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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-VIII

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1) In ΔABC the following relationship holds:

$$\frac{24r^2}{R} \le \frac{a^2}{m_a} + \frac{b^2}{m_a} + \frac{c^2}{m_c} \le \frac{4R^2 - 2Rr}{r}$$

Proposed by George Apostolopoulos-Greece

Solution.

For RHS using $m_a \ge \frac{b^2+c^2}{4R}$ $(Tereshin)$, we get:

$$\sum_{cyc} \frac{a^2}{m_a} \le \sum_{cyc} \frac{a^2}{\frac{b^2+c^2}{4R}} = 4R \sum_{cyc} \frac{a^2}{b^2+c^2} \overset{AM-GM}{\le} 4R \sum_{cyc} \frac{a^2}{2bc} = \frac{2R}{abc} \sum_{cyc} a^3 =$$

$$= \frac{2R}{4Rrs} \cdot 2s(s^2 - 3r^2 - 6Rr) = \frac{1}{r}(s^2 - 3r^2 - 6Rr) \overset{Gerretsen}{\ge}$$

$$\ge \frac{1}{r}(4R^2 + 4Rr + 3r^2 - 3r^2 - 6Rr) = \frac{1}{r}(4R^2 - 2Rr) = \frac{4R^2 - 2Rr}{r}$$

Equality holds if and only if triangle is equilateral.

For LHS using Bergstrom Inequality, we have:

$$\sum_{cyc} \frac{a^2}{m_a} \ge \frac{(\sum a)^2}{\sum m_a} \ge \frac{(2s)^2}{4R+r} = \frac{4s^2}{4R+r} \overset{(1)}{\ge} \frac{24r^2}{R}; (1) \Leftrightarrow \frac{4s^2}{4R+r} \ge \frac{24r^2}{R} \Leftrightarrow$$

$Rs^2 \ge 6r^2(4R+r)$, which follows from $s^2 \ge 16Rr - 5r^2 (Gerretsen)$.

2) In ΔABC the following relationship holds:

$$\frac{4r(4R+r)}{R+r} \le \frac{a^2}{m_a} + \frac{b^2}{m_a} + \frac{c^2}{m_c} \le \frac{2R}{r}(2R-r) - \frac{3r(R-2r)}{2(2R-r)}$$

Proposed by Marin Chirciu-Romania

Solution.

For RHS using $m_a \ge \frac{b^2+c^2}{4R}$ $(Tereshin)$, we get:

$$\sum_{cyc}\frac{a^2}{m_a}\le\sum_{cyc}\frac{a^2}{\frac{b^2+c^2}{4R}}=4R\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{AM-GM}{\le}4R\sum_{cyc}\frac{a^2}{2bc}=\frac{2R}{abc}\sum_{cyc}a^3=$$

$$=\frac{2R}{4Rrs}\cdot 2s(s^2-3r^2-6Rr)=\frac{1}{r}(s^2-3r^2-6Rr)\overset{Blundon-Gerretsen}{\le}$$

$$\le\frac{1}{r}\left(\frac{R(4R+r)^2}{2(2R-r)}-3r^2-6Rr\right)=\frac{16R^3-16R^2r+Rr^2+6r^3}{2r(2R-r)}=$$

$$=\frac{4R(2R-r)^2-3r^2(R-2r)}{2r(2R-r)}=\frac{2R}{r}(2R-r)-\frac{3r(R-2r)}{2(2R-r)}$$

Equality holds if and only if triangle is equilateral.

For LHS using Bergstrom inequality, we get:

$$\sum_{cyc}\frac{a^2}{m_a}\ge\frac{(\sum a)^2}{\sum m_a}\ge\frac{(2s)^2}{4R+r}=\frac{4s^2}{4R+r}\overset{(1)}{\ge}\frac{4r(4R+r)}{R+r};(1)\Leftrightarrow\frac{4s^2}{4R+r}\ge\frac{4r(4R+r)}{R+r}\Leftrightarrow$$

$\frac{s^2}{4R+r}\ge\frac{r(4R+r)}{R+r}$, which follows from $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}(Gerretsen)$

$$\frac{s^2}{4R+r}\ge\frac{\frac{r(4R+r)^2}{R+r}}{4R+r}=\frac{r(4R+r)}{R+r}.$$

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\frac{24r^2}{R}\le\frac{4r(4R+r)}{R+r}\le\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}<-\frac{2R}{r}(2R-r)-\frac{3r(R-2r)}{2(2R-r)}\le\frac{4R^2-2Rr}{r}$$

Proposed by Marin Chirciu-Romania

Solution. See inequality 2) and $R\ge 2r(Euler)$. Equality holds if and only if triangle is equilateral.

REFERENCE:

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