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1301. **If $y'+\alpha y=|x^2|-[e^{|x-\beta|}]+1, y(0)=0$ then find the possible expression for $y(x)$ where $[\cdot]$ is the floor function, and $\alpha,\beta>0$. You are not to use Laplace transform.**

Proposed by Tobi Joshua-Nigeria

Solution by Akerele Olofin-Nigeria

Et us define $K(\alpha,\beta,x)=\int[e^{|x-\beta|}]e^{\alpha x}dx$. Recall if $\frac{dy}{dx}+Py=Q\to$

$$y\cdot e^{\alpha x}=\int|x^2|e^{\alpha x}dx-\int[e^{|x-\beta|}]e^{\alpha x}dx+\int e^{\alpha x}dx\to$$

$y=\frac{\alpha^2x^2-2\alpha x+2}{\alpha^3}-\frac{K(\alpha,\beta,x)}{e^{\alpha x}}+\frac{1}{\alpha}+\frac{C}{e^{\alpha x}}$, where C –is a constant.

Recall $y(0)=0\to C=K(\alpha,\beta,0)+\frac{\alpha^2-2}{\alpha^3}$

$$y=\frac{\alpha^2x^2-2\alpha x+2}{\alpha^3}-\frac{K(\alpha,\beta,x)}{e^{\alpha x}}+\frac{1}{\alpha}+\frac{K(\alpha,\beta,0)}{e^{\alpha x}}+\frac{\alpha^2-2}{\alpha^3e^{\alpha x}}$$

1302. **Show that:**

$$\sum_{n=0}^{\infty}\frac{\left((\sqrt{5}-2)\varphi^{n-1}+\varphi^{3n-1}\right)^2}{\varphi^{7n}}=\frac{\sqrt{5}+7}{\sqrt{5}-7}\sum_{n=0}^{\infty}\frac{(-1)^n\left((\sqrt{5}-2)\varphi^{n-1}+\varphi^{3n-1}\right)^2}{\varphi^{7n-1}}$$

-where φ –Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Dawid Bialek-Poland

$$S_1=\sum_{n=0}^{\infty}\frac{\left((\sqrt{5}-2)\varphi^{n-1}+\varphi^{3n-1}\right)^2}{\varphi^{7n}}=\sum_{n=0}^{\infty}\frac{a^2\cdot\varphi^{2n-2}+2a\cdot\varphi^{4n-2}+\varphi^{6n-2}}{\varphi^{7n}}=$$

$$=a^2\sum_{n=0}^{\infty}\varphi^{-5n-2}+2a\sum_{n=0}^{\infty}\varphi^{-3n-2}+\sum_{n=0}^{\infty}\varphi^{-n-2}, where\ a=\sqrt{5}-2$$

$$S_1=\frac{a^2}{\varphi^2}\sum_{n=0}^{\infty}\left(\frac{1}{\varphi^5}\right)^n+\frac{2a}{\varphi^2\sum_{n=0}^{\infty}\left(\frac{1}{\varphi^3}\right)^n}+\frac{1}{\varphi^2}\sum_{n=0}^{\infty}\left(\frac{1}{\varphi}\right)^n\overset{(*)}{=}\frac{a^2}{\varphi^2}\cdot\frac{1}{1-\frac{1}{\varphi^5}}+\frac{2a}{\varphi^2}\cdot\frac{1}{1-\frac{1}{\varphi^3}}+$$

$$+\frac{1}{\varphi^2}\cdot\frac{1}{1-\frac{1}{\varphi}}, where\ (*)\sum_{n=0}^{\infty}x^n \overset{def}{=} \frac{1}{1-x}, |x|<1$$

$$S_1=\frac{a^2}{\varphi^2}\cdot\frac{\varphi^5}{\varphi^5-1}+\frac{2a}{\varphi^2}\cdot\frac{\varphi^3}{\varphi^3-1}+\frac{1}{\varphi^2}\cdot\frac{\varphi}{\varphi-1}+2a\cdot\frac{\varphi}{\varphi^3-1}+\frac{1}{\varphi(\varphi-1)}=$$

$$=a^2\cdot\frac{\varphi^3}{\varphi^5-1}+2a\cdot\frac{\varphi}{\varphi^3-1}+\frac{1}{\varphi(\varphi-1)}=$$

$$=a^2\cdot\frac{\varphi^3}{\varphi^5-1}+2a\cdot\frac{\varphi}{\underbrace{(\varphi-1)}_{\frac{1}{\varphi}}\underbrace{(\varphi^2+\varphi+1)}_{2\varphi^2}}+\frac{1}{\underbrace{\varphi(\varphi-1)}_{\frac{1}{\varphi}}}$$

$$S_1=a^2\cdot\frac{\varphi^3}{\varphi^5-1}+a+1\overset{a=\sqrt{5}-2}{=}\left(\sqrt{5}-2\right)^2\cdot\frac{\varphi^3}{\varphi^5-1}+\sqrt{5}-2+1=$$

$$=\left(9-4\sqrt{5}\right)\cdot\frac{\varphi^3}{\varphi^5-1}+\sqrt{5}-1\overset{\varphi=\frac{1+\sqrt{5}}{2}}{=}\left(9-4\sqrt{5}\right)\cdot\frac{\left(\frac{1+\sqrt{5}}{2}\right)^3}{\left(\frac{1+\sqrt{5}}{2}\right)^5-1}+\sqrt{5}-1=$$

$$=\left(9-4\sqrt{5}\right)\cdot\frac{2+\sqrt{5}}{\frac{9+5\sqrt{5}}{2}}+\sqrt{5}-1=\frac{2\left(9-4\sqrt{5}\right)\left(2+\sqrt{5}\right)}{9+5\sqrt{5}}+\sqrt{5}-1=$$

$$=\frac{2\sqrt{5}-4}{9+5\sqrt{5}}+\sqrt{5}-1=\frac{2\left(\sqrt{5}-2\right)}{9+5\sqrt{5}}\cdot\frac{9-5\sqrt{5}}{9-5\sqrt{5}}+\sqrt{5}-1=$$

$$=-\frac{1}{22}\cdot\left(19\sqrt{5}-43\right)+\sqrt{5}-1=\frac{21+3\sqrt{5}}{22}\to S_1=\frac{3}{22}\cdot\left(7+\sqrt{5}\right);(1)$$

$$S_2=\frac{\sqrt{5}+7}{\sqrt{5}-7}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\left(\left(\sqrt{5}-2\right)\varphi^{n-1}+\varphi^{3n-1}\right)^2}{\varphi^{7n-1}}=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(a^2\cdot\varphi^{2n-2}+2a\cdot\varphi^{4n-2}+\varphi^{6n-2}}{\varphi^{7n-1}}, where\ a=\sqrt{5}-2$$

$$S_2=\frac{27+7\sqrt{5}}{22}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{cdot}\left(a^2\cdot\varphi^{-5n-1}+2a\cdot\varphi^{-3n-1}+\varphi^{-n-1}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\frac{a^2}{\varphi}\sum_{n=0}^{\infty}\left(-\frac{1}{\varphi^5}\right)^n+\frac{2a}{\varphi}\sum_{n=0}^{\infty}\left(-\frac{1}{\varphi}\right)^n+\frac{1}{\varphi}\sum_{n=0}^{\infty}\left(-\frac{1}{\varphi}\right)^n\right)\overset{(**)}{=}$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\frac{a^2}{\varphi}\cdot\frac{1}{1+\frac{1}{\varphi^5}}+\frac{2a}{\varphi}\cdot\frac{1}{1+\frac{1}{\varphi^3}}+\frac{1}{\varphi}\cdot\frac{1}{1+\frac{1}{\varphi}}\right),$$

$$where\ \sum_{n=0}^{\infty}(-x)^n\overset{def}{=}\frac{1}{1+x},|x|<1$$

$$S_2=\frac{27+7\sqrt{5}}{22}\cdot\left(\frac{a^2}{\varphi}\cdot\frac{\varphi^5}{\varphi^5+1}+\frac{2a}{\varphi}\cdot\frac{\varphi^3}{\varphi^3+1}+\frac{1}{\varphi}\cdot\frac{\varphi}{\varphi+1}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(a^2\cdot\frac{\varphi^4}{\varphi^5+1}+2a\cdot\frac{\varphi^2}{\varphi^3+1}+\frac{1}{\varphi+1}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(a^2\cdot\frac{\varphi^4}{\varphi^5+1}+2a\cdot\frac{\varphi^2}{\underbrace{(\varphi+1)}_{\varphi^2}\underbrace{(\varphi^2-\varphi+1)}_{2}}+\frac{1}{\underbrace{\varphi+1}_{\varphi^2}}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(a^2\cdot\frac{\varphi^4}{\varphi^5+1}+a+\frac{1}{\varphi^2}\right)\overset{a=\sqrt{5}-2}{=}$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\left(\sqrt{5}-2\right)^2\cdot\frac{\varphi^4}{\varphi^5+1}+\sqrt{5}-2+\frac{1}{\varphi^2}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\left(9-4\sqrt{5}\right)\cdot\frac{\varphi^4}{\varphi^5+1}+\sqrt{5}-2+\frac{1}{\varphi^2}\right)\overset{\varphi=\frac{1+\sqrt{5}}{2}}{=}$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\left(9-4\sqrt{5}\right)\cdot\frac{\left(\frac{1+\sqrt{5}}{2}\right)^4}{\left(\frac{1+\sqrt{5}}{2}\right)^5+1}+\sqrt{5}-2+\frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)^2}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\left(9-4\sqrt{5}\right)\cdot\frac{7+3\sqrt{5}}{13+5\sqrt{5}}+\sqrt{5}-2+\frac{3-\sqrt{5}}{2}\right)$$

$$S_2=\frac{27+7\sqrt{5}}{22}\cdot\left(\frac{3-\sqrt{5}}{13+5\sqrt{5}}+\sqrt{5}-2+\frac{3-\sqrt{5}}{2}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\frac{3-\sqrt{5}}{13+5\sqrt{5}}\cdot\frac{13-5\sqrt{5}}{13-5\sqrt{5}}+\sqrt{5}-2+\frac{3+\sqrt{5}}{2}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\left(\frac{32-14\sqrt{5}}{22}+\sqrt{5}-2+\frac{3-\sqrt{5}}{2}\right)=$$

$$=\frac{27+7\sqrt{5}}{22}\cdot\frac{21-3\sqrt{5}}{22}=\frac{21+3\sqrt{5}}{22}=\frac{3}{22}(7+\sqrt{5});(2)$$

From (1),(2) it follows that:

$$\sum_{n=0}^{\infty}\frac{\left((\sqrt{5}-2)\varphi^{n-1}+\varphi^{3n-1}\right)^2}{\varphi^{7n}}=\frac{\sqrt{5}+7}{\sqrt{5}-7}\sum_{n=0}^{\infty}\frac{(-1)^n\left((\sqrt{5}-2)\varphi^{n-1}+\varphi^{3n-1}\right)^2}{\varphi^{7n-1}}$$

1303. If we have the integrals:

$$\int_0^{\infty}\frac{tanh\left(\frac{x}{2}\right)+tanh(2x)}{x}\left(e^{\frac{3x}{2}}-1\right)^2e^{-4x}dx=log(A)$$

$$\int_0^{\infty}\frac{tanh\left(\frac{x}{2}\right)+tanh(2x)}{x}\left(e^{\frac{3x}{2}}+1\right)^2e^{-4x}dx=log(B)$$

then prove that:

$$AB=\left(\frac{\sqrt{\pi}\Gamma\left(\frac{1}{8}\right)}{3\Gamma\left(\frac{5}{8}\right)}\right)^4$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega(a,n)=\int_0^{\infty}\frac{tanh(ax)}{x}e^{-nx}dx\Rightarrow\frac{d\Omega}{da}=\int_0^{\infty}sech^2(ax)e^{-nx}dx=\left[\frac{1}{a}tanh(ax)e^{-nx}\right]_0^{\infty}$$

$$=\frac{n}{a}\int_0^{\infty}\left(\frac{1-e^{-2ax}}{1+e^{-2ax}}\right)e^{-nx}dx=\frac{n}{2a^2}\int_0^1\left(\frac{1-t}{1+t}\right)t^{\frac{n}{2a}-1}dt=$$

$$=\frac{n}{2a}\left(\int_0^1\frac{t^{\frac{n}{2a}-1}}{1+t}dt-\int_0^1\frac{t^{\frac{n}{2a}}}{1+t}dt\right)=\frac{n}{4a^2}\left(2\psi\left(\frac{n}{2a}+\frac{1}{2}\right)-\psi\left(\frac{n}{2a}\right)-\psi\left(\frac{n}{4a}+1\right)\right)$$

$$\because \int_0^1 \frac{t^m}{1+t}dt = \frac{1}{2}\left(\psi\left(\frac{m}{2}+1\right) - \psi\left(\frac{m}{2}+\frac{1}{2}\right)\right)$$

$$\Omega(a,n) = \int_0^a \frac{n}{4a^2}\left(2\psi\left(\frac{n}{4a}+\frac{1}{2}\right) - \psi\left(\frac{n}{4a}\right) - \psi\left(\frac{n}{4a}+1\right)\right)da \overset{z=\frac{n}{4a}}{=}$$

$$= \int_{\frac{n}{4a}}^{\infty}\left(2\psi\left(z+\frac{1}{2}\right) - \psi(z) - \psi(z+1)\right)dz =$$

$$= \left[2log\Gamma\left(z+\frac{1}{2}\right) - log\Gamma(z) - log\Gamma(z+1)\right]_{\frac{n}{4a}}^{\infty} = log\left(\frac{\Gamma\left(\frac{n}{4a}\right)\Gamma\left(\frac{n}{4a}+1\right)}{\Gamma^2\left(\frac{n}{4a}+\frac{1}{2}\right)}\right) =$$

$$= log\left(\frac{n}{4a}\cdot\frac{\Gamma^2\left(\frac{n}{4a}\right)}{\Gamma^2\left(\frac{n}{4a}+\frac{1}{2}\right)}\right) \Rightarrow \int_0^\infty \frac{tanh(ax)}{x}e^{-nx}dx = log\left(\frac{n}{4a}\cdot\frac{\Gamma^2\left(\frac{n}{4a}\right)}{\Gamma^2\left(\frac{n}{4a}+\frac{1}{2}\right)}\right)$$

$$\int_0^\infty \frac{tanh\left(\frac{x}{2}\right)+tanh(2x)}{x}\left(e^{\frac{3x}{2}}-1\right)^2 e^{-4x}dx,$$

$$J = \int_0^\infty \frac{tanh\left(\frac{x}{2}\right)+tanh(2x)}{x}\left(e^{\frac{3x}{2}}+1\right)^2 e^{-4x}dx$$

$$I+J = \int_0^\infty \frac{tanh\left(\frac{x}{2}\right)+tanh(2x)}{x}\left[\left(e^{\frac{3x}{2}}-1\right)^2+\left(e^{\frac{3x}{2}}+1\right)^2\right]e^{-4x}dx =$$

$$= 2\int_0^\infty \frac{tanh\left(\frac{x}{2}\right)+tanh(2x)}{x}(e^{-x}+e^{-4x})dx =$$

$$= 2\left(\int_0^\infty \frac{tanh\left(\frac{x}{2}\right)}{x}e^{-x}dx + \int_0^\infty \frac{tanh\left(\frac{x}{2}\right)}{x}e^{-4x}dx + \int_0^\infty \frac{tanh(2x)}{x}e^{-x}dx + \int_0^\infty \frac{tanh(2x)}{x}e^{-x}dx\right) =$$

$$= 2\left[\Omega\left(\frac{1}{2},1\right) + \Omega\left(\frac{1}{2},4\right) + \Omega(2,1) + \Omega(2,4)\right] =$$

$$= 2\left[log\left\{\frac{1}{2}\cdot\frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma^2(1)}\right\} + log\left\{2\cdot\frac{\Gamma^2(2)}{\Gamma^2\left(\frac{5}{2}\right)}\right\} + log\left\{\frac{1}{8}\cdot\frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma^2(1)}\right\}\right] =$$

$$= 2log\left\{\frac{1}{16}\cdot\frac{\Gamma^4\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{5}{2}\right)}\cdot\frac{\Gamma^2\left(\frac{1}{8}\right)}{\Gamma^2\left(\frac{5}{8}\right)}\right\}; \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

$$I+J=\log\left\{\frac{\pi}{9}\cdot\frac{\Gamma^2\left(\frac{1}{8}\right)}{\Gamma\left(\frac{5}{8}\right)}\right\}^2=\log\left(\frac{\sqrt{\pi}}{3}\cdot\frac{\Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{5}{8}\right)}\right)^4$$

$$\therefore AB=\left(\frac{\sqrt{\pi}}{3}\cdot\frac{\Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{5}{8}\right)}\right)^4$$

$$\int_0^\infty\frac{\tanh\left(\frac{x}{2}\right)+\tanh(2x)}{x}\left(e^{\frac{3x}{2}}-1\right)^2e^{-4x}dx+\int_0^\infty\frac{\tanh\left(\frac{x}{2}\right)+\tanh(2x)}{x}\left(e^{\frac{3x}{2}}+1\right)^2e^{-4x}dx=$$

$$=\log\left(\frac{\sqrt{\pi}}{3}\cdot\frac{\Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{5}{8}\right)}\right)^4=4\log\left(\frac{\sqrt{\pi}}{3}\cdot\frac{\Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{5}{8}\right)}\right)$$

1304. Prove that:

$$\sum_{k=1}^{\infty}(-1)^k\prod_{n=1}^{k}\cot\left(\frac{n\pi}{2k+1}\right)=\frac{1}{2}\left(\zeta\left(\frac{1}{2},\frac{5}{4}\right)-\zeta\left(\frac{1}{2},\frac{3}{4}\right)\right)$$

-where $\zeta(s,a)$ is Hurwitz function.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Izumi Ainsworth-Lima-Peru

$$\sum_{k=1}^{\infty}(-1)^k\prod_{n=1}^{k}\cot\left(\frac{n\pi}{2k+1}\right)=\sum_{k=1}^{\infty}(-1)^k\frac{\prod_{n=1}^{k}\cos\left(\frac{n\pi}{2k+1}\right)}{\prod_{n=1}^{k}\sin\left(\frac{n\pi}{2k+1}\right)}=$$

$$=\sum_{k=1}^{\infty}(-1)^k\frac{\frac{1}{2^k}}{\frac{\sqrt{2k+1}}{2^k}}=\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{\frac{1}{2}}}-1=\beta\left(\frac{1}{2}\right)-1=$$

$$=4^{-\frac{1}{2}}\left(\zeta\left(\frac{1}{2},\frac{1}{4}\right)-\zeta\left(\frac{1}{2},\frac{3}{4}\right)\right)-1=\frac{1}{2}\left(2+\zeta\left(\frac{1}{2},\frac{5}{4}\right)-\zeta\left(\frac{1}{2},\frac{3}{4}\right)\right)-1=$$

$$=\frac{1}{2}\left(\zeta\left(\frac{1}{2},\frac{5}{4}\right)-\zeta\left(\frac{1}{2},\frac{3}{4}\right)\right)$$

$\beta(z)$ –Dirichlet beta function, $\zeta(a,b)$ –Hurwitz zeta function.

