A NEW PROOF FOR MITRINOVIC'S INEQUALITY USING TOSCANO'S IDENTITY

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ABSTRACT. In this paper it's proved Mitrinovic's inequality using a famous identity.

Notations:

r - inradii; R - circumradii; r_a, r_b, r_c - exradii; s - semiperimeter; F - area.

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{s-a}{F}+\frac{s-b}{F}+\frac{s-c}{F}=\frac{3s-(a+b+c)}{F}=$$

$$=\frac{3s-2s}{F}=\frac{s}{F}=\frac{1}{\frac{F}{s}}=\frac{1}{r}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \tag{1}$$

In the well known identity:

$$(x+y+z)^3=x^3+y^3+z^3+3(x+y)(y+z)(z+x); x,y,z\in\mathbb{R}$$

We take:

$$x=\frac{1}{r_a}; y=\frac{1}{r_b}; z=\frac{1}{r_c}$$

$$\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)^3=\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}+3\prod_{cyc}\left(\frac{1}{r_a}+\frac{1}{r_b}\right)$$

By (1):

$$\left(\frac{1}{r}\right)^3=\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}+3\prod_{cyc}\left(\frac{s-a}{F}+\frac{s-b}{F}\right)$$

$$\frac{1}{r^3}-\frac{1}{r_a^3}-\frac{1}{r_b^3}-\frac{1}{r_c^3}=\frac{3}{F^3}\prod_{cyc}(s-a+s-b)$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{3}{F^3}\prod_{cyc}(a+b+c-a-b)$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{3}{F^3}\cdot abc$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{3\cdot 4RF}{F^3}$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{12R}{F^2} \tag{2}$$

Relation (2) it's called Toscano's identity.
Let be $f:(0,\infty)\to\mathbb{R}; f(x)=x^3; f'(x)=3x^2; f''(x)=6x>0; f$ convexe.
By Jensen's inequality:

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)$$

Again: $x=\frac{1}{r_a}; y=\frac{1}{r_b}; z=\frac{1}{r_c}$.

$$f\left(\frac{1}{r_a}\right)+f\left(\frac{1}{r_b}\right)+f\left(\frac{1}{r_c}\right)\geq 3f\left(\frac{1}{3}\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\right)$$

By (1):

$$\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\geq 3f\left(\frac{1}{3r}\right)=3\cdot\frac{1}{27r^3}=\frac{1}{9r^3}$$

$$-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)\leq -\frac{1}{9r^3}$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)\leq \frac{1}{r^3}-\frac{1}{9r^3}$$

By (2):

$$\frac{12R}{F^2}\leq\frac{1}{r^3}-\frac{1}{9r^3}$$

$$\frac{12R}{r^2s^2}\leq\frac{8}{9r^3}$$

$$\frac{3R}{s^2}\leq\frac{2}{9r}$$

$$2s^2\geq 27Rr\overset{\text{EULER}}{\geq}27\cdot 2r\cdot r=54r^2$$

$$2s^2\geq 54r^3\Rightarrow s^2\geq 27r^2$$

$$s\geq 3\sqrt{3}r$$

which it's Mitrinovic's inequality.

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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