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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-XIX

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1) In ΔABC the following relationship holds:

$$\prod_{cyc}\frac{m_b+m_c}{m_a}+\frac{9F^2}{m_am_bm_c(m_a+m_b+m_c)}\ge 9$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution.

Using $R_m=\frac{m_am_bm_c}{3F}, r_m=\frac{3F}{2(m_a+m_b+m_c)}\to\frac{R_m}{r_m}=\frac{2m_am_bm_c(m_a+m_b+m_c)}{9F^2}$, it follows that:

$\prod_{cyc}\frac{m_b+m_c}{m_a}+\frac{2r_m}{R_m}\ge 9; (1)$. It is enough to prove that:

$\prod_{cyc}\frac{b+c}{a}+\frac{2r}{R}\ge 9; (2)$, which follows from:

$$\prod_{cyc}\frac{b+c}{a}+\frac{2r}{R}\ge 9\Leftrightarrow\frac{\prod(b+c)}{abc}+\frac{2r}{R}\ge 9\Leftrightarrow\frac{2s(s^2+r^2+2Rr)}{4Rrs}+\frac{2r}{R}\ge 9\Leftrightarrow$$

$\Leftrightarrow s^2\ge 16Rr-5r^2(Gerretsen)$. Equality holds if and only if triangle is equilateral.

2) In ΔABC the following relationship holds:

$$\prod_{cyc}\frac{m_b+m_c}{m_a}+\frac{\lambda F^2}{m_am_bm_c(m_a+m_b+m_c)}\ge 8+\frac{\lambda}{9},\qquad \lambda\le 9$$

Proposed by Marin Chirciu-Romania

Solution.

Using $R_m=\frac{m_am_bm_c}{3F}, r_m=\frac{3F}{2(m_a+m_b+m_c)}\to\frac{R_m}{r_m}=\frac{2m_am_bm_c(m_a+m_b+m_c)}{9F^2}$, it follows that:

$\prod_{cyc}\frac{m_b+m_c}{m_a}+\frac{2\lambda r_m}{R_m}\ge 8+\frac{\lambda}{9}; (1)$. It is enough to prove that:

$\prod_{cyc}\frac{b+c}{a}+\frac{\lambda r}{R}\ge 8+\frac{\lambda}{9}; (2)$, which follows from:

$$\prod_{cyc}\frac{b+c}{a}+\frac{2\lambda r}{R}\ge 8+\frac{\lambda}{9}\Leftrightarrow\frac{\prod(b+c)}{abc}+\frac{2\lambda r}{R}\ge 8+\frac{\lambda}{9}\Leftrightarrow$$

$$\frac{2s(s^2+r^2+2Rr)}{4Rrs}+\frac{2\lambda r}{R}\geq 8+\frac{\lambda}{9}\Leftrightarrow 9s^2\geq(2\lambda+126)Rr-(4\lambda+9)r^2$$

-which follows from $s^2\geq 16Rr-5r^2(Gerretsen)$. Remains to prove that:

$$9(16Rr-5r^2)\geq(2\lambda+126)Rr-(4\lambda+9)r^2\Leftrightarrow(R-2r)(9-\lambda)\geq 0$$

-which is true from $R\geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

Note.

For $\lambda=9$ we obtain Inequality in triangle-1779, proposed by Adil Abdullayev-Baku-Azerbaijan in RMM 8/2020.

REFERENCE:

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