1305. Prove the relation:

$$\sum_{n=1}^{\infty}\frac{\left((3n^3+2n^2-n)\bmod 5\right)}{3n^3+2n^2-n}=$$

$$=\int_0^1\frac{t^{\frac{2}{3}}+3t^{\frac{2}{5}}+4t^{\frac{4}{15}}-4t^{\frac{7}{15}}-16t^{\frac{1}{15}}+12}{20(1-t)t^{\frac{13}{15}}}dt=$$

$$=\frac{4\psi^{(0)}\left(\frac{1}{5}\right)}{5}+\frac{\psi^{(0)}\left(\frac{3}{5}\right)}{5}-\frac{3\psi^{(0)}\left(\frac{2}{15}\right)}{5}-\frac{\psi^{(0)}\left(\frac{2}{5}\right)}{5}-\frac{\psi^{(0)}\left(\frac{4}{5}\right)}{20}-\frac{3\psi^{(0)}\left(\frac{8}{15}\right)}{20}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\sum_{n=1}^{\infty}\frac{\left((3n^3+2n^2-n)\bmod 5\right)}{3n^3+2n^2-n}=\sum_{n=1}^{\infty}\frac{n(n+1)(3n-1)\bmod 5}{n(n+1)(3n-1)}$$

$$n(n+1)(3n-1)\bmod 5=\begin{cases}4, for\ n=5n+1\\1, for\ n=5n+3\\0, otherwise\end{cases}$$

$$\Omega=\sum_{n=0}^{\infty}\frac{4}{(5n+1)(5n+2)(15n+2)}+\sum_{n=0}^{\infty}\frac{1}{(5n+3)(5n+4)(15n+8)}=$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{5n+2}+\frac{9}{15n+2}-\frac{4}{5n+1}+\frac{1}{4(5n+4)}+\frac{9}{4(15n+8)}-\frac{1}{5n+3}\right)=$$

$$=\sum_{n=0}^{\infty}\left(\frac{\frac{1}{5}}{n+\frac{2}{5}}+\frac{\frac{3}{5}}{n+\frac{2}{15}}-\frac{\frac{4}{5}}{n+\frac{1}{5}}+\frac{\frac{1}{20}}{n+\frac{4}{5}}+\frac{\frac{3}{20}}{n+\frac{8}{15}}-\frac{\frac{1}{5}}{n+\frac{3}{5}}\right)$$

$$\sum_{n=1}^{\infty}\frac{\left((3n^3+2n^2-n)\bmod 5\right)}{3n^3+2n^2-n}=$$

$$=\frac{4\psi^{(0)}\left(\frac{1}{5}\right)}{5}+\frac{\psi^{(0)}\left(\frac{3}{5}\right)}{5}-\frac{3\psi^{(0)}\left(\frac{2}{15}\right)}{5}-\frac{\psi^{(0)}\left(\frac{2}{5}\right)}{5}-\frac{\psi^{(0)}\left(\frac{4}{5}\right)}{20}-\frac{3\psi^{(0)}\left(\frac{8}{15}\right)}{20}$$

$$\Omega=\sum_{n=1}^{\infty}\left(\frac{1}{5n-3}+\frac{9}{15n-13}-\frac{4}{5n-4}+\frac{1}{4(5n-1)}+\frac{9}{4(15n-7)}-\frac{1}{5n-2}\right)$$

Using $MAZ_{\sum}$ identity: $\sum_{n=1}^{\infty}f(n)=\int_0^{\infty}\frac{L^{-1}\{f(s)\}}{e^t-1}dt$

$$\Omega=\int_0^1\frac{L^{-1}\left\{\frac{4}{s-\frac{3}{5}}+\frac{12}{s-\frac{13}{15}}-\frac{16}{s-\frac{4}{5}}+\frac{1}{s-\frac{1}{5}}+\frac{3}{s-\frac{7}{15}}-\frac{4}{s-\frac{2}{5}}\right\}}{20(e^t-1)}dt=$$

$$=\int_0^{\infty}\frac{4e^{\frac{3t}{5}}+12e^{\frac{13t}{15}}-16e^{\frac{4t}{5}}+e^{\frac{t}{5}}+3e^{\frac{7t}{15}}-4e^{\frac{2t}{5}}}{20(e^t-1)}\left(\frac{e^{-t}}{e^{-t}}\right)dt=$$

$$\overset{x=e^{-t}}{=}\int_0^1\frac{4x^{-\frac{3}{5}}+12e^{\frac{13}{15}}-16e^{\frac{4t}{5}}+e^{\frac{t}{5}}+3e^{\frac{7t}{15}}-4e^{\frac{2t}{5}}}{20(1-x)}\left(\frac{x^{\frac{13}{15}}}{x^{\frac{13}{15}}}\right)dx$$

$$\sum_{n=1}^{\infty}\frac{\left((3n^3+2n^2-n)mod\ 5\right)}{3n^3+2n^2-n}=$$

$$=\int_0^1\frac{t^{\frac{2}{3}}+3t^{\frac{2}{5}}+4t^{\frac{4}{15}}-4t^{\frac{7}{15}}-16t^{\frac{1}{15}}+12}{20(1-t)t^{\frac{13}{15}}}dt$$

Solution 2 by Probal Chakraborty-India

$$\sum_{n=1}^{\infty}\frac{\left((3n^3+2n^2-n)mod\ 5\right)}{3n^3+2n^2-n}=\sum_{n=1}^{\infty}\frac{n(n+1)(3n-1)mod\ 5}{n(n+1)(3n-1)}$$

$$n(n+1)(3n-1)mod\ 5=\begin{cases}4, for\ n=5n+1\\1, for\ n=5n+3\\0, otherwise\end{cases}$$

$$\Omega=\sum_{n=0}^{\infty}\frac{4}{(5n+1)(5n+2)(15n+2)}+\sum_{n=0}^{\infty}\frac{1}{(5n+3)(5n+4)(15n+8)}=$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{5n+2}+\frac{9}{15n+2}-\frac{4}{5n+1}+\frac{1}{4(5n+4)}+\frac{9}{4(15n+8)}-\frac{1}{5n+3}\right)=$$

$$=\sum_{n=0}^{\infty}\left[\frac{1}{5\left(n+\frac{2}{5}\right)}+\frac{3}{5\left(n+\frac{2}{15}\right)}-\frac{4}{5\left(n+\frac{1}{5}\right)}+\frac{1}{20\left(n+\frac{4}{5}\right)}+\frac{3}{20\left(n+\frac{8}{15}\right)}-\frac{1}{5\left(n+\frac{3}{5}\right)}\right]$$

$$=\frac{4\psi^{(0)}\left(\frac{1}{5}\right)}{5}+\frac{\psi^{(0)}\left(\frac{3}{5}\right)}{5}-\frac{3\psi^{(0)}\left(\frac{2}{15}\right)}{5}-\frac{\psi^{(0)}\left(\frac{2}{5}\right)}{5}-\frac{\psi^{(0)}\left(\frac{4}{5}\right)}{20}-\frac{3\psi^{(0)}\left(\frac{8}{15}\right)}{20}=$$

$$=\int_0^1\left[\frac{t^{\frac{2}{5}-1}}{5(1-t)}dx+\frac{3t^{\frac{2}{15}-1}}{5(1-t)}-\frac{4t^{\frac{1}{5}-1}}{5(1-t)}+\frac{t^{\frac{4}{5}-1}}{20(1-t)}+\frac{3t^{\frac{8}{15}-1}}{20(1-t)}-\frac{t^{\frac{3}{5}-1}}{5(1-t)}\right]dt=$$

$$\because \psi(z)=-\int_0^1\frac{t^{z-1}}{1-t}dt$$

$$=\int_0^1\frac{\left(4t^{-\frac{3}{5}}+12t^{-\frac{13}{15}}-16t^{-\frac{4}{5}}+t^{-\frac{1}{5}}+3t^{-\frac{7}{15}}-4t^{-\frac{2}{5}}\right)t^{\frac{13}{15}}}{20(1-t)t^{\frac{13}{15}}}dt=$$

$$=\int_0^1\frac{t^{\frac{2}{3}}+3t^{\frac{2}{5}}+4t^{\frac{4}{15}}-4t^{\frac{7}{15}}-16t^{\frac{1}{15}}+12}{20(1-t)t^{\frac{13}{15}}}dt$$

1306. Prove that:

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log\left(1+\sqrt{2}\right)$$

where $\psi(*)$ –is the digamma function.

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Probal Chakraborty-India

$$\because \psi(z+1)=-\gamma+\int_0^1\frac{1-t^z}{1-t}dt$$

$$\psi\left(1-\frac{3}{8}\right)=-\gamma+\int_0^1\frac{1-t^{-\frac{7}{8}}}{1-t}dt;\ \ \psi\left(\frac{5}{8}\right)=-\gamma+\int_0^1\frac{1-t^{-\frac{7}{8}}}{1-t}dt$$

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\int_0^1\frac{1-t^{-\frac{3}{8}}-1+t^{-\frac{7}{8}}}{1-t}dt=\int_0^1\frac{t^{-\frac{7}{8}}-t^{-\frac{3}{8}}}{1-t}dt\overset{z^8=t}{=}8\int_0^1\frac{z^{-7}-z^{-3}}{1-z^8}z^7dz=$$

$$=8\int_0^1\frac{1-z^4}{1-z^8}dz=8\left(\int_0^1\frac{1}{1+z^4}dz+\int_0^1\frac{\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz\right)=\frac{8}{2}\int_0^1\frac{\frac{1}{z^2}-1+1+\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz=$$

$$=\frac{8}{2}\left(\int_0^1 \frac{d\left(z-\frac{1}{z}\right)}{\left(z-\frac{1}{z}\right)^2+2}\right)=-\frac{8}{2}\int_0^1 \frac{d\left(z+\frac{1}{z}\right)}{\left(z+\frac{1}{z}\right)^2-2}=$$

$$=\frac{8}{2\sqrt{2}}\tan^{-1}\left|\frac{z-\frac{1}{z}}{\sqrt{2}}\right|_0^1+\frac{8}{2\sqrt{2}}\log\left|\frac{z+\frac{1}{z}-\sqrt{2}}{z+\frac{1}{z}+\sqrt{2}}\right|_0^1=\pi\sqrt{2}+2\sqrt{2}\log(1+\sqrt{2})$$

Solution 2 by Akerele Olofin-Nigeria

$$\psi\left(\frac{r}{m}\right)=-\gamma-\log(2m)-\frac{\pi}{2}\cot\left(\frac{r\pi}{m}\right)+2\sum_{n=1}^{\left[\frac{m-1}{2}\right]}\cos\left(\frac{2\pi nr}{m}\right)\log\left(\sin\left(\frac{\pi n}{m}\right)\right),$$

$r<m,[*]-GIF$. Now,

$$\psi\left(\frac{5}{8}\right)=-\gamma-\log16-\frac{\pi}{2}\cot\left(\frac{5\pi}{8}\right)+2\sum_{n=1}^{3}\cos\left(\frac{10n\pi}{8}\right)\log\left(\sin\left(\frac{\pi n}{8}\right)\right)=$$

$$=-\gamma-4\log2-\frac{\pi}{2}(1-\sqrt{2})+\frac{2}{\sqrt{2}}\log(1+\sqrt{2})=$$

$$=-\gamma-4\log2-\frac{\pi}{2}+\frac{\sqrt{2}}{2}\pi+\frac{2}{\sqrt{2}}\log(1+\sqrt{2})$$

$$\psi\left(\frac{1}{8}\right)=-\gamma-\log16-\frac{\pi}{2}\cot\left(\frac{\pi}{8}\right)+2\sum_{n=1}^{3}\cos\left(\frac{2n\pi}{8}\right)\log\left(\sin\left(\frac{\pi n}{8}\right)\right)=$$

$$=-\gamma-4\log2-\frac{\pi}{2}(1+\sqrt{2})+\frac{2}{\sqrt{2}}\log(\sqrt{2}-1)=$$

$$=-\gamma-4\log2-\frac{\pi}{2}-\frac{\sqrt{2}}{2}\pi+\frac{2}{\sqrt{2}}\log(\sqrt{2}-1)\Rightarrow$$

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=$$

$$=-\gamma-4\log2-\frac{\pi}{2}+\frac{\sqrt{2}}{2}\pi+\frac{2}{\sqrt{2}}\log(1+\sqrt{2})-\left(-\gamma-4\log2-\frac{\pi}{2}-\frac{\sqrt{2}}{2}\pi+\frac{2}{\sqrt{2}}\log(\sqrt{2}-1)\right)$$

$$=\sqrt{2}\pi+\frac{2}{\sqrt{2}}\left(\log(1+\sqrt{2})-\log(\sqrt{2}-1)\right)=\pi\sqrt{2}+2\sqrt{2}\log(1+\sqrt{2})$$

Therefore,

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}\log(1+\sqrt{2})$$

Solution 3 by Arslan Ahmed-Yemen

$$\Omega=\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=-\int_0^1\frac{x^{-\frac{3}{8}}-x^{-\frac{7}{8}}}{1-x}dx\overset{u^8=x}{=}8\int_0^1\frac{du}{1+u^4}=$$

$$=\frac{4}{\sqrt{2}}\int_0^1\frac{u+\sqrt{2}}{u^2+\sqrt{2}u+1}du-\frac{4}{\sqrt{2}}\int_0^1\frac{u-2}{u^2-\sqrt{2}u+1}du=$$

$$=\left[\sqrt{2}log\left(u^2+\sqrt{2}u+1\right)-\sqrt{2}log\left(u^2-\sqrt{2}u+1\right)\right]_0^1-$$

$$-\left[2\sqrt{2}tan^{-1}\left(\sqrt{2}u+1\right)+2\sqrt{2}tan^{-1}\left(\sqrt{2}u-1\right)\right]_0^1=\sqrt{2}(log\left(\frac{\left(2+\sqrt{2}\right)^2}{2}+\pi\right)$$

Therefore,

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log\left(1+\sqrt{2}\right)$$

Solution 4 by Serlea Kabay-Liberia

Recall $\psi(x)=-\gamma+\sum_{k=1}^{\infty}\frac{x}{n(n+x)}$. Let

$$\omega=\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=-\gamma+\gamma+\sum_{n=1}^{\infty}\frac{\frac{5}{8}}{n\left(n+\frac{5}{8}\right)}-\sum_{n=1}^{\infty}\frac{\frac{1}{8}}{n\left(n+\frac{1}{8}\right)}$$

$$\omega=\frac{1}{8}\sum_{n=1}^{n}\left(\frac{5}{n\left(n+\frac{5}{8}\right)}-\frac{1}{n\left(n+\frac{1}{8}\right)}\right)=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{64}{8n+1}-\frac{64}{8n+5}\right)=$$

$$=8\sum_{n=1}^{\infty}\left(\frac{1}{8n+1}-\frac{1}{8n+5}\right)=8\int_0^1\frac{dx}{1+x^4}=$$

$$=\frac{8}{2\sqrt{2}}\int_0^1\left(\frac{x+\sqrt{2}}{x^2+x\sqrt{2}+1}-\frac{x-\sqrt{2}}{x^2-x\sqrt{2}+1}\right)dx=$$

$$=\frac{2}{\sqrt{2}}log\left(2+\sqrt{2}\right)-\frac{1}{\sqrt{2}}log\left(2-\sqrt{2}\right)+\frac{2}{\sqrt{2}}\int_0^1\frac{\sqrt{2}}{x^2+x\sqrt{2}+1}dx+$$

$$+\frac{2}{\sqrt{2}}\int_0^1\frac{\sqrt{2}}{x^2-x\sqrt{2}+1}dx=\frac{2}{\sqrt{2}}log\left(3+2\sqrt{2}\right)+\frac{2\pi}{\sqrt{2}}$$

