R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT A RMM INEQUALITY-IX

By Marin Chirciu-Romania

1) In ΔABC the following inequality holds:

$$\frac{a^4}{r_a r_b}+\frac{b^4}{r_b r_c}+\frac{c^4}{r_c r_a}\ge\frac{16F}{\sqrt{3}}$$

D.M. Bătinețu-Giurgiu, Flaviu Cristian Verde

Solution:

We have

$$RHS=\sum\frac{a^4}{r_a r_b}=\frac{1}{r_a r_b r_c}\sum a^4 r_c=\frac{1}{r_a r_b r_c}\sum\frac{a^4}{\frac{1}{r_c}}\overset{Holder}{\ge}\frac{1}{rp^2}\cdot\frac{(\sum a)^4}{9\sum\frac{1}{r_a}}=$$

$=\frac{1}{rp^2}\cdot\frac{(2p)^4}{9\cdot\frac{1}{r}}=\frac{16p^2}{9}\overset{(1)}{\ge}\frac{16pr}{\sqrt{3}}=\frac{16F}{\sqrt{3}}=Md$, where (1) $\Leftrightarrow p\ge 3r\sqrt{3}$, (Mitrinovic inequality)

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

2) In ΔABC the following relationship holds:

$$48r^2\le\frac{a^4}{r_b r_c}+\frac{b^4}{r_c r_a}+\frac{c^4}{r_a r_b}\le\frac{16}{r}(R^3-5r^3)$$

Marin Chirciu

Solution: We prove:

Lemma:

3) In ΔABC the following relationship holds:

$$\sum\frac{a^4}{r_b r_c}=\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]$$

Proof:

$$\sum\frac{a^4}{r_b r_c}=\sum\frac{a^4}{\frac{S}{s-b}\frac{S}{p-c}}=\frac{1}{S^2}\sum s^4(s-b)(s-c)=$$

$$=\frac{1}{s^2r^2}\cdot 4rp^2[p^2(R-2r)+r^2(5R+2r)]=$$

$=\frac{4}{r}[s^2(R-2r)+r^2(5R+2r)]$, which follows from

$$\sum a^4(s-b)(s-c)=4rp^2[p^2(R-2r)+r^2(5R+2r)]$$

Back to the main problem.
RHS inequality.

$$\sum \frac{a^4}{r_br_c}=\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]\overset{Gerretsen}{\leq}$$
$$\leq \frac{4}{r}[(4R^2+4Rr+3r^2)(R-2r)+r^2(5R+2r)]=$$
$$=\frac{4}{r}(4R^3-4R^2r-4r^3)=\frac{16}{r}(R^3-R^2r-r^3)\overset{Euler}{\leq}\frac{16}{r}(R^3-5r^3)$$

Equality holds if and only if the triangle is equilateral.
LHS inequality

$$\sum \frac{a^4}{r_br_c}=\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]\overset{Gerretsen}{\geq}$$
$$\geq \frac{4}{r}[(16Rr-5r^2)(R-2r)+r^2(5R+2r)]=$$
$$=4(16R^2-32Rr+12r^2)\overset{Euler}{\geq}48r^2$$

Equality holds if and only if the triangle is equilateral.
Remark: In the same way:

4) In ΔABC the following relationship holds:

$$24Rr\leq \frac{a^4}{h_bh_c}+\frac{b^4}{h_ch_a}+\frac{c^4}{h_ah_b}\leq 4R^2\left(\frac{2R}{r}-1\right)$$

Marin Chirciu

Solution: We prove

Lemma:

5) In ΔABC the following relationship holds:

$$\sum \frac{a^4}{h_bh_c}=\frac{2R}{r}(p^2-3r^2-6Rr)$$

Proof:

$$\sum \frac{a^4}{h_bh_c}=\sum \frac{a^4}{\frac{2S}{b}\frac{2S}{c}}=\frac{1}{4S^2}\sum a^4bc=\frac{1}{4p^2r^2}\cdot abc\sum a^3=$$
$$=\frac{1}{4p^2r^2}\cdot 4Rrp\cdot 2p(p^2-3r^2-6Rr)=$$

$=\frac{2R}{r}(p^2-3r^2-6Rr)$, which follows from $\sum a^3=2p(p^2-3r^2-6Rr)$

Back to the main problem.
RHS inequality

$$\sum\frac{a^4}{h_bh_c}=\frac{2R}{r}(s^2-3r^2-6Rr)\overset{Gerretsen}{\leq}\frac{2R}{r}(4R^2+4Rr+3r^2-3r^2-6Rr)=$$

$$=\frac{2R}{r}(4R^2-2Rr)=\frac{4R^2}{r}(2R-r)=4R^2\left(\frac{2R}{r}-1\right)$$

Equality holds if and only if the triangle is equilateral.
LHS inequality

$$\sum\frac{a^4}{h_bh_c}=\frac{2R}{r}(p^2-3r^2-6Rr)\overset{Gerretsen}{\geq}\frac{2R}{r}(16Rr-5r^2-3r^2-6Rr)=$$

$$=\frac{2R}{r}(10Rr-8r^2)=4R(5R-4r)\overset{Euler}{\geq}24Rr$$

Equality holds if and only if the triangle is equilateral.
Remark: Between the sums $\sum\frac{a^4}{h_bh_c}$ and $\sum\frac{a^4}{r_br_c}$ the following relationship holds:

6) In ΔABC the following relationship holds:

$$\sum\frac{a^4}{h_bh_c}\leq\sum\frac{a^4}{r_br_c}$$

Marin Chirciu

Solution:

Using the sums $\sum\frac{a^4}{h_bh_c}=\frac{2R}{r}(p^2-3r^2-6Rr)$ and $\sum\frac{a^4}{r_br_c}=\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]$, the inequality can be written:

$$\frac{2R}{r}(p^2-3r^2-6Rr)\leq\frac{4}{r}[p^2(R-2r)+r^2(5R+2r)]\Leftrightarrow$$

$$\Leftrightarrow R(p^2-3r^2-6Rr)\leq 2[p^2(R-2r)+r^2(5R+2r)]\Leftrightarrow$$

$$\Leftrightarrow p^2(R-4r)+r(6R^2+13Rr+4r^2)\geq 0$$

We distinguish the following cases:
Case 1). If $(R-4r)\geq 0$, the inequality is obvious.
Case 2). If $(R-4r)<0$, the inequality can be rewritten:
$r(6R^2+13Rr+4r^2)\geq p^2(4r-R)$, which follows from Gerretsen's inequality:

$$p^2\leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$$r(6R^2+13Rr+4r^2)\geq(4R^2+4Rr+3r^2)(4r-R)\Leftrightarrow 2R^3-3R^2r-4r^3\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R^2+Rr+2r^2)\geq 0$, obviously from Euler's inequality $R\geq 2r$.
Equality holds if and only if the triangle is equilateral.

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro