2+y^2} \geq \frac{1}{\sqrt{2}}(x+y)$$

Let us denote $x = (b+c)cos\frac{A}{2}, y = \frac{|b-c|sinA}{2}$, then we have:

$$2m_a \geq \frac{1}{\sqrt{2}}\left((b+c)cos\frac{A}{2} + \frac{|b-c|sinA}{2}\right)$$

Therefore, we get a new inequality:

$$m_a \geq \frac{1}{2\sqrt{2}}\left((b+c)cos\frac{A}{2} + \frac{|b-c|sinA}{2}\right)$$

$$w_a = \frac{2bc}{b+c}cos\frac{A}{2}, a = 4Rsin\frac{A}{2}cos\frac{A}{2}$$

$$m_a w_a \geq \frac{1}{2\sqrt{2}} \cdot 2\left((b+c)cos^2\frac{A}{2} \cdot \frac{bc}{b+c} + \frac{|b-c|sinA}{2} \cdot \frac{bc}{b+c}cos\frac{A}{2}\right)$$

$$m_a w_a \geq \frac{1}{\sqrt{2}}\left(bccos^2\frac{A}{2} + \frac{|b-c|}{b+c} \cdot bc \cdot \frac{a}{4R}\right), m_a w_a \geq \frac{1}{\sqrt{2}}\left(bc \cdot \frac{s(s-a)}{bc} + \frac{|b-c|}{b+c} \cdot \frac{abc}{4R}\right)$$

$$m_a w_a \geq \frac{1}{\sqrt{2}}\left(s(s-a) + \frac{|b-c|}{b+c} \cdot F\right)$$

Therefore, we get a new inequality:

$$\sum_{cyc} m_a w_a \ge \frac{1}{\sqrt{2}}\left(s^2 + F\sum_{cyc}\frac{|b-c|}{b+c}\right)$$

$$s^2 = n_a^2 + 2r_a h_a, 3(m_a w_a + m_b w_b + m_c w_c) \ge \frac{1}{\sqrt{2}}\left(3s^2 + 3F\sum_{cyc}\frac{|b-c|}{b+c}\right)$$

Therefore, we get a new inequality:

$$3(m_a w_a + m_b w_b + m_c w_c) \ge \frac{1}{\sqrt{2}}\left(\sum_{cyc} n_a^2 + 2\sum_{cyc} r_a h_a + 3F\sum_{cyc}\frac{|b-c|}{b+c}\right)$$

But $x^2 + y^2 + z^2 \ge xy + yz + zx, \forall x, y, z \in \mathbb{R} \Rightarrow n_a^2 + n_b^2 + n_c^2 \ge n_a n_b + n_b n_c + n_c n_a$

Hence, we get:

$$3(m_a w_a + m_b w_b + m_c w_c) \ge \frac{1}{\sqrt{2}}\left(\sum_{cyc} n_a n_b + 2\sum_{cyc} r_a h_a + 3F\sum_{cyc}\frac{|b-c|}{b+c}\right)$$

We known that: $AI = \frac{r}{\sin\frac{A}{2}} = \frac{s-a}{\cos\frac{A}{2}}(and\ analogs).$

$$m_a \cdot AI \ge \frac{1}{2\sqrt{2}}\left[(b+c)\cos\frac{A}{2}\cdot\frac{s-a}{\cos\frac{A}{2}} + \frac{|b-c|\sin A}{2}\cdot\frac{r}{\sin\frac{A}{2}}\right]$$

Therefore, we get a new inequality:

$$m_a \cdot AI \ge \frac{1}{2\sqrt{2}}[(b+c)(s-a) + r|b-c|]$$

But $(b+c)(s-a) = \frac{(b+c)(b+c-a)}{2} = \frac{(b+c)^2 - a(b+c)}{2}$

$$\sum_{cyc}(b+c)(s-a) = \frac{(b+c)^2 - a(b+c) + (a+c)^2 - b(a+c) + (a+b)^2 - c(a+b)}{2}$$

$$= a^2 + b^2 + c^2$$

Therefore, we get a new inequality:

$$\sum_{cyc} m_a \cdot AI \ge \frac{1}{2\sqrt{2}}\left(a^2 + b^2 + c^2 + r\sum_{cyc}|b-c|\right)$$

$AI^2 = bc - 4Rr,\ bc = 2Rh_a,\quad AI^2 = 2R(h_a - 2r) \Rightarrow AI = \sqrt{2R(h_a - 2r)}$

Therefore, we get a new inequality:

$$\sum_{cyc} m_a \cdot \sqrt{h_a - 2r} \geq \frac{1}{4\sqrt{R}}\left(a^2 + b^2 + c^2 + r\sum_{cyc}|b - c|\right)$$

We prove it that:

$$m_a \geq \frac{1}{2\sqrt{2}}\left[(b + c)\cos\frac{A}{2} + \frac{|b - c|\sin A}{2}\right]$$

$$2m_a \geq \frac{1}{\sqrt{2}}\left[(b + c)\cos\frac{A}{2} + \frac{|b - c|\sin A}{2}\right],\ n_a + g_a \geq 2m_a$$

Therefore, we get a new inequality:

$$n_a g_a \geq \frac{1}{\sqrt{2}}\left[(b + c)\cos\frac{A}{2} + \frac{|b - c|\sin A}{2}\right]$$

But $n_a g_a \geq m_a w_a \Rightarrow \frac{n_a g_a}{w_a} \geq m_a$**.Therefore, we get a new inequality:**

$$\frac{n_a g_a}{w_a} \geq \frac{1}{2\sqrt{2}}\left[(b + c)\cos\frac{A}{2} + \frac{|b - c|\sin A}{2}\right]$$

Reference:

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