Therefore,

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log\left(1+\sqrt{2}\right)$$

Solution 5 by Mohammad Rostami-Kabul-Afghanistan

$$\because \sum_{n=0}^{\infty}\frac{1}{n+a}=-\psi(a),\qquad \Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt$$

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=-\sum_{n=0}^{\infty}\frac{1}{n+\frac{5}{8}}+\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{8}}=-\Phi\left(1,1,\frac{5}{8}\right)+\Phi\left(1,1,\frac{1}{8}\right)=$$

$$=-\frac{1}{\Gamma(1)}\int_0^{\infty}\frac{e^{-\frac{5}{8}t}}{1-e^{-t}}dt+\frac{1}{\Gamma(1)}\int_0^{\infty}\frac{e^{-\frac{1}{8}t}}{1-e^{-t}}dt\overset{e^{-t}=u}{=}$$

$$=-\int_1^0\frac{u^{\frac{5}{8}}}{1-u}\left(-\frac{du}{u}\right)+\int_1^0\frac{u^{\frac{1}{8}}}{1-u}\left(-\frac{du}{u}\right)=\int_0^1\frac{-u^{-\frac{3}{8}}+u^{-\frac{7}{8}}}{1-u}du\overset{y^8=u}{=}$$

$$=\int_0^1\frac{-y^{-3}+y^{-7}}{1-y^8}(8y^7dy)=8\int_0^1\frac{1-y^4}{1-y^8}dy=8\int_0^1\frac{1}{1+y^4}dy=$$

$$=-4\int_0^1\frac{-\frac{2}{y^2}}{\frac{1}{y^2}+y^2}dy=-4\int_0^1\frac{\left(-\frac{1}{y^2}+1\right)+\left(-\frac{1}{y^2}-1\right)}{\frac{1}{y^2}+y^2}dy=$$

$$=-4\int_0^1\frac{d\left(\frac{1}{y}+y\right)}{\left(\frac{1}{y}+y\right)^2-2}-4\int_0^1\frac{d\left(\frac{1}{y}-y\right)}{\left(\frac{1}{y}-y\right)^2+2}=$$

$$=-4\left[\frac{1}{2\sqrt{2}}log\left(\frac{\frac{1}{y}+y-\sqrt{2}}{\frac{1}{y}+y+\sqrt{2}}\right)\right]_0^1-4\left[\frac{1}{\sqrt{2}}tan^{-1}\left(\frac{\frac{1}{y}-y}{\sqrt{2}}\right)\right]_0^1=$$

$$=\frac{2}{\sqrt{2}}log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)+\frac{4}{\sqrt{2}}\cdot\frac{\pi}{2}=\frac{2}{\sqrt{2}}\pi+\frac{2}{\sqrt{2}}log\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)=\pi\sqrt{2}+2\sqrt{2}log(1+\sqrt{2})$$

Therefore,

$$\psi\left(\frac{5}{8}\right)-\psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}log(1+\sqrt{2})$$

1307. If $2<a\le b<3$ then:

$$\int_a^b\frac{sin^{-1}(2x-5)\cdot logx}{x}dx+2\int_a^b\frac{tan^{-1}\left(\sqrt{\frac{3-x}{x-2}}\right)\cdot logx}{x}dx=\frac{\pi}{8}log(ab)log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$f:(2,3)\to\mathbb{R}.\ f(x)=\sin^{-1}(2x-5)+2\tan^{-1}\left(\sqrt{\frac{3-x}{x-2}}\right)$$

$$f'(x)=0\to f(x)=constant,\ f\left(\frac{5}{2}\right)=\frac{\pi}{4}\to f(x)=\frac{\pi}{4},\forall x\in(2,3)$$

$$\int_a^b\frac{\sin^{-1}(2x-5)\cdot\log x}{x}dx+2\int_a^b\frac{\tan^{-1}\left(\sqrt{\frac{3-x}{x-2}}\right)\cdot\log x}{x}dx=$$

$$=\int_a^b\frac{f(x)\cdot\log x}{x}dx=\frac{\pi}{4}\int_a^b\frac{\log x}{x}dx=\frac{\pi}{8}(\log^2b-\log^2a)=$$

$$=\frac{\pi}{8}\log(ab)\log\left(\frac{b}{a}\right)$$

1308. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function at $x=0$ such that:

$$f(x)+f\left(\frac{x}{x+1}\right)=x^2,\forall x\in\mathbb{R}.$$

Then find the value of $f(1)$.

Proposed by Surjeet Singhania-India

Solution by Ravi Prakash-New Delhi-India

$$f(0)+f(0)=0\Rightarrow f(0)=0$$

$$f(x)+f\left(\frac{x}{x+1}\right)=x^2\Rightarrow f\left(\frac{1}{n}\right)+f\left(\frac{1}{n+1}\right)=\frac{1}{n^2},\forall n\in\mathbb{N}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\left[f\left(\frac{1}{k}\right)+f\left(\frac{1}{k+1}\right)\right]=\sum_{k=1}^{n}(-1)^{k-1}\cdot\frac{1}{k^2}\Rightarrow$$

$$f(1)+(-1)^{n-1}f\left(\frac{1}{n+1}\right)=\sum_{k=1}^{n}(-1)^{k-1}\cdot\frac{1}{k^2};(1)$$

As f –continuous at 0, $f\left(\frac{1}{k+1}\right)\to f(0)=0$, as $k\to\infty\Rightarrow(-1)^{k-1}f\left(\frac{1}{k+1}\right)\to 0$, as $k\to\infty$

Taking limit as $n\to\infty$ on as$k\to\infty$ in (1), we get:

$$f(1)+0=\sum_{k=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n^2}=\frac{\pi^2}{12}$$

1309. Prove the integral:

$$\int_0^1 xLi_2\left(\frac{(1-x)^4}{(1+x)^4}\right)dx = \frac{\pi^2}{12} + 2\pi + 4 - 16log2$$

Li_2 —is Poly-log function.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Naren Bhandari-Bajura-Nepal

Note that for all $x \in (0,1)$

$$\frac{d}{dx}f(x) = \frac{d}{dx}Li_2\left(\frac{(1-x)^4}{(1+x)^4}\right) = -\frac{8log8\left(logx + log(1+x^2) - 4log(1+x)\right)}{x^2-1}; (1)$$

$$\int_0^1 xf(x)\,dx = \frac{f(x)}{2}\int_0^1 \frac{d}{dx}(x^2-1)\,dx - \frac{1}{2}\int_0^1\left(f'(x)\int\frac{d}{dx}(x^2-1)dx\right)dx$$

Now, using the result

$$= \frac{1}{2}Li_2(1) + 4\int_0^1 log8\left(logx + log(1+x^2) - 4log(1+x)\right)dx = \frac{\pi^2}{12} + I$$

For all $x \in (0,1)$ the integrals are convergent so, we apply the linearity giving us

$$\int_0^1 logx\,dx = (xlogx - x)_0^1 = -1; (2)$$

$$\int_0^1 log(1+x^2)\,dx = xlog(1+x^2)\Big|_0^1 - \int_0^1 \frac{2x^2}{1+x^2}dx = log2 + 2(tan^{-1}x - x)\Big|_0^1$$

$$= \frac{\pi}{2} - 2 + log2; (3)$$

$$\int_0^1 4log(1+x)dx = 4(xlog(1+x) + log(1+x) - x)|_0^1 = 4log4 - 4; (4)$$

Combining the results (2),(3),(4) we have our desired result:

$$\int_0^1 xLi_2\left(\frac{(1-x)^4}{(1+x)^4}\right)dx = \frac{\pi^2}{12} + 2\pi + 4 - 16log2$$

The last two integral also can be evaluated by the MacLaurin series of $log(1+x)$ and $log(1+x^2)$.

1310. Prove that:

$$\int_0^1 \frac{(1-2x)}{\sqrt{x(1-x)}} tan^{-1}\left(\sqrt{\frac{1}{x}-1}\right)\frac{dx}{x^2-x+1} = \frac{8}{3}G - \frac{\pi}{3}log\left(2+\sqrt{3}\right)$$

,where G –denotes Catalan's constant.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$I = \int_0^1 \frac{(1-2x)}{\sqrt{x(1-x)}} tan^{-1}\left(\sqrt{\frac{1}{x}-1}\right)\frac{dx}{x^2-x+1} \overset{x=cos^2t}{=} -2\int_0^{\frac{\pi}{2}} \frac{tcos2t}{cos^4t+sin^2t}dt =$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{t(tan^2t-1)}{cos^2t+tan^2t}dt$$

$$I \overset{x=tan\,t}{=} 2\int_0^{\infty} tan^{-1}x\left(\frac{x^2-1}{\frac{1}{1+x^2}+x^2}\right)\frac{dx}{1+x^2} = 2\int_0^{\infty}\frac{x^2-1}{x^4+x^2+1}tan^{-1}x\,dx \overset{IBP}{=}$$

$$= \left[tan^{-1}xlog\left(\frac{x^2-x+1}{x^2+x+1}\right)\right]_0^{\infty} + \int_0^{\infty}\frac{1}{1+x^2}log\left(\frac{x^2-x+1}{x^2+x+1}\right)dx \overset{x=tan\,t}{=}$$

$$= \int_0^{\frac{\pi}{2}} log\left(\frac{1+\frac{1}{2}sin2t}{1-\frac{1}{2}sin2t}\right)dt = \frac{1}{2}\int_0^{\pi} log\left(\frac{1+\frac{1}{2}sint}{1-\frac{1}{2}sint}\right)dt = \int_0^{\frac{\pi}{2}} log\left(\frac{1+\frac{1}{2}sint}{1-\frac{1}{2}sint}\right)dt =$$

$$= 2\int_0^{\frac{\pi}{2}} sint\int_0^{\frac{1}{2}}\frac{1}{1-y^2sin^2t}dy\,dt = 2\int_0^{\frac{1}{2}}\int_0^{\frac{\pi}{2}}\frac{sint}{1-y^2sin^2t}dtdy =$$

$$= 2\int_0^{\frac{1}{2}}\int_0^{\frac{\pi}{2}} \frac{sint}{1-y^2+y^2cos^2t}dtdy \overset{z=cost}{=} 2\int_0^{\frac{1}{2}} \frac{1}{y^2}\int_0^1 \frac{dz}{\left(\frac{1-y^2}{y^2}\right)+z^2}dy =$$

$$= 2\int_0^{\frac{1}{2}} \frac{1}{y\sqrt{1-y^2}}tan^{-1}\left(\frac{yz}{\sqrt{1-y^2}}\right)\Big|_0^1 dy = 2\int_0^{\frac{1}{2}} \frac{tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)}{y\sqrt{1-y^2}}dy \overset{y=sin\theta}{=}$$

$$= 2\int_0^{\frac{\pi}{6}} \frac{\theta}{sin\theta}d\theta = 2\left[\theta log\left(tan\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{6}} - 2\int_0^{\frac{\pi}{6}} log\left(tan\frac{\theta}{2}\right)d\theta =$$

$$= \frac{\pi}{3}log\left(tan\frac{\pi}{12}\right) - 4\underbrace{\int_0^{\frac{\pi}{12}} log(tan\theta)\,d\theta}_{-\frac{2}{3}G} = \frac{\pi}{3}log\left(\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}\right) + \frac{8}{3}G$$

$$= \frac{8}{3}G - \frac{\pi}{3}log(2+\sqrt{3})$$

Therefore,

$$\int_0^1 \frac{(1-2x)}{\sqrt{x(1-x)}}tan^{-1}\left(\sqrt{\frac{1}{x}-1}\right)\frac{dx}{x^2-x+1} = \frac{8}{3}G - \frac{\pi}{3}log(2+\sqrt{3})$$

1311. Find:

$$\Omega(a) = \int_0^{loga} x\cdot(1+a^x loga)\cdot a^{x+a^x}\,dx, a > 1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammad Rostami-Kabul-Afghanistan

$$\Omega(a) = \int_0^{loga} x\cdot(1+a^x loga)\cdot a^{x+a^x}\,dx \overset{x=u,dv=(1+a^x log)^{x+a^x}}{=} uv|_0^{loga} - \int_0^{loga} v\,dv$$

$$x = u \Rightarrow dx = du, \frac{1}{loga}\int (1+a^x loga)a^{x+a^x}\cdot loga\,dx = \int dv \Rightarrow \frac{1}{loga}a^{x+a^x} = v$$

$$\Omega(a)=\frac{1}{loga}xa^{x+a^x}\Big|_0^{loga}-\frac{1}{loga}\int_0^{loga}a^{x+a^x}\,dx=a^{loga+a^{loga}}-\frac{1}{loga}\underbrace{\int_0^{loga}a^{a^x}a^x\,dx}_{I}$$

$$\begin{cases}a^x=t\\x=loga\end{cases}\Rightarrow\begin{cases}a^xloga=dt\\t=a^{loga}\end{cases}\Rightarrow\begin{cases}a^xdx=\dfrac{dt}{loga}\\x=0\Rightarrow t=1\end{cases}$$

$$I=\int_0^{a^{loga}}a^t\frac{dt}{loga}=\frac{a^t}{log^2a}\Big|_1^{a^{loga}}=\frac{a^{a^{loga}}-a}{log^2a}$$

Therefore,

$$\Omega(a)=a^{loga+a^{loga}}-\frac{1}{loga}\left(\frac{a^{a^{loga}}-a}{log^2a}\right)=a^{loga+a^{loga}}-\frac{1}{log^3a}\left(a^{a^{loga}}-a\right)$$

Solution 2 by Gabi Brehuescu-Romania

$$\Omega(a)=\int_0^{loga}x\cdot(1+a^xloga)\cdot a^{x+a^x}\,dx=\int_0^{loga}x(x+a^x)'a^{x+a^x}\,dx=$$

$$=\int_0^{loga}x\left(a^{x+a^x}\right)'\cdot\frac{dx}{loga}=\frac{1}{loga}\underbrace{\int_0^{loga}x\left(a^{x+a^x}\right)'dx}_{I_1};(*)$$

$$I_1=\int_0^{loga}x\left(a^{x+a^x}\right)'dx=xa^{x+a^x}\Big|_0^{loga}-\int_0^{loga}a^{x+a^x}dx=$$

$$=loga\cdot a^{loga+a^{loga}}-\underbrace{\int_0^{loga}a^{x+a^x}dx}_{I_2};(**)$$

$$I_2=\int_0^{loga}a^{x+a^x}dx\overset{a^x=y\Rightarrow x=log_ay}{=}\int_1^{a^{loga}}ya^y\cdot\frac{dy}{yloga}=\frac{1}{loga}\int_1^{a^{loga}}a^y\,dy=\frac{a^{a^{loga}}-a}{log^2a}\overset{(**)}{\Longrightarrow}$$

$$I_1=loga\cdot a^{loga+a^{loga}}-\frac{a^{a^{loga}}-a}{log^2a}\overset{(*)}{\Rightarrow}$$

$$\Omega(a)=a^{loga+a^{loga}}-\frac{1}{log^3a}(a^{a^{loga}}-a)$$

Solution 3 by Oyebamiji Oluwasey-Nigeria

$$\Omega(a)=\int_0^{loga}x\cdot(1+a^xloga)\cdot a^{x+a^x}\,dx=\int_0^{loga}x\cdot\frac{d(a^{x+a^x})}{loga}=$$

$$=\frac{1}{loga}xa^{x+a^x}\Big|_0^{loga}-\frac{1}{loga}\int_0^{loga}a^{x+a^x}dx$$

$$\int_0^{loga}a^{x+a^x}dx=\int_0^{loga}a^xa^{a^x}dx\overset{a^x=u}{=}\int_1^{a^{loga}}ua^u\frac{du}{uloga}=\frac{1}{loga}\int_1^{a^{loga}}a^udu$$

$$=\frac{1}{log^2u}\left(a^{a^{loga}}-a\right)$$

$$\Omega(a)=a^{loga+a^{loga}}-\frac{1}{loga}\left(\frac{a^{a^{loga}}-a}{log^2a}\right)=\frac{1}{log^3a}\left[a^{a^{loga}}\left(a^{loga}log^3a-1\right)+a\right]=$$

$$=a^{loga+a^{loga}}-\frac{1}{log^3a}(a^{a^{loga}}-a)$$

Solution 4 by Kaushik Mahanta-Assam-India

$$\int(1+a^xloga)a^{x+a^x}dx\ \overset{\substack{a^{x+a^x}=t\\ loga(1+a^xloga)a^{x+a^x}dx=dt}}{=}\ \int\frac{dt}{loga}=\frac{t}{loga}+C=\frac{a^{x+a^x}}{loga}+C$$

$$\Omega(a)=\int_0^{loga}x\cdot(1+a^xloga)\cdot a^{x+a^x}\,dx\overset{IBP}{=}x\frac{a^{x+a^x}}{loga}\Big|_0^{loga}-\int_0^{loga}\frac{a^{x+a^x}}{loga}dx=$$

$$=a^{loga+a^{loga}}-\frac{1}{loga}\int_0^{loga}a^{a^x}a^x\,dx\ \overset{\substack{a^{a^x}=t\\ a^{a^x+x}dx=\frac{dt}{log^2a}}}{=}\ a^{loga+a^{loga}}-\frac{1}{log^3a}a^{a^x}\Big|_0^{loga}=$$

$$=a^{loga+a^{loga}}-\frac{1}{log^3a}(a^{a^{loga}}-a)$$

1312. Find without any software:

$$\Omega=\int\frac{(3-3sinh^2x-sinh^4x+sinh^{10}x)coshx}{(cosh^2x-2)^2}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\Omega=\int\frac{(3-3sinh^2x-sinh^4x+sinh^{10}x)coshx}{(cosh^2x-2)^2}dx\overset{sinhx=t}{=}$$

$$=\int\frac{(3-3t^2-t^4+t^{10}}{(t^2-1)^2}dt=\int\frac{3(1-t^2)-t^4(1-t^6)}{(t^2-1)^2}dt=$$

$$=\int\frac{3-t^4(1+t^2+t^4)}{1-t^2}dt=\int\frac{(1-t^4)+(1-t^6)+(1-t^8)}{1-t^2}dt=$$

$$=\int[(1+t^2)+(1+t^2+t^4)+(1+t^2+t^4+t^6)]\,dt=$$

$$=3t+t^3+\frac{2}{5}t^5+\frac{1}{7}t^7+C=$$

$$=3sinhx+sinh^3x+\frac{2}{5}sinh^5x+\frac{1}{7}sinh^7x+C$$

Solution 2 by Hafiz Iqbal-Situbondo-Indonesia

$$\Omega=\int\frac{(3-3sinh^2x-sinh^4x+sinh^{10}x)coshx}{(cosh^2x-2)^2}dx=$$

$$=\int\frac{e^{-7x}(e^{2x}+1)(e^{12x}+2e^{10x}+31e^{8x}+129e^{6x}+31e^{9x}+2e^{2x}+1)}{128}dx$$

$$=\frac{1}{128}\int e^{7x}dx+\frac{1}{128}\int e^{-5x}dx+\frac{33}{128}\int e^{3x}dx+\frac{155}{128}\int e^{x}dx+\frac{155}{128}\int e^{-x}dx$$

$$+\frac{33}{128}\int e^{-3x}dx+\frac{3}{128}\int e^{-5x}dx+\frac{1}{128}\int e^{-7x}dx=$$

$$=\frac{1}{128}\left(\frac{5e^{7x}+21e^{5x}+385e^{3x}+5425e^{x}-5425e^{-x}-385e^{-3x}-231e^{-5x}-5e^{-7x}}{35}\right)+C$$

Solution 3 by Probal Chakraborty-India

$$\Omega=\int\frac{(3-3sinh^2x-sinh^4x+sinh^{10}x)coshx}{(cosh^2x-2)^2}dx=$$

$$=\int\frac{(3-3sinh^2x-sinh^4x+sinh^{10}x)}{(sinh^2x-1)^2}coshx\,dx\overset{sinhx=z}{=}$$

$$=\int\frac{3-3z^2-z^4+z^{10}}{(1-z^2)^2}dz=3\int\frac{1-z^2}{(1-z^2)^2}dz-\int\frac{z^4(1-z^6)}{(1-z^2)^2}dz=$$

$$= 3\int\frac{dz}{1-z^2} - \int\frac{z^4(1+z^2+z^4)}{1-z^2}dz = \int(z^6+2z^4+2z^2+3)\,dz =$$

$$= \frac{\sinh^7 x}{7} + \frac{2\sinh^5 x}{5} + \frac{2\sinh^3 x}{3} + 3\sinh x + C$$

Solution 4 by Samar Das-Kolkata-India

$$\Omega = \int\frac{(3-3\sinh^2 x-\sinh^4 x+\sinh^{10}x)\cosh x}{(\cosh^2 x-2)^2}dx =$$

$$= \int\frac{(3-3\sinh^2 x-\sinh^4 x+\sinh^{10}x)\cosh x}{(1+\sinh^2 x-2)^2}dx \overset{y=\sinh x}{=}$$

$$= \int\frac{(3-3y^2)-y^4+y^{10}}{(y^2-1)^2}dy = 3\int\frac{dy}{1-y^2} + \int\frac{(y^4(1+y^2+y^4)}{y^2-1}dy =$$

$$= 3\int\frac{dy}{1-y^2} + \int\left(y^6+2y^4+3y^2+3+\frac{3}{y^2-1}\right)dy =$$

$$= \frac{y^7}{7}+\frac{2y^5}{5}+y^3+3y+C = \frac{\sinh^7 x}{7}+\frac{2\sinh^5 x}{5}+\frac{2\sinh^3 x}{3}+3\sinh x+C$$

1313. Find:

$$\Omega(a) = \int_0^a\frac{\cos x\cdot\sin^7 x}{1+\sin^2 x+\sin^4 x}dx, \qquad a>0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Noor Alam-India

$$\Omega(a) = \int\frac{\cos x\cdot\sin^7 x}{1+\sin^2 x+\sin^4 x}dx \overset{\sin x=t}{=} \int\frac{t^7}{1+t^2+t^4}dt =$$

$$= \int(t^3-t)dt + \int\frac{t}{1+t^2+t^4}dt = \frac{t^4}{4}-\frac{t^2}{2}+C_1+I_1$$

$$I_1 = \int\frac{t}{1+t^2+t^4}dt \overset{t^2=u}{=} \frac{1}{2}\int\frac{du}{u^2+u+1} = \frac{1}{2}\int\frac{du}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t^2+1}{\sqrt{3}}\right)+C_2$$

$$\Omega(a)=\left[\frac{\sin^4 x}{4}-\frac{\sin^2 x}{2}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sin^2 x+1}{\sqrt{3}}\right)\right]_0^a=$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sin^2 a+1}{\sqrt{3}}\right)-\frac{\pi}{6\sqrt{3}}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\Omega(a)=\int_0^a \frac{\cos x\cdot\sin^7 x}{1+\sin^2 x+\sin^4 x}dx \overset{\sin^2 x=t}{=} \frac{1}{2}\int_0^{\sin^2 a}\frac{t^3}{1+t+t^2}dt=\frac{1}{2}\int_0^{\sin^2 a}\frac{t^3-1+1}{1+t+t^2}dt=$$

$$=\frac{1}{2}\int_0^{\sin^2 x}\left[(t-1)+\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]dt=\frac{1}{2}\left[\frac{t^2}{2}-t+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^{\sin^2 a}=$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sin^2 a+1}{\sqrt{3}}\right)-\frac{\pi}{6\sqrt{3}}$$

Solution 3 by Mikael Bernardo-Mozambique

$$\Omega(a)=\int_0^a \frac{\cos x\cdot\sin^7 x}{1+\sin^2 x+\sin^4 x}dx \overset{\sin x=u}{=} \int_0^{\sin a}\frac{u^7}{1+u^2+u^4}du=$$

$$=\int_0^{\sin a}\left(u^3-\frac{u^5+u^3}{1+u^2+u^4}\right)du=\frac{u^4}{4}\Big|_0^{\sin a}-\int_0^{\sin a}\left(u-\frac{u}{1+u^2+u^4}\right)du=$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\int_0^{\sin a}\frac{u}{1+u^2+u^4}du=$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\frac{1}{2}\int_0^{\sin a}\frac{2u}{\left(u^2+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du=$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(u^2+\frac{1}{2}\right)\right)\Big|_0^{\sin a}=$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sin^2 a+1}{\sqrt{3}}\right)-\frac{\pi}{6\sqrt{3}}$$

Solution 4 by Gabi Brehuescu-Romania

$$\Omega(a)=\int_0^a \frac{cosx\cdot sin^7x}{1+sin^2x+sin^4x}dx=$$

$$=\int_0^a \frac{cosx\cdot sin^7x-sinxcosx}{1+sin^2x+sin^4x}dx+\int_0^a \frac{sinxcosx}{1+sin^2x+sin^4x}dx=$$

$$=\int_0^a \frac{sinxcosx(sin^6x-1)}{1+sin^2x+sin^4x}dx+\frac{1}{2}\int_0^a \frac{(sin^2x)'}{1+sin^2+sin^4x}dx=I_1+I_2$$

$$I_1=\int_0^a \frac{sinxcosx(sin^6x-1)}{1+sin^2x+sin^4x}dx=\int_0^a \frac{sinxcosx(sin^2x-1)(sin^4x+sin^2x+1)}{1+sin^2x+sin^4x}dx=$$

$$=\int_0^a cos^3x(cosx)'dx=\frac{cos^4x}{4}\bigg|_0^a=\frac{cos^4a-1}{4};(*)$$

$$I_2=\int_0^a \frac{(sin^2x)'}{sin^4x+sin^2x+1}dx=\frac{2}{\sqrt{3}}tan^{-1}\left(\frac{2sin^2x+1}{\sqrt{3}}\right)\bigg|_0^a=$$

$$=\frac{2}{\sqrt{3}}\left(tan^{-1}\left(\frac{2sin^2a+1}{\sqrt{3}}\right)-\frac{\pi}{6}\right);(**)$$

From $(*)$, $(**)$, we get:

$$\Omega(a)=\frac{cos^4a-1}{4}+\frac{1}{\sqrt{3}}\left(tan^{-1}\left(\frac{2sin^2a+1}{\sqrt{3}}\right)-\frac{\pi}{6}\right)$$

Solution 5 by Probal Chakraborty-Kolkata-India

$$\Omega(a)=\int_0^a \frac{cosx\cdot sin^7x}{1+sin^2x+sin^4x}dx\overset{sinx=z}{=}\int_0^{sina}\frac{z^7}{1+z^2+z^4}dz=$$

$$=\int_0^{sina}\frac{z^7+z^5+z^3-z^3-z^5}{z^4+z^2+1}dz=\int_0^{sina}\left[\frac{z^3(z^4+z^2+1)}{z^4+z^2+1}-\frac{z(z^4+z^2+1)-z}{z^4+z^2+1}\right]dz=$$

$$=\int_0^{sina}(z^3-z)\,dz+\int_0^{sina}\frac{z}{z^4+z^2+1}dz=$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sin^2 a+1}{\sqrt{3}}\right)-\frac{\pi}{6\sqrt{3}}$$

Solution 6 by Arslan Ahmed-Yemen

$$\Omega(a)=\int_0^a \frac{\cos x\cdot\sin^7 x}{1+\sin^2 x+\sin^4 x}dx=\int_0^a \frac{\cos x\cdot\sin x\cdot\sin^6 x}{1+\sin^2 x+\sin^4 x}dx=$$

$$\left(\sin x\cos x=\frac{1}{2}\sin 2x;\frac{1}{2}(\cos 2x-1)=\sin^2 x\right)$$

$$=-\frac{1}{4}\int_0^a \frac{\left(\frac{1}{2}(\cos 2x-1)\right)^3(-2\sin 2x)}{1+\frac{1}{2}(\cos 2x-1)+\left(\frac{1}{2}(\cos 2x-1)\right)^2}dx \overset{\substack{u=\frac{1}{2}(\cos 2x-1)\\ du=-2\sin 2x dx}}{=}$$

$$=-\frac{1}{4}\int_0^{\frac{1}{2}(\cos 2a-1)}\frac{u^3}{1+u+u^2}du=-\frac{1}{4}\left(\int_0^{\frac{1}{2}(\cos 2a-1)}\frac{du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}+\int_0^{\frac{1}{2}(\cos 2a-1)}(u-1)du\right)$$

$$=-\frac{1}{4}\left[\frac{2\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)}{\sqrt{3}}+\frac{u^2}{2}-u\right]_0^{\frac{1}{2}(\cos 2a-1)}=$$

$$=-\frac{1}{4}\left[\frac{1}{6}\left(4\sqrt{3}\tan^{-1}\left(\frac{2\sqrt{3}\left(\frac{1}{2}(\cos 2a-1)\right)+\sqrt{3}}{3}+3(\frac{1}{2}(\cos 2a-1)^2-6(\frac{1}{2}(\cos 2a-1)\right)\right)-\frac{\pi}{3\sqrt{3}}\right]$$

$$=\frac{\sin^4 a}{4}-\frac{\sin^2 a}{2}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sin^2 a+1}{\sqrt{3}}\right)-\frac{\pi}{6\sqrt{3}}$$

1314. Solve for real numbers:

$$\int_x^{2x}\log(1+\tan 3x\cdot\tan t)dt\cdot\int_{2x}^{3x}\log(1+\tan 5x\cdot\tan t)dt=0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Gabi Brehuescu-Romania

$$\int_x^{2x} \log(1+\tan 3x\cdot \tan t)dt\cdot \int_{2x}^{3x}\log(1+\tan 5x\cdot \tan t)dt=0;(1)$$

From (1), we get:

$$\int_x^{2x}\log(1+\tan 3x\cdot \tan t)dt=0;(I)$$

$$\int_{2x}^{3x}\log(1+\tan 5x\cdot \tan t)dt=0;(II)$$

$$\int_x^{2x}\log(1+\tan 3x\cdot \tan t)dt=\int_x^{2x}\log\left(1+\frac{\sin 3x\sin t}{\cos 3x\cos t}\right)dt=$$

$$=\int_x^{2x}\log\left(\frac{\cos 3x\cos t+\sin 3x\sin t}{\cos 3x\cos t}\right)dt=\int_x^{2x}\log\left(1+\frac{\cos(3x-t)}{\cos 3x\cos t}\right)dt=I_1$$

$$3x-t=y, dt=-dy$$

$$I_1=\int_{2x}^{x}\log\left(\frac{\cos y}{\cos 3x\cos(3x-y)}\right)(-dy)=\int_x^{2x}\log\left(\frac{\cos t}{\cos 3x\cos(3x-t)}\right)dt=J_1$$

$$I_1+J_1=\int_x^{2x}\log\left(\frac{\cos(3x-t)}{\cos 3x\cos t}\right)dt+\int_x^{2x}\log\left(\frac{\cos t}{\cos 3x\cos(3x-t)}\right)dt=$$

$$=\int_x^{2x}\log\left(\frac{1}{\cos^2 3x}\right)dt=x\log\left(\frac{1}{\cos^2 3x}\right)$$

$$\begin{cases}\cos 3x\neq 0\\ \cos 5x\neq 0\end{cases}\Rightarrow I_1=\frac{x}{2}\log\left(\frac{1}{\cos^2 3x}\right)$$

$$I_1=0\Leftrightarrow \frac{x}{2}\log\left(\frac{1}{\cos^2 3x}\right)=0\Rightarrow x=0 \text{ or } \log\left(\frac{1}{\cos^2 3x}\right)=0\Leftrightarrow \cos^2 3x=1\Leftrightarrow$$

$$x\in\left\{\frac{k\pi}{3}\middle| k\in\mathbb{Z}\right\}$$

$$I_2=\int_{2x}^{3x}log(1+tan5x\cdot tant)dt=\int_{2x}^{3x}log\left(\frac{cos(5x-t)}{cos5xcost}\right)dt\overset{5x=t=y}{=}$$

$$=\int_{3x}^{2x}log\left(\frac{cosy}{cos5xcos(5x-y)}\right)(-dy)=\int_{2x}^{3x}log\left(\frac{cost}{cos5xcos(5x-t)}\right)dt$$

$$I_2+J=\int_{2x}^{3x}log\left(\frac{1}{cos^2 5x}\right)dt\Rightarrow I_2=\frac{x}{2}log\left(\frac{1}{cos^2 5x}\right)$$

$$I_2=0\Leftrightarrow x=0\ or\ log\left(\frac{1}{cos^2 5x}\right)=0\Leftrightarrow cos^2 5x=0\Leftrightarrow x\in\left\{\frac{k\pi}{5}\middle|k\in\mathbb{Z}\right\}$$

Therefore,

$$x\in\left\{\frac{k\pi}{3}\middle|k\in\mathbb{Z}\right\}\cup\left\{\frac{k\pi}{5}\middle|k\in\mathbb{Z}\right\}$$

Solution 2 by Mohammad Rostami-Kabul-Afghanistan

$$\int_{x}^{2x}log(1+tan3x\cdot tant)dt=I_1;\int_{2x}^{3x}log(1+tan5x\cdot tant)dt=I_2$$

$$I_1=\int_{x}^{2x}log(1+tan3x\cdot tant)dt=\int_{x}^{2x}log\left(\frac{cot3x+tant}{cot3x}\right)dt=$$

$$=\int_{x}^{2x}log\left(\frac{cos3x}{sin3x}+\frac{sint}{cost}\right)dt-\int_{x}^{2x}log(cot3x)\,dt=$$

$$=\int_{x}^{2x}log\left(\frac{cos(3x-t)}{sin3xcost}\right)dt-tlog(cot3x)|_x^{2x}=$$

$$=\overbrace{\int_{x}^{2x}log\big(cos(3x-t)\big)dt}^{A}-\int_{x}^{2x}log(sinx)\,dt-\overbrace{\int_{x}^{2x}log(cost)\,dt}^{-A}-xlog(cot3x)=$$

$$=[tlog(sin3x)]_x^{2x}-xlog(cot3x)=-xlog(cos3x)$$

$$I_2=\int_{2x}^{3x}log(1+tan5x\cdot tant)dt=\int_{2x}^{3x}log\left(\frac{cot5x+tant}{cot5x}\right)dt=$$

$$=\int_{2x}^{3x}\log\left(\frac{\cos5x}{\sin5x}+\frac{\sin t}{\cos t}\right)dt-\int_{2x}^{3x}\log(\cot5x)\,dt=$$

$$=\int_{2x}^{3x}\log\left(\frac{\cos(5x-t)}{\sin5x\cos t}\right)dt-t\log(\cot5x)|_{2x}^{3x}=$$

$$=\overbrace{\int_{2x}^{3x}\log\big(\cos(5x-t)\big)dt}^{B}-\int_{2x}^{3x}\log(\sin x)\,dt-\overbrace{\int_{2x}^{3x}\log(\cos t)\,dt}^{-B}-x\log(\cot5x)=$$

$$=[t\log(\sin5x)]_{2x}^{3x}-x\log(\cot5x)=-x\log(\cos5x)$$

$$I_1=0\Leftrightarrow x\log(\cos3x)=0\Leftrightarrow x=0\ or\ \cos3x=1\Leftrightarrow x\in\left\{\frac{2k\pi}{3}\middle|k\in\mathbb{Z}\right\}$$

$$I_2=0\Leftrightarrow x\log(\cos5x)=0\Leftrightarrow x=0\ or\ \cos5x=1\Leftrightarrow x\in\left\{\frac{2k'\pi}{5}\middle|k'\in\mathbb{Z}\right\}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$I_1=\int_{x}^{2x}\log(1+\tan3x\cdot\tan t)dt=\int_{x}^{2x}\log\left(1+\frac{\sin3x\sin t}{\cos3x\cos t}\right)dt=$$

$$=\int_{x}^{2x}\log\left(\frac{\cos(3x-t)}{\cos3x\cos t}\right)dt=$$

$$=\int_{x}^{2x}\log\big(\cos(3x-t)\big)\,dt-\log(\cos3x)\int_{x}^{2x}dt-\int_{x}^{2x}\log(\cos t)dt=$$

$$=\int_{x}^{2x}\log\Big(\cos\big(3x-(x+2x-t)\big)\Big)\,dt-x\log(\cos3x)-\int_{x}^{2x}\log(\cos t)dt=$$

$$=\int_{x}^{2x}\log(\cos t)\,dt-\int_{x}^{2x}\log(\cos t)dt-x\log(\cos3x)=-x\log(\cos3x)$$

Next,

$$I_2=\int_{x2}^{3x}\log(1+\tan5x\cdot\tan t)dt=\int_{2x}^{3x}\log\left(1+\frac{\sin5x\sin t}{\cos5x\cos t}\right)dt=$$

$$=\int_{2x}^{3x}log\left(\frac{cos(5x-t)}{cos5xcost}\right)dt=$$

$$=\int_{2x}^{3x}log\big(cos(5x-t)\big)\,dt-log(cos5x)\int_{2x}^{3x}dt-\int_{2x}^{3x}log(cost)dt=$$

$$=\int_{2x}^{3x}log\Big(cos\big(5x-(x+4x-t)\big)\Big)\,dt-xlog(cos5x)-\int_{2x}^{3x}log(cost)dt=$$

$$=\int_{2x}^{3x}log(cost)\,dt-\int_{2x}^{3x}log(cost)dt-xlog(cos5x)=-xlog(cos5x)$$

Now, $I_1I_2=0\Leftrightarrow 3x^2log(cos3x)log(cos5x)=0\Leftrightarrow$

$$x=0\ or\ cos3x=1\ or\ cos5x=1\Leftrightarrow$$

$$x\in\left\{0,\frac{2n\pi}{3},\frac{2m\pi}{3}\middle|m,n\in\mathbb{Z}\right\}$$

1315. Find:

$$\Omega=\int\left[2^{\left(1-\frac{3}{\sqrt[3]{(x+4)^3-64-12(x+3)\sqrt[3]{(x+4)^3-64-12(x+3)\sqrt[3]{\dots}}}}\right)}\right]$$

Proposed by Paul Olowo-Nigeria

Solution by Samar Das-India

$$\text{Let } p=\sqrt[3]{(x+4)^3-64-12(x+3)\sqrt[3]{(x+4)^3-64-12(x+3)\sqrt[3]{\dots}}}$$

$$p=\sqrt[3]{(x+4)^3-64-12(x+4)p}\to p^3+12(x+4)p-(x+4)^3+64=0$$

$$\to p^3-x^3+12xp+48p-12x^2-48x=0$$

$$\to(p-x)(p^2+xp+x^2+12x+48)=0$$

Therefore $p=x$ but $\Delta=b^2-4ac<0\to p^2+xp+x^2+12x+48=0$.

$$\int 2^{1-\frac{3}{x}}dx = 2\int 2^{-\frac{3}{x}}dx \overset{y=-\frac{3}{x}}{=} 2\int 2^y\left(\frac{3}{y^2}dy\right) = 6\int \frac{2^y}{y^2}dy = 6\int \frac{e^{ylog2}}{y^2}dy =$$

$$= 6\int \frac{1+\frac{ylog2}{1!}+\frac{(ylog2)^2}{2!}+\frac{(ylog3)^3}{3!}+\cdots}{y^2}dy \ =$$

$$= 6\left\{-\frac{1}{y}+\frac{log2}{1!}\cdot log|y|+\frac{log^2 2}{2!}\cdot y+\frac{log^3 2}{3!}\cdot\frac{y^2}{2}+\cdots\right\} =$$

$$= 6\left(-\frac{1}{y}+\frac{log2}{1!}\cdot|y|+\sum_{r=1}^{\infty}\left(\frac{log^{r+1}2}{(r+1)!}\cdot\frac{y^r}{r}\right)\right);\because y=-\frac{3}{x}$$

1316. Prove that:

$$\int_0^1 \frac{log^3(1-x)log^2x}{1-x}dx - 3\int_0^1 \frac{log(1-x)Li_2^2(1-x)}{1-x}dx$$
$$= 6\zeta^2(3) - 8\zeta(6)$$

-where $Li_2(x)$ is dilogarithm function and $\zeta(x)$ is Riemann zeta function.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Probal Chakraborty-India

$$\Omega = \int_0^1 \frac{log^3(1-x)log^2x}{1-x}dx - 3\int_0^1 \frac{log(1-x)Li_2^2(1-x)}{1-x}dx \overset{1-x=t}{=}$$

$$= \int_0^1 \frac{log^3xlog^2(1-x)}{x}dx - 3\int_0^1 \frac{logxLi_2^2(x)}{x}dx$$

$$I_1 = \int_0^1 \frac{log^3xlog^2(1-x)}{x}dx = log^4xlog^2(1-x)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{log^4xlog(1-x)}{1-x}dx$$

$$\because \frac{log(1-x)}{1-x} = -\sum_{n=1}^{\infty} H_n x^n = -\sum_{n=1}^{\infty}\frac{H_n}{2}\int_0^1 x^n log^4x\,dx = -\sum_{n}^{\infty}\frac{H_n}{2}\cdot\frac{\partial^4}{\partial s}\Big|_{s=4}\int_0^1 x^{n+s}dx =$$

$$= -\sum_{n=1}^{\infty}\frac{H_n}{2}\cdot\frac{4!}{(n+1)^5} = 12\zeta^2(3) - 18\zeta(6)$$

$$I_2 = 3\int_0^1 \frac{logxLi_2^2(x)}{x}dx = \frac{3}{2}\left[logxLi_2^2(x)\right]_0^1 - 3\int_0^1 \frac{Li_2(x)logxlog(1-x)}{x}dx =$$

$$= -3\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n-1} logxlog(1-x)}{n^2} dx = -3\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\partial^2}{\partial s \partial a} \frac{\Gamma(n+s)\Gamma(a-1)}{\Gamma(n+s+a-1)}\bigg|_{s=0,a=0} =$$

$$= -3\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\partial}{\partial s}\left[\frac{\Gamma(n+s)\Gamma(-1)}{\Gamma(n+s-1)} - \frac{\Gamma(n+s)\Gamma(-1)\psi(n+s-1)}{\Gamma(n+s-1)}\right]_{s=0}$$

$$= -10\zeta(6) + 6\zeta^2(3)$$

Therefore,

$$\int_0^1 \frac{log^3(1-x)log^2 x}{1-x} dx - 3\int_0^1 \frac{log(1-x)Li_2^2(1-x)}{1-x} dx = 6\zeta^2(3) - 8\zeta(6)$$

1317. Find without any software:

$$\Omega(n,x) = \int \frac{csc(2x)}{(1+tan^3 x)^n} dx\,, n \in \mathbb{N}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Samar Das-West Bengal-India

$$\Omega(n,x) = \int \frac{csc(2x)}{(1+tan^3 x)^n} dx = \int \frac{(1+tan^2 x)dx}{2tanx(1+tan^3 x)^n} \overset{y=tanx}{=} \frac{1}{2}\int \frac{y^2 dy}{y^3(1+y^3)^n} \overset{z=1+y^2}{=}$$

$$= \frac{1}{6}\int \frac{dz}{(z-1)z^n} = -\frac{1}{6}\int \frac{dz}{z^{n+1}\left(1-\frac{1}{z}\right)};\quad \left(\because \left|\frac{1}{z}\right| < 1 \therefore z = 1+y^3 = 1+tan^3 x\right)$$

$$= -\frac{1}{6}\int \frac{\left(1-\frac{1}{z}\right)^{-1}}{z^{n+1}} dz = -\frac{1}{6}\int \frac{1}{z^{n+1}}\left\{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right\}dz =$$

$$= -\frac{1}{6}\int (z^{-n-1} + z^{-n-2} + z^{-n-3} + \cdots)dz =$$

$$= -\frac{1}{6}\left(\frac{z^{-n}}{-n} + \frac{z^{-n-1}}{-n-1} + \cdots\right) = \frac{1}{6}\cdot\sum_{r=1}^{\infty} \frac{z^{-n-r}}{n+r-1} + C =$$

$$= \frac{1}{6}\sum_{r=1}^{\infty} \frac{1}{(n+r-1)(1+tan^3 x)^{n+r}} + C$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega(n,x) = \int \frac{csc(2x)}{(1+tan^3 x)^n} dx\,; \Omega(0) = \int csc(2x)\, dx = \frac{1}{2} log|tanx| + C$$

$$\Omega(1)=\int\frac{csc(2x)}{1+tan^3x}dx\overset{t=tanx}{=}\frac{1}{2}\int\frac{dt}{t(1+t^3)}\overset{z=t^3}{=}\frac{1}{6}\int\frac{dz}{z(1+z)}=\frac{1}{6}log\left|\frac{z}{1+z}\right|+C$$

For $n\geq 2$:

$$\Omega(n)=\int\frac{csc(2x)}{(1+tan^3x)^n}dx\overset{t=tanx}{=}\frac{1}{2}\int\frac{dt}{t(1+t^3)^n}\overset{z=t^3}{=}\frac{1}{6}\int\frac{dz}{z(1+z)^n}=$$

$$=\frac{1}{6}\int\left(\frac{1}{z}-\frac{1}{1+z}-\frac{1}{(1+z)^2}-\cdots-\frac{1}{(1+z)^n}\right)dz$$

$$\Omega(n)=\frac{1}{6}log\left|\frac{z}{1+z}\right|-\frac{1}{6}\sum_{k=2}^{n}\int\frac{dz}{(1+z)^k}=\frac{1}{6}log\left|\frac{z}{1+z}\right|+\frac{1}{6}\sum_{k=2}^{n}\frac{1}{(k-1)(1+z)^{k-1}}$$

$$\Omega(n)=\int\frac{csc(2x)}{(1+tan^3x)^n}dx=\frac{1}{6}\left[log\left|\frac{tan^3x}{1+tan^3x}\right|+\sum_{k=2}^{n}\frac{1}{(k-1)(1+tan^3x)^{k-1}}\right]+C$$

Solution 3 by Mikael Bernardo-Mozambique

$$\Omega(n,x)=\int\frac{csc(2x)}{(1+tan^3x)^n}dx=\frac{1}{2}\int\frac{cscxsecx}{(1+tan^3x)^n}dx=\frac{1}{2}\int\frac{sec^2x}{tanx(1+tan^3x)^n}dx=$$

$$\overset{tanx=u}{=}\frac{1}{2}\int\frac{du}{u(1+u^3)^n}\overset{u^3=y}{=}\frac{1}{6}\int\frac{y^{\frac{1}{3}-1}}{y^{\frac{1}{3}}(1+y)^n}dy=\frac{1}{6}\int\frac{dy}{y(1+y)^n}=\frac{1}{6}\int\frac{1+y-y}{y(1+y)^n}dy=$$

$$=\frac{1}{6}\int\left(\frac{1}{y(1+y)^{n-1}}-\frac{1}{(1+y)^n}\right)dy=\frac{1}{6}\int\left(\frac{1+y-y}{y(1+y)^{n-1}}-\frac{1}{(1+y)^n}\right)dy=$$

$$=\frac{1}{6}\int\left(\frac{1}{y(1+y)^{n-2}}-\frac{1}{(1+y)^{n-1}}-\frac{1}{(1+y)^n}\right)dy=$$

$$=\frac{1}{6}\int\left(\frac{1}{y}-\frac{1}{1+y}-\frac{1}{(1+y)^2}-\cdots-\frac{1}{(1+y)^{n-2}}-\frac{1}{(1+y)^{n-1}}-\frac{1}{(1+y)^n}\right)dy=$$

$$=\frac{1}{6}\left(log|y|-log|1+y|+\frac{1}{1+y}+\cdots+\frac{1}{(n-3)(1+y)^{n-3}}+\frac{1}{(n-2)(1+y)^{n-2}}+\frac{1}{(n-1)(1+y)^{n-1}}\right)$$

$$=\frac{1}{6}\left(log\left|\frac{u^3}{1+u^3}\right|+\frac{1}{1+u^3}+\cdots+\frac{1}{(n-3)(1+u^3)^{n-3}}+\frac{1}{(n-2)(1+u^3)^{n-2}}+\frac{1}{(n-1)(1+u^3)^{n-1}}\right)$$

$$\Omega(n,x)=\frac{1}{6}\left(log\left|\frac{tan^3x}{1+tan^3x}\right|+\frac{1}{1+tan^3x}+\cdots+\frac{1}{(n-2)(1+tan^3x)^{n-2}}+\frac{1}{(n-1)(1+tan^3x)^{n-1}}\right)+C$$

Solution 4 Serlea Kabay-Liberia

$$\Omega(n,x)=\int\frac{csc(2x)}{(1+tan^3x)^n}dx\overset{u=tan^3x}{=}\int\frac{csc\left(2tan^{-1}(\sqrt[3]{x})\right)}{3u^{\frac{2}{3}}\left(u^{\frac{2}{3}}+1\right)(1+u)^n}du=$$

$$= \int \frac{du}{\sin\left(\tan^{-1}(\sqrt[3]{u})\right)\cos\left(\tan^{-1}(\sqrt[3]{u})\right)6u^{\frac{2}{3}}\left(u^{\frac{2}{3}}+1\right)(1+u)^n}$$

Since $\sin\left(\tan^{-1}(\sqrt[3]{u})\right) = \frac{\sqrt[3]{u}}{\sqrt{u^{\frac{2}{3}}+1}}$ and $\cos\left(\tan^{-1}(\sqrt[3]{u})\right) = \frac{1}{\sqrt{u^{\frac{2}{3}}+1}}$, thus

$$\Omega(n,x) = \frac{1}{6}\int \frac{du}{\frac{\sqrt[3]{u}}{u^{\frac{2}{3}}+1}\cdot 6u^{\frac{2}{3}}\left(u^{\frac{2}{3}}+1\right)(1+u)^n} = \frac{1}{6}\int \frac{du}{u^{\frac{2}{3}}\cdot\sqrt[3]{u}(1+u)^n} = \frac{1}{6}\int \frac{du}{u(1+u)^n}$$

Now, $f(n) = \frac{1}{u(1+u)^n} = \frac{\alpha}{n} + \frac{\alpha_1}{1+u} + \frac{\alpha_2}{(1+u)^2} + \cdots + \frac{\alpha_n}{(1+u)^n} \to$

$$\alpha = \lim_{u\to 0} uf(n) = 1;\ \alpha_n = \lim_{u\to -1}(1+u)^n f(n) = -1$$

Now, $f(n) = \frac{1}{u(1+u)^n} = \frac{\alpha}{n} + \frac{\alpha_1}{1+u} + \frac{\alpha_2}{(1+u)^2} + \cdots + \frac{\alpha_{n-1}}{(1+u)^{n-1}} - \frac{1}{(1+u)^n}$

$$\lim_{n\to\infty}(1+u)f(n) =$$

$$= \lim_{n\to\infty}\left((1+u)\left(\frac{1}{u} + \frac{\alpha_1}{1+u} + \frac{\alpha_2}{(1+u)^2} + \cdots + \frac{\alpha_{n-1}}{(1+u)^{n-1}} - \frac{1}{(1+u)^n}\right)\right)$$

$$\alpha_1 + 1 = 0 \to \alpha_1 = -1$$

$$\Omega(n,x) = \int\left(\frac{1}{u} - \frac{1}{1+u} - \frac{1}{(1+u)^2} - \cdots - \frac{1}{(1+u)^n}\right)du$$

$$\Omega(n,m) = \frac{1}{6}\left(\log|\tan^3 x| - \log|1+\tan^3 x| + \sum_{i=2}^{n}\frac{1}{(1+\tan^3 x)(i-1)}\right)$$

Solution 5 by Arslan Ahmed-Yemen

$$\Omega(n,x) = \int \frac{\csc(2x)}{(1+\tan^3 x)^n}dx = \frac{1}{2}\int \frac{dx}{\sin x\cos x(1+\tan^3 x)^n} =$$

$$= \frac{1}{2}\int \frac{\sec x}{\sin x(1+\tan^3 x)^n}\cdot\frac{\sec x}{\sec x}dx \overset{u=\tan x}{=} \frac{1}{2}\int \frac{du}{u(1+u^3)^n} =$$

$$= \frac{1}{6}\int \frac{-3u^2du}{-u^3(1+u^3)^n} \overset{-z=u^3}{=} \frac{1}{6}\int \frac{dz}{z(1-z)^n} =$$

$$= \frac{(1-z)^{1-n}}{6z(1-n)} + \frac{1}{6(1-n)}\int \frac{(1-z)^{1-n}}{z^2}dz$$

$$\omega = \int \frac{(1-z)^{1-n}}{z^2}dz = 2\int \frac{\frac{1}{2\sqrt{z}}(1-\sqrt{z})^{1-n}(1+\sqrt{z})^{1-n}}{\sqrt{z^3}}dz \overset{\sqrt{z}=s}{\Rightarrow}$$

$$\xi = 2\int \frac{(1-s)^{1-n}(1+s)^{1-n}}{s^3}ds = 2\int s^{-2-1}(1-s)^{1-n}\big(1-(-s)\big)^{1-n} \to$$

$$\xi = -\frac{1}{s^2}2F_1[-2;n-1;n-1;-1;s;-s]+C$$

$$\omega = -\frac{1}{z}2F_1\left[-2;n-1;n-1;-1;\sqrt{z};-\sqrt{z}\right]+C$$

$$\Omega(n,z) = \frac{(1-z)^{1-n}}{6z(1-n)} - \frac{1}{u^3 6(1-n)}2F_1\left[-2;n-1;n-1;-1;-\sqrt{z};\sqrt{z}\right]+C$$

$$\Omega(n,u) = -\frac{(1+u^3)^{1-n}}{6u^3(1-n)} + \frac{1}{u^3 6(1-n)}2F_1\left[-2;n-1;n-1;-1;-\sqrt{u^3};\sqrt{u^3}\right]+C$$

$$\Omega(n,x) = -\frac{(1+tan^3x)^{1-n}}{6tan^3x(1-n)} + \frac{1}{6tan^3x(1-n)}2F_1\left[-2;n-1;n-1;-1;-\sqrt{tan^3x};\sqrt{tan^3x}\right]+C$$

Solution 6 by Akerele Olofin-Nigeria

$$\text{Let } csc(2x) = \frac{1}{2cosxtanx} \to$$

$$\Omega(n,x) = \int sec^2x \cdot \frac{dx}{2tanx(1+tan^3x)^n} \overset{u=tanx}{=} \frac{1}{2}\int \frac{du}{u(1+u^3)^n} =$$

$$= \frac{1}{2}\int 3u^2 \cdot \frac{du}{3u^3(u^3+1)^n} \overset{v=u^3}{=} \frac{1}{6}\int \frac{dv}{v(v+1)^n} =$$

$$= \frac{1}{6}\cdot\frac{(v+1)^{1-n}\ {}_2F_1(1;1-n;2-n;v+1)}{n-1}+C$$

Therefore,

$$\Omega(n,x) = \frac{1}{6}\left(\frac{(1+tan^3x)^{1-n}\ {}_2F_1(1;1-n;2-n;1+tan^3x)}{n-1}\right)+C$$

1318. Find:

$$\Omega = \int_0^\infty \frac{x^2 log(1+x)}{x^4+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Zaharia Burghelea-Romania

$$I(t)=\int_0^\infty \frac{x^2 log(1+tx)}{1+x^4}dx \to I'(t)=\int_0^\infty \frac{x^3}{(1+x^4)(1+tx)}dx=$$

$$=\frac{t}{1+t^4}\int_0^\infty \frac{1+t^2x^2}{1+x^4}dx-\frac{t^2}{1+t^4}\int_0^\infty \frac{x}{1+x^4}dx+\frac{1}{1+t^4}\int_0^\infty\left(\frac{x^3}{1+x^4}-\frac{t}{1+tx}\right)dx$$

$$\int_0^\infty \frac{1}{1+x^4}dx \overset{x\to\frac{1}{x}}{=} \int_0^\infty \frac{x^2}{1+x^4}dx \to \int_0^\infty \frac{1+t^2x^2}{1+x^4}dx=\frac{1+t^2}{2}\int_0^\infty \frac{1+x^2}{1+x^4}dx=$$

$$=\frac{1+t^2}{2}\int_0^\infty \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx \overset{x-\frac{1}{x}=t}{=} \frac{1+t^2}{2}\int_{-\infty}^\infty \frac{1}{t^2+2}dt=\frac{\pi(1+t^2)}{2\sqrt{2}}$$

$$\int_0^\infty \frac{x}{1+x^4}dx=\frac{1}{2}tan^{-1}(x^2)\Big|_0^\infty=\frac{\pi}{4}$$

$$\int_0^\infty\left(\frac{x^3}{1+x^4}-\frac{t}{1+tx}\right)dx=log\left(\frac{\sqrt[4]{1+x^4}}{1+tx}\right)\Bigg|_0^\infty=-logt\to$$

$$I'(t)=\frac{\pi}{2\sqrt{2}}\cdot\frac{t(1+t^2)}{1+t^4}-\frac{\pi}{4}\cdot\frac{t^2}{1+t^4}-\frac{logt}{1+t^4}$$

$$I(0)=0\to\int_0^\infty \frac{x^2 log(1+x)}{1+x^4}dx=\int_0^1 I'(t)\,dt=$$

$$=\frac{\pi}{2\sqrt{2}}\cdot\int_0^1 \frac{t(1+t^2)}{1+t^4}dt-\frac{\pi}{4}\int_0^1 \frac{t^2}{1+t^4}dt-\int_0^1 \frac{logt}{1+t^4}dt$$

$$\int_0^1 \frac{t+t^3}{1+t^4}dt=\left(\frac{1}{2}tan^{-1}(t^2)+\frac{1}{4}log(1+t^4)\right)\Bigg|_0^1=\frac{\pi}{8}+\frac{1}{4}log2$$

$$\int_0^1 \frac{t^2}{1+t^4}dt=\frac{1}{2}\int_0^1 \frac{1+t^2-1+t^2}{1+t^4}dt=\frac{1}{2}\int_0^1 \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2-2}dt=$$

$$=\frac{1}{2}\int_{-\infty}^0 \frac{1}{x^2+2}dx+\frac{1}{2}\int_2^\infty \frac{1}{x^2-2}dx=\frac{\pi}{4\sqrt{2}}-\frac{log(1+\sqrt{2})}{2\sqrt{2}}$$

$$\int_0^1 \frac{logt}{1+t^4}dt=\sum_{n=0}^\infty(-1)^n\int_0^1 t^n logt\,dt=\sum_{n=0}^\infty \frac{(-1)^{n+1}}{(4n+1)^2}=$$

$$=\sum_{n=0}^{\infty}\frac{1}{(8n+5)^2}-\sum_{n=0}^{\infty}\frac{1}{(8n+1)^2}=\frac{1}{64}\left(\Psi_1\left(\frac{5}{8}\right)-\Psi_1\left(\frac{1}{8}\right)\right)$$

Therefore,

$$\Omega=\int_0^{\infty}\frac{x^2log(1+x)}{x^4+1}dx=\frac{\pi log2}{8\sqrt{2}}+\frac{\pi log(1+\sqrt{2})}{8\sqrt{2}}+=\frac{1}{64}\left(\Psi_1\left(\frac{5}{8}\right)-\Psi_1\left(\frac{1}{8}\right)\right)$$

1319. For all $n\geq 0$, prove that:

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\int_0^1 log^2(1+x)x^k\,dx=\frac{2^{n+1}-1}{(n+1)^3}+\frac{2^n}{n+1}log^2 2-\frac{2^{n+1}}{(n+1)^2}log2$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Gabriel Brehuescu-Romania

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\int_0^1 log^2(1+x)x^k\,dx=\frac{1}{2}\int_0^1 log^2(1+x)\sum_{k=0}^{n}\binom{n}{k}x^k dx=$$

$$=\frac{1}{2}\int_0^1 log^2(1+x)(1+x)^n\,dx:=I$$

$$I=\frac{1}{2}\cdot\frac{1}{n+1}\int_0^1 log^2(1+x)((1+x)^{n+1})'dx\overset{IBP}{=}$$

$$=\frac{1}{2(n+1)}(1+x)^{n+1}log^2(1+x)\Big|_0^1-\frac{1}{(n+1)}\int_0^1(1+x)^n log(1+x)dx=$$

$$=\frac{1}{2(n+1)}\left(2^{n+1}log^2 2-2\int_0^1 log(1+x)((1+x)^{n+1})'\,dx\right)\overset{IBP}{=}$$

$$=\frac{1}{2(n+1)}\left(2^{n+1}log^2 2-\frac{2}{n+1}\left((1+x)^{n+1}log(1+x)\Big|_0^1-\int_0^1(1+x)^n dx\right)\right)=$$

$$=\frac{1}{2(n+1)}\left[2^{n+1}log^2 2-\frac{2}{n+1}\left(2^{n+1}log2-\int_0^1(1+x)^n dx\right)\right]=$$

$$=\frac{1}{2(n+1)}\left[2^{n+1}log^2 2-\frac{2}{n+1}\left(2^{n+1}log2-\frac{(1+x)^{n+1}}{n+1}\Big|_0^1\right)\right]=$$

$$=\frac{1}{2(n+1)}\left[2^{n+1}log^2 2-\frac{2}{n+1}\left(2^{n+1}log2-\frac{2^{n+1}-1}{n+1}\right)\right]=$$

$$=\frac{2^n}{n+1}\log^2 2-\frac{2^{n+1}}{(n+1)^2}\log 2+\frac{2^{n+1}-1}{(n+1)^3}$$

Solution 2 by Surjeet Singhania-India

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1\log^2(1+x)x^k\,dx=\int_0^1\log^2(1+x)\sum_{k=0}^{n}\binom{n}{k}x^k dx=$$

$$=\int_0^1\log^2(1+x)(1+x)^n\,dx\overset{x\to x+1}{=}\int_1^2 x^n\log^2 x\,dx\overset{IBP}{=}$$

$$=\frac{x^{n+1}}{n+1}\log^2 x\Big|_1^2-\frac{2}{n+1}\int_1^2 x^{n+1}\cdot\frac{\log x}{x}dx=$$

$$=\frac{2^{n+1}}{n+1}\log^2 2-\frac{2}{n+1}\left[\frac{x^{n+1}}{n+1}\cdot\log x\Big|_1^2-\frac{1}{n+1}\int_1^2 x^n\,dx\right]=$$

$$=\frac{2^{n+1}}{n+1}\log^2 2-\frac{2^{n+2}}{(n+1)^2}\log 2+\frac{2}{(n+1)^2}\int_1^2 x^n\,dx=$$

$$=\frac{2^{n+1}}{n+1}\log^2 2-\frac{2^{n+2}}{(n+1)^2}\log 2+\frac{2}{(n+1)^3}(2^{n+1}-1)$$

Therefore,

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\int_0^1\log^2(1+x)x^k\,dx=\frac{2^{n+1}-1}{(n+1)^3}+\frac{2^n}{n+1}\log^2 2-\frac{2^{n+1}}{(n+1)^2}\log 2$$

Solution 3 by Akerele Olofin-Nigeria

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1\log^2(1+x)x^k\,dx=\int_0^1\log^2(1+x)\sum_{k=0}^{n}\binom{n}{k}x^k dx=$$

$$=\int_0^1\log^2(1+x)(1+x)^n\,dx\overset{x\to x+1}{=}\int_1^2 x^n\log^2 x\,dx=$$

$$=\frac{\partial^2}{\partial a^2}\Big|_{a=0}\int_1^2 x^{a+n}\,dx=\frac{\partial^2}{\partial a^2}\Big|_{a=0}\frac{2^{a+n+1}-1}{a+n+1}=$$

$$=\frac{2^{n+1}-1}{(n+1)^3}+\frac{2^n}{n+1}\log^2 2-\frac{2^{n+1}}{n+1}\log 2$$

Therefore,

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\int_0^1 log^2(1+x)x^k\,dx = \frac{2^{n+1}-1}{(n+1)^3}+\frac{2^n}{n+1}log^2 2-\frac{2^{n+1}}{(n+1)^2}log2$$

1320. Find a closed form:

$$\Omega = \lim_{n\to\infty}\left(logn - \frac{1}{\pi}\sum_{k=1}^{n}\int_{-\frac{1}{k}}^{\frac{1}{k}}(x^8+x^4+1)cos^{-1}(kx)\,dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Serlea Kabay-Liberia

$$Let\ I = \int_{-\frac{1}{k}}^{\frac{1}{k}}(x^8+x^4+1)cos^{-1}(kx)\,dx \overset{u=kx}{=} \frac{1}{k}\int_{-1}^{1}\left(\frac{u^8}{k^8}+\frac{u^4}{k^4}+1\right)cos^{-1}u\,du \overset{IBP}{=}$$

$$= \frac{1}{k}\left(\frac{u^9}{9k^8}+\frac{u^5}{5k^4}+u\right)cos^{-1}u\Big|_{-1}^{1} + \frac{1}{k}\int_{-1}^{1}\left(\frac{u^9}{9k^8}+\frac{u^5}{5k^4}+u\right)\frac{1}{\sqrt{1-u^2}}du =$$

$$= \frac{\pi}{k}\left(\frac{1}{9k^8}+\frac{1}{5k^4}+1\right) + \underbrace{\frac{1}{k}\int_{-1}^{1}\left(\frac{u^9}{9k^8}+\frac{u^5}{5k^4}+u\right)\frac{1}{\sqrt{1-u^2}}du}_{I_1}$$

$$I_1 = \int_{-1}^{1}\left(\frac{u^9}{9k^8}+\frac{u^5}{5k^4}+u\right)\frac{1}{\sqrt{1-u^2}}du = 0 \quad \because \int_{-a}^{a}f(x)dx = 0, f - odd\ function.$$

$$\Omega = \lim_{n\to\infty}\left(logn - \frac{1}{\pi}\sum_{k=1}^{n}\frac{\pi}{k}\left(\frac{1}{9k^8}+\frac{1}{5k^4}+1\right)\right) = \lim_{n\to\infty}\left(logn - \sum_{k=1}^{n}\left(\frac{1}{9k^9}+\frac{1}{5k^5}+\frac{1}{k}\right)\right) =$$

$$= \lim_{n\to\infty}\left(logn - \frac{H_n^{(9)}}{9} - \frac{H_n^{(5)}}{5} - H_n\right)$$

$$\because H_n^{(a)} = \zeta(\text{a}) + (-1)^{a-1}\frac{\Psi_{a-1}(k+1)}{(P-1)!}$$

$$\Omega = \lim_{n\to\infty}(logn - H_n) - \frac{1}{9}\lim_{n\to\infty}\left(\zeta(9) - \frac{\Psi_8(n+1)}{8!}\right) - \frac{1}{5}\lim_{n\to\infty}\left(\zeta(5) - \frac{\Psi_4(n+1)}{4!}\right)$$

Therefore,

$$\Omega = -\gamma - \frac{\zeta(5)}{5} - \frac{\zeta(9)}{9}$$

Where γ, Ψ, H_n, ζ —denote the Euler constant, Polygamma function, Harmonic Series and zeta function respectively.

Generalization by Ay Men-Algerie

$$\Omega = \lim_{n\to\infty}\left(\log n - \frac{1}{\pi}\sum_{k=1}^{n}\int_{-\frac{1}{k}}^{\frac{1}{k}}\sum_{m=0}^{p} x^{2m}\cos^{-1}(kx)\,dx\right) =$$

$$= \lim_{n\to\infty}\left(\log n - \frac{1}{\pi}\sum_{m=0}^{p}\sum_{k=1}^{n}\left(\int_{-\frac{1}{k}}^{\frac{1}{k}}\frac{kx^{2m+1}}{(2m+1)\sqrt{1-k^2x^2}}dx - \frac{\pi}{2m+1}\left(-\frac{1}{k}\right)^{2m+1}\right)\right) =$$

$$= \lim_{n\to\infty}\left(\log n - \sum_{m=0}^{p}\frac{1}{2m+1}\sum_{k=}^{n}\left(\frac{1}{k}\right)^{2m+1}\right) =$$

$$= \lim_{n\to\infty}\left(\log n - \left(\sum_{m=1}^{p}\frac{1}{2m+1}\sum_{k=1}^{n}\left(\frac{1}{k}\right)^{2m+1} + \sum_{k=1}^{n}\frac{1}{k}\right)\right) =$$

$$= -\lim_{n\to\infty}\left(\sum_{m=1}^{p}\frac{1}{2m+1}\sum_{k=1}^{n}\frac{1}{k^{2m+1}} + \sum_{k=1}^{n}\frac{1}{k} - \log n\right) =$$

$$= -\left(\sum_{m=1}^{p}\frac{1}{2m+1}\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}} + \gamma\right) = -\left(\sum_{m=1}^{p}\frac{\zeta(2m+1)}{2m+1} + \gamma\right)$$

1321. Find a closed form:

$$\Omega(a,b) = \int_a^b \log\left(\frac{\left(\frac{x}{a}\right)^{\frac{1}{x}\tan^{-1}\left(\frac{b}{x}\right)}}{\left(\frac{b}{x}\right)^{\frac{1}{x}\tan^{-1}\left(\frac{x}{a}\right)}}\right)dx, 1 < a < b$$

Proposed by Daniel Sitaru-Romania

Solution by Mikael Bernardo-Mozambique

$$\Omega(a,b) = \int_a^b \log\left(\frac{\left(\frac{x}{a}\right)^{\frac{1}{x}\tan^{-1}\left(\frac{b}{x}\right)}}{\left(\frac{b}{x}\right)^{\frac{1}{x}\tan^{-1}\left(\frac{x}{a}\right)}}\right)dx =$$

$$=\int_a^b \frac{1}{x}tan^{-1}\left(\frac{b}{x}\right)log\left(\frac{x}{a}\right)dx-\int_a^b \frac{1}{x}tan^{-1}\left(\frac{x}{a}\right)log\left(\frac{b}{x}\right)dx=\Omega_1-\Omega_2$$

$$\Omega_1:\frac{b}{x}=\frac{t}{a}\to x=\frac{ab}{t}\to dx=-\frac{ab}{t^2}dt$$

$$\Omega_1=\int_a^b \frac{t}{ab}tan^{-1}\left(\frac{t}{a}\right)log\left(\frac{b}{t}\right)\frac{ab}{t^2}dt=\int_a^b \frac{1}{t}tan^{-1}\left(\frac{t}{a}\right)log\left(\frac{b}{t}\right)dt=\Omega_2\to$$

$$\Omega(a,b)=\Omega_1-\Omega_2=0$$

1322. Find a closed form:

$$\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)}{x^2\sqrt{1+2x^2}}dx, a>0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Igor Soposki-Skopje-Macedonia

$$\Omega(a)=\int\frac{tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)}{x^2\sqrt{1+2x^2}}dx\overset{u=tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)}{=}-\int\frac{t}{\sqrt{t^2+2}}dt\overset{t^2+2=u^2}{=}-u=$$

$$=-\sqrt{t^2+2}=-\sqrt{\frac{1}{x^2}+2}=-\frac{\sqrt{1+2x^2}}{x}$$

$$\Omega=-tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)\cdot\frac{\sqrt{1+2x^2}}{x}-\int\frac{2x}{(x^2+1)\sqrt{1+2x^2}}\cdot\frac{\sqrt{1+2x^2}}{x}dx$$

$$\Omega(a)=-\left[tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)\cdot\frac{\sqrt{1+2x^2}}{x}+2tan^{-1}x\right]_{\frac{1}{a}}^{a}=$$

$$=-\left[tan^{-1}\left(\frac{\sqrt{1+2a^2}}{a^2}\right)\cdot\frac{\sqrt{1+2a^2}}{a}+2tan^{-1}a\right]+\left[tan^{-1}\left(\frac{\sqrt{1+2\left(\frac{1}{a}\right)^2}}{\left(\frac{1}{a}\right)^2}\right)\cdot\frac{\sqrt{1+2\left(\frac{1}{a}\right)^2}}{\frac{1}{a}}+2tan^{-1}\left(\frac{1}{a}\right)\right]$$

Solution 2 by Samar Das-India

$$\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)}{x^2\sqrt{1+2x^2}}dx\overset{y=\frac{1}{x}}{=}\int_a^{\frac{1}{a}}\frac{tan^{-1}\left(\frac{\sqrt{1+\frac{2}{y^2}}}{\frac{1}{y^2}}\right)}{\frac{1}{y^2}\sqrt{1+\frac{2}{y^2}}}\left(-\frac{1}{y^2}\right)dy=$$

$$= \int_{\frac{1}{a}}^{a} \frac{\tan^{-1}\left(y\sqrt{y^2+2}\right)}{\sqrt{y^2+2}} y dy = \int_{\frac{1}{a}}^{a} \frac{\tan^{-1}\left(x\sqrt{x^2+1}\right)}{\sqrt{x^2+2}} x dx \overset{x^2+2=z^2}{=}$$

$$= \int_{\frac{\sqrt{1+2a^2}}{a}}^{\sqrt{a^2+2}} \frac{\tan^{-1}\left(z\sqrt{z^2-2}\right)}{z} z dz = \int_{\frac{\sqrt{1+2a^2}}{a}}^{\sqrt{a^2+2}} \tan^{-1}\left(z\sqrt{z^2-2}\right) dz$$

$$\left(\because \int \tan^{-1}\left(z\sqrt{z^2-2}\right) = \right) z\tan^{-1}\left(z\sqrt{z^2-2}\right) - \int \frac{2(z^2-1)zdz}{(z^2-1)^2\sqrt{z^2-2}} =$$

$$= z\tan^{-1}\left(z\sqrt{z^2-2}\right) - 2\tan^{-1}\sqrt{z^2-2} + C$$

$$\Omega(a) = \left[z\tan^{-1}\left(z\sqrt{z^2-2}\right) - 2\tan^{-1}\sqrt{z^2-2}\right]_{\frac{\sqrt{1+2a^2}}{a}}^{\sqrt{a^2+2}} =$$

$$= \sqrt{a^2+2}\tan^{-1}\left(a\sqrt{a^2+2}\right) - 2\tan^{-1}a - \frac{\sqrt{1+2a^2}}{a}\tan^{-1}\left(\frac{1}{a}\cdot\frac{\sqrt{1+2a^2}}{a}\right)$$

$$+ 2\tan^{-1}\left(\frac{1}{a}\right) =$$

$$\sqrt{a^2+2}\tan^{-1}\left(a\sqrt{a^2+2}\right) - 2\tan^{-1}a - \frac{\sqrt{1+2a^2}}{a}\tan^{-1}\left(\frac{\sqrt{1+2a^2}}{a^2}\right) - 2(\frac{\pi}{2}$$

$$- 2\cot^{-1}a)$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega(a) = \int_{\frac{1}{a}}^{a} \frac{\tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)}{x^2\sqrt{1+2x^2}} dx = \int_{\frac{1}{a}}^{a} \frac{\tan^{-1}\sqrt{\frac{1}{x^2}+2}}{x^3\sqrt{\frac{1}{x^2}+2}} dx \overset{\sqrt{\frac{1}{x^2}+2}=\tan\theta,\, b=\tan^{-1}\left(\sqrt{a^2+2}\right),\, c=\tan^{-1}\left(\sqrt{\frac{1}{a^2}+2}\right)}{=}$$

$$= \int_b^c \frac{\theta}{\tan\theta}(-\tan\theta)\sec^2\theta d\theta = \int_c^b \theta\sec^2\theta d\theta =$$

$$= [\theta\tan\theta]_c^b - \int_c^b \tan\theta d\theta = b\tan b - c\tan c - [\log(\sec\theta)]_c^b =$$

$$= \sqrt{a^2+2}\tan^{-1}\sqrt{a^2+2} - \frac{1}{a}\sqrt{1+2a^2}\tan^{-1}\sqrt{\frac{1+2a^2}{a^2}} + \log\sqrt{\frac{1+\tan^2 c}{1+\tan^2 b}} =$$

$$= \sqrt{a^2+2}tan^{-1}\sqrt{a^2+2} - \frac{1}{a}\sqrt{1+2a^2}tan^{-1}\sqrt{\frac{1+2a^2}{a^2}} + \frac{1}{2}log\left(\frac{3a^2+1}{a^2(3+a^2)}\right)$$

Solution 4 by Probal Chakraborty-India

$$\Omega(a) = \int_{\frac{1}{a}}^{a} \frac{tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)}{x^2\sqrt{1+2x^2}}dx \overset{IBP}{=}$$

$$= \left[tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)\int \frac{dx}{x^2\sqrt{1+2x^2}} - \int\left(\frac{d}{dx}\left(tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)\int \frac{dx}{x^2\sqrt{1+2x^2}}\right)\right)dx\right]_{\frac{1}{a}}^{a}$$

$$\because \int \frac{dx}{x^2\sqrt{1+2x^2}} \overset{x=\frac{1}{t}}{=} -\int \frac{dt}{\sqrt{1+\frac{2}{t^2}}} = -\frac{1}{2}\int \frac{2tdt}{\sqrt{t^2+2}} = -\sqrt{t^2+2} = -\frac{\sqrt{1+2x^2}}{x}$$

$$\because \frac{d}{dx}\left[tan^{-1}\left(\frac{\sqrt{1+2x^2}}{x^2}\right)\right] = -\frac{2x}{(x^2+1)\sqrt{1+2x^2}}$$

$$\Omega(a) = -tan^{-1}\left(\frac{\sqrt{1+2a^2}}{a^2}\right)\frac{\sqrt{1+2a^2}}{a} + tan^{-1}\left(a\sqrt{1+2a^2}\right)$$

$$-\int_{\frac{1}{a}}^{a} \frac{2x\sqrt{1+2x^2}}{(x^2+1)x\sqrt{1+2x^2}}dx =$$

$$= -tan^{-1}\left(\frac{\sqrt{1+2a^2}}{a^2}\right)\frac{\sqrt{1+2a^2}}{a} + tan^{-1}\left(a\sqrt{1+2a^2}\right) - \int_{\frac{1}{a}}^{a} \frac{2}{x^2+1}dx =$$

$$= -tan^{-1}\left(\frac{\sqrt{1+2a^2}}{a^2}\right)\frac{\sqrt{1+2a^2}}{a} + tan^{-1}\left(a\sqrt{1+2a^2}\right) - 2\left(tan^{-1}a - tan\wedge -1\left(\frac{1}{a}\right)\right)$$

1323. Prove or disprove:

$$\int_0^{\infty} \frac{sin(a\pi x)}{(1-x^2)(2^2-x^2)\cdot \ldots \cdot (n^2-x^2)}\frac{dx}{x} = \frac{\pi 2^{2n-1}}{(n!)^2}(1-(-1)^a)$$

Proposed by Ghazaly Abiodun-Nigeria

Solution by Syed Shahabudeen-India

Let us check for $a \in 2k+1, k \in \mathbb{N}$ and $n = 2$.

$$LHS=\int_0^{\infty}\frac{\sin(a\pi x)}{x(1-x^2)(4-x^2)}dx=$$

$$=\lim_{z\to\infty}\int_0^{z}\sin(a\pi x)\left(\frac{1}{4x}-\frac{1}{6(x+1)}+\frac{1}{6(1-x)}+\frac{1}{24(2-x)}+\frac{1}{24(x+2)}\right)dx=$$

$$=\lim_{z\to\infty}\left(\frac{1}{4}\int_0^{a\pi z}\frac{\sin t}{t}dt+\frac{1}{6}\int_{a\pi}^{a\pi(1+z)}\frac{\sin t}{t}dt-\frac{1}{6}\int_{a\pi}^{a\pi(1-z)}\frac{\sin t}{t}dt-\frac{1}{24}\int_{2a\pi}^{a\pi(2-z)}\frac{\sin t}{t}dt\right.$$
$$\left.+\frac{1}{24}\int_{2a\pi}^{a\pi(2+z)}\frac{\sin t}{t}dt\right)=$$

$$=\lim_{z\to\infty}\frac{1}{24}(6Si(a\pi z)+4Si(a\pi+\pi z)-4Si(a\pi-\pi z)-Si(2a\pi-2z)+Si(2a\pi$$
$$+2z))=\frac{2^3\pi}{4!}=\frac{\pi}{3}$$

If we check the RHS then it gives: $RHS=\frac{2^4\pi}{(2!)^2}\Rightarrow LHS\neq LHS$

If we observe the pattern in LHS for different values of $n\in\mathbb{N}$ it should be

$$\frac{\pi 2^{2n-1}}{(n!)^2}(1-(-1)^a)$$

Therefore,

$$\int_0^{\infty}\frac{\sin(a\pi x)}{(1-x^2)(2^2-x^2)\cdot\ldots\cdot(n^2-x^2)}\frac{dx}{x}=\frac{\pi 2^{2n-2}}{(2n!)^2}(1-(-1)^a)$$

1324. Show that:

$$\Omega=\int_0^{\frac{\pi}{4}}\tan^{-1}\sqrt{\frac{\cos(2\theta)}{2\cos^2\theta}}d\theta=\frac{\pi^2}{24}$$

Proposed by Simon Peter-Vangaindrano-Madagascar

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{4}}\tan^{-1}\sqrt{\frac{\cos(2\theta)}{2\cos^2\theta}}d\theta=\int_0^{\frac{\pi}{4}}\int_0^1\frac{\sqrt{\frac{\cos(2\theta)}{2\cos^2\theta}}}{1+\frac{\cos(2\theta)}{2\cos^2\theta}x^2}dxd\theta$$

$$\Omega=\int_0^1\int_0^{\frac{\pi}{4}}\frac{\sqrt{1-2\sin^2\theta}}{2-2\sin^2\theta+(1-2\sin^2\theta)x^2}\sqrt{2}\cos\theta d\theta dx\overset{\sqrt{2}\sin\theta=\sin\varphi}{=}$$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{2\cos^2\varphi + \sin^2\varphi + \cos^2\varphi x^2} d\varphi dx =$$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sin^2\varphi + \cos^2\varphi(x^2+2)} d\varphi dx = \int_0^1 \int_0^{\frac{\pi}{2}} \frac{1}{\tan^2\varphi + x^2 + 2} d\varphi dx$$

$$\Omega \overset{y=\tan\varphi}{=} \int_0^1 \int_0^{\infty} \frac{1}{(x^2+y^2+2)(1+y^2)} dy dx$$

$$= \int_0^1 \frac{1}{1+x^2} \int_0^{\infty} \left(\frac{1}{y^2+1} - \frac{1}{x^2+y^2+2}\right) dy dx$$

$$\Omega = \int_0^1 \frac{1}{1+x^2} \left[\tan^{-1}y - \frac{1}{\sqrt{x^2+2}} \tan^{-1}\left(\frac{y}{\sqrt{x^2+2}}\right)\right]_0^{\infty} dx =$$

$$= \frac{\pi}{2} \int_0^1 \left(\frac{1}{1+x^2} - \frac{1}{(x^2+1)\sqrt{x^2+2}}\right) dx$$

$$\Omega \overset{x=\tan z}{=} \frac{\pi^2}{8} - \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \frac{\cos z}{\sqrt{1+\cos^2 z}} dz = \frac{\pi^2}{8} - \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \frac{\cos z}{\sqrt{2-\sin^2 z}} dz \overset{\sqrt{2}t=\sin z}{=}$$

$$= \frac{\pi^2}{8} - \frac{\pi}{2} \int_0^{\frac{1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Omega = \frac{\pi^2}{8} - \frac{\pi}{2} \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi^2}{8} - \frac{\pi^2}{12} = \frac{\pi^2}{24}$$

1325. Find:

$$\Omega = \int x^{-2} \cdot \tan\left(\frac{1}{2x}\right) \cdot \tan\left(\frac{1}{3x}\right) \cdot \tan\left(\frac{1}{6x}\right) dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Yen Tung Chung-Taichung-Taiwan

$$\Omega = \int x^{-2} \cdot \tan\left(\frac{1}{2x}\right) \cdot \tan\left(\frac{1}{3x}\right) \cdot \tan\left(\frac{1}{6x}\right) dx \overset{\theta=\frac{1}{6x}}{=}$$

$$= \int \tan(3\theta)\tan(2\theta)\tan\theta \cdot \left(-\frac{1}{6}d\theta\right) = -\frac{1}{6}\int \tan(3\theta)\tan(2\theta)\tan\theta \, d\theta =$$

$$\left(\begin{array}{c} \because \tan(3\theta) = \tan(2\theta+\theta) = \dfrac{\tan(2\theta)+\tan\theta}{1-\tan 2\theta \tan\theta} \Rightarrow \\ \tan(3\theta)\tan(2\theta)\tan\theta = \tan(3\theta) - \tan(2\theta) - \tan\theta \end{array}\right)$$

$$=-\frac{1}{6}\int(tan(3\theta)-tan(2\theta)-tan\theta)d\theta=$$

$$=-\frac{1}{18}log|sec3\theta|+\frac{1}{12}log|sec2\theta|+\frac{1}{6}log|sec\theta|+C=$$

$$=-\frac{1}{18}log\left|sec\left(\frac{1}{2x}\right)\right|+\frac{1}{12}log\left|sec\left(\frac{1}{3x}\right)\right|+\frac{1}{6}log\left|sec\left(\frac{1}{6x}\right)\right|+C$$

Solution 2 by Hussain Reza Zadah-Afghanistan

$$\Omega=\int x^{-2}\cdot tan\left(\frac{1}{2x}\right)\cdot tan\left(\frac{1}{3x}\right)\cdot tan\left(\frac{1}{6x}\right)dx\overset{u=\frac{1}{x}}{=}$$

$$=-\int tan\left(\frac{u}{2}\right)tan\left(\frac{u}{3}\right)tan\left(\frac{u}{6}\right)du\overset{\frac{u}{6}=t}{=}$$

$$=-6\int tan(3t)tan(2t)tant\,dt=$$

$$(\because tan(3t)tan(2t)tant=tan(3t)-tan(2t)-tant)$$

$$=-6\int(tan(3t)-tan(2t)-tant)dt=$$

$$=-6\left(-\frac{1}{3}log|cos3t|+\frac{1}{2}log|cos(2t)|+log|cost|\right)+C=$$

$$=2log\left|cos\left(\frac{1}{2x}\right)\right|-3log\left|cos\left(\frac{1}{3x}\right)\right|-6log\left|cos\left(\frac{1}{6x}\right)\right|+C$$

Solution 3 by Mohammad Rostami-Kabul-Afghanistan

$$\frac{1}{2x}-\frac{1}{3x}=\frac{1}{6x}\Rightarrow tan\left(\frac{1}{2x}-\frac{1}{3x}\right)=tan\left(\frac{1}{6x}\right)\Rightarrow$$

$$\frac{tan\left(\frac{1}{2x}\right)-tan\left(\frac{1}{3x}\right)}{1+tan\left(\frac{1}{2x}\right)tan\left(\frac{1}{3x}\right)}=tan\left(\frac{1}{6x}\right)$$

$$\left(\because tan\left(\frac{1}{2x}\right)tan\left(\frac{1}{3x}\right)tan\left(\frac{1}{6x}\right)=tan\left(\frac{1}{2x}\right)-tan\left(\frac{1}{3x}\right)-tan\left(\frac{1}{6x}\right)\right)$$

$$\Omega=\int\frac{1}{x^2}\left(tan\left(\frac{1}{2x}\right)-tan\left(\frac{1}{3x}\right)-tan\left(\frac{1}{6x}\right)\right)dx\overset{\frac{1}{x}=y}{=}$$

$$=-\int tan\left(\frac{y}{2}\right)dy+\int tan\left(\frac{y}{3}\right)dy+\int tan\left(\frac{y}{6}\right)dy=$$

$$=-2\int\frac{1}{2}tan\left(\frac{y}{2}\right)dy+3\int\frac{1}{3}tan\left(\frac{y}{3}\right)dy+6\int\frac{1}{6}tan\left(\frac{y}{6}\right)dy=$$

$$= 2log\left|cos\frac{y}{2}\right| - 3log\left|cos\frac{y}{3}\right| - 6log\left|cos\frac{y}{6}\right| + C$$

1326. Evaluate the integral in a closed –form:

$$\int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)} + \frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1} + \frac{16sinx}{4cosx+1}\right)dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)} + \frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1} + \frac{16sinx}{4cosx+1}\right)dx = \int_0^{\frac{\pi}{3}}\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}dx +$$

$$+\left[6\sqrt{2}log\left(2sin\left(\frac{3x}{4}\right)+1\right) - 4log(1+4cosx)\right]_0^{\frac{\pi}{3}}$$

$$\Omega = 6\sqrt{2}log(1+\sqrt{2}) + 4log\left(\frac{5}{3}\right) + \underbrace{\int_0^{\frac{\pi}{3}}\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}dx}_{I}$$

$$I = \int_0^{\frac{\pi}{3}}\frac{sin^2x}{cos^2xcos^3\left(\frac{x}{2}\right)}dx = 4\int_0^{\frac{\pi}{3}}\frac{sin^2\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right)}{cos^2xcos^2\left(\frac{x}{2}\right)}dx =$$

$$= 4\int_0^{\frac{\pi}{3}}\frac{sin^2\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right)}{\left(1-sin^2\left(\frac{x}{2}\right)\right)\left(1-2sin^2\left(\frac{x}{2}\right)\right)^2}$$

$$I \overset{t=sin\frac{x}{2}}{=} 8\int_0^{\frac{1}{2}}\frac{t^2}{(1-t^2)(1-2t^2)^2}dt = 8\int_0^{\frac{1}{2}}\left(\frac{2}{2t^2-1} + \frac{1}{(2t^2-1)^2} - \frac{1}{t^2-1}\right)dt =$$

$$= 8\left[\frac{3\sqrt{2}}{8}log\left(\frac{\sqrt{2}-2t}{2t+\sqrt{2}}\right) + \frac{t}{2(1-2t^2)} + \frac{1}{2}log\left(\frac{1+t}{1-t}\right)\right]_0^{\frac{1}{2}} =$$

$$= 3\sqrt{2}log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) + 4 + 4log3 = 4 - 6\sqrt{2}log(1+\sqrt{2}) + 4log3 =$$

$$= 6\sqrt{2}log(1+\sqrt{2}) + 4log\left(\frac{5}{3}\right) + 4 - 6\sqrt{2}log(1+\sqrt{2}) + 4log3 = 4(1+log5)$$

Therefore,

$$\int_0^{\frac{\pi}{3}}\left(\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1}+\frac{16\sin x}{4\cos x+1}\right)dx=4(1+\log 5)$$

Solution 2 by Samar Das-India

$$\int_0^{\frac{\pi}{3}}\left(\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1}+\frac{16\sin x}{4\cos x+1}\right)dx=$$

$$=\int_0^{\frac{\pi}{3}}\left(\frac{4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\left(1-\sin^2\left(\frac{x}{2}\right)\right)\left(1-2\sin^2\left(\frac{x}{2}\right)\right)^2}+\frac{32\sin\frac{x}{2}\cos\frac{x}{2}}{5-8\sin^2\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{2}\right)}{1+2\sin\left(\frac{3x}{2}\right)}\right)dx=$$

$$\overset{y=\sin\frac{x}{2}}{=}\int_0^{\frac{1}{2}}\frac{8y^2}{(1-y^2)(1-2y^2)^2}dy+64\int_0^{\frac{1}{2}}\frac{ydy}{5-8y^2}+6\sqrt{2}\log\left|1+2\sin\left(\frac{3x}{4}\right)\right|_0^{\frac{\pi}{3}}=$$

$$=8\int_0^{\frac{1}{2}}\frac{1}{1-2y^2}\left(\frac{1}{y^2-1}-\frac{1}{2y^2-1}\right)dy-\frac{64}{16}\int_0^{\frac{1}{2}}\frac{ydy}{5-8y^2}+6\sqrt{2}\log(1+\sqrt{2})=$$

$$=-8\int_0^{\frac{1}{2}}\left(\frac{1}{y^2-1}-\frac{1}{y^2-\frac{1}{2}}\right)dy+\int_0^{\frac{1}{2}}\left(\frac{1}{y-\frac{1}{\sqrt{2}}}-\frac{1}{y+\frac{1}{\sqrt{2}}}\right)^2dy$$

$$-4\left(\log\left|5-\frac{8}{4}\right|-\log 5\right)+6\sqrt{2}\log(1+\sqrt{2})=$$

$$=-4\log\left|\frac{y-1}{y+1}\right|_0^{\frac{1}{2}}+\frac{8\sqrt{2}}{2}\log\left|\frac{y-\frac{1}{\sqrt{2}}}{y+\frac{1}{\sqrt{2}}}\right|_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}}\frac{dy}{\left(y-\frac{1}{\sqrt{2}}\right)^2}+\int_0^{\frac{1}{2}}\frac{dy}{\left(y+\frac{1}{\sqrt{2}}\right)^2}-$$

$$-2\int_0^{\frac{1}{2}}\frac{dy}{y^2-\left(\frac{1}{\sqrt{2}}\right)^2}-4\log\left(\frac{3}{5}\right)+6\sqrt{2}\log(1+\sqrt{2})=$$

$$=-4\log\left(\frac{1}{3}\right)-4\log\left(\frac{3}{5}\right)+4\sqrt{2}\log\left|\frac{\frac{1}{2}-\frac{1}{\sqrt{2}}}{\frac{1}{2}+\frac{1}{\sqrt{2}}}\right|+\left(\frac{1}{y-\frac{1}{\sqrt{2}}}\right)_0^{\frac{1}{2}}+2\cdot\frac{\sqrt{2}}{2}\log\left|\frac{y-\frac{1}{\sqrt{2}}}{y+\frac{1}{\sqrt{2}}}\right|_0^{\frac{1}{2}}$$

$$+6\sqrt{2}\log(1+\sqrt{2})$$

Therefore,

$$\int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}+\frac{16sinx}{4cosx+1}\right)dx=4(1+log5)$$

Solution 3 by Timson Azeez Folorunsho-Nigeria

$$\int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}+\frac{16sinx}{4cosx+1}\right)dx=$$

$$=\int_0^{\frac{\pi}{3}}\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}dx+\int_0^{\frac{\pi}{3}}\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}dx+\int_0^{\frac{\pi}{3}}\frac{16sinx}{4cosx+1}dx=I_1+I_2+I_3$$

$$I_1=\int_0^{\frac{\pi}{3}}\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}dx=\int_0^{\frac{\pi}{3}}\frac{sin^2x}{cos^2x\,cos^3\left(\frac{x}{2}\right)}dx\overset{y=\frac{x}{2}}{\triangleq}2\int_0^{\frac{\pi}{6}}\frac{sin^2(2y)dy}{cos^2(2y)cos^3y}=$$

$$=8\int_0^{\frac{\pi}{6}}\frac{sin^2y(1-sin^2y)cosy}{(1-2sin^2y)^2(1-sin^2y)^2}dy=8\int_0^{\frac{\pi}{6}}\frac{cosysin^2y}{(1-2sin^2y)(1-sin^2y)}dy\overset{u=siny}{\triangleq}$$

$$=8\int_0^{\frac{1}{2}}\left(\frac{2}{2u^2-1}+\frac{1}{(2u^2-1)^2}+\frac{1}{2(u+1)}-\frac{1}{2(u-1)}\right)du=$$

$$=8\int_0^{\frac{1}{2}}\left(\frac{1}{\sqrt{2}u-1}-\frac{1}{\sqrt{2}u+1}+\frac{1}{(2u^2-1)^2}+\frac{1}{2(u+1)}-\frac{1}{2(u-1)}\right)du=$$

$$=8\int_0^{\frac{1}{2}}\left(\frac{1}{\sqrt{2}u-1}-\frac{1}{\sqrt{2}u+1}+\frac{1}{4(\sqrt{2}u+1)}+\frac{1}{4(\sqrt{2}u+1)^2}-\frac{1}{4(\sqrt{2}u-1)}\right.$$

$$\left.+\frac{1}{4(\sqrt{2}u-1)^2}+\frac{1}{2(u+1)}-\frac{1}{2(u-1)}\right)du=$$

$$=3\sqrt{2}log\left|\frac{\sqrt{2}-2}{\sqrt{2}+2}\right|+\frac{1}{2}log3+5$$

$$I_2=\int_0^{\frac{\pi}{3}}\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}dx\overset{v=\frac{3x}{4}}{\triangleq}\frac{4}{3}\int_0^{\frac{\pi}{4}}\frac{9\sqrt{2}cosv}{2sinv+1}dv\overset{y=2sinv+1}{\triangleq}$$

$$=\frac{4}{3}\int_1^{1+\sqrt{2}}\frac{9\sqrt{2}}{2y}dy=6\sqrt{2}\int_1^{1+\sqrt{2}}\frac{dy}{y}=6\sqrt{2}log(1+\sqrt{2})$$

$$I_3=\int_0^{\frac{\pi}{3}}\frac{16sinx}{4cosx+1}dx\overset{y=4xosx+1}{\cong}-4\int_5^3\frac{dy}{y}=4log\left(\frac{5}{3}\right)$$

Therefore,

$$\int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}+\frac{16sinx}{4cosx+1}\right)dx=4(1+log5)$$

1327. Prove that:

$$\int_{-1}^1\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)dx=\pi$$

Proposed by Mohammed Bouras-Morocco

Solution 1 by Rana Ranino-Setif-Algerie

$$\int_{-1}^1\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)dx=$$

$$=\int_{-1}^0\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)+\int_0^1\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)dx=$$

$$=\int_0^1\left[\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)-\frac{x}{\sqrt{1-x^2}}log\left(\frac{1-x}{1+\sqrt{1-x^2}}\right)\right]dx=$$

$$=\int_0^1\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1-x}\right)dx$$

$$\Omega=\underbrace{\left[-\sqrt{1-x^2}log\left(\frac{1+x}{1-x}\right)\right]_0^1}_{=0}+2\int_0^1\frac{\sqrt{1-x^2}}{1-x^2}dx\overset{x=sin\theta}{\cong}2\int_0^{\frac{\pi}{2}}\frac{cos^2\theta}{cos^2\theta}d\theta=\pi$$

Therefore,

$$\int_{-1}^1\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)dx=\pi$$

Solution 2 by Serlea Kabay-Liberia

$$\Omega=\int_{-1}^1\frac{x}{\sqrt{1-x^2}}log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)dx=$$

$$= \underbrace{\int_{-1}^{1} \frac{x\log(1+x)}{\sqrt{1-x^2}}dx}_{I_1} - \underbrace{\int_{-1}^{1} \frac{x\log\left(1+\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}dx}_{I_2}$$

Now,

$$I_1 = \int_{-1}^{1} \frac{x\log(1+x)}{\sqrt{1-x^2}}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\log(1+\sin\theta)\,d\theta =$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{1+\sin\theta}d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin\theta)\,d\theta =$$

$$= \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2} + \cos\left(-\frac{\pi}{2}\right)\right) = \pi$$

$$I_2 = \int_{-1}^{1} \frac{x\log\left(1+\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}dx = \int_{-1}^{1} f(x)\,dx; \because f(-x) = f(x) \Rightarrow$$

$$2I_2 = \int_{-1}^{1} f(x)\,dx - \int_{-1}^{1} f(x)\,dx = 0 \Rightarrow I_2 = 0$$

Therefore,

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \log\left(\frac{1+x}{1+\sqrt{1-x^2}}\right)dx = \pi$$

1328. Prove without any software:

$$\int_0^1 \frac{x^3}{1+x^3+x^{12}}dx + \int_0^1 \frac{x^5}{1+x^5+x^{10}}dx + \int_0^1 \frac{x^7}{1+x^7+x^8}dx < 1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM-GM inequality: $\forall x \in [0,1]$, we have:

$$1 + x^7 + x^8 \geq 3\sqrt[3]{x^{15}} = 3x^5 \to \frac{x^7}{1+x^7+x^8} \leq \frac{x^2}{3}$$

$$1 + x^5 + x^{10} \geq 3\sqrt[3]{x^{15}} = 3x^5 \to \frac{x^5}{1+x^5+x^{10}} \leq \frac{1}{3}$$

$$1 + x^3 + x^{12} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + x^3 + x^{12} \geq 5\sqrt[5]{\frac{x^{15}}{27}} \to \frac{x^3}{1+x^3+x^{12}} \leq \frac{\sqrt[5]{27}}{5} \to$$

$$\int_0^1 \frac{x^3}{1+x^3+x^{12}}dx + \int_0^1 \frac{x^5}{1+x^5+x^{10}}dx + \int_0^1 \frac{x^7}{1+x^7+x^8}dx \le$$

$$\le \frac{\sqrt[5]{27}}{5} + \frac{4}{9} \overset{(*)}{\le} 1; (*) \Leftrightarrow \sqrt[5]{27} \le \frac{25}{9}$$

-which is true because, $\sqrt[5]{27} < \sqrt[5]{32} = 2 < \frac{25}{9}$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$x \in [0,1] \to 1-x^2 \in [0,1] \to x^3(1-x^2) \ge x^{10}(1-x^2) \to$$

$$1+x^3+x^{12} \ge 1+x^{10}+x^5 \to \frac{1}{1+x^3+x^{12}} \le \frac{1}{1+x^5+x^{10}}; (1)$$

$$x \in [0,1] \to 1+x^5+x^{10} \ge 1+x^7+x^{12} \to \frac{1}{1+x^3+x^{12}} \le \frac{1}{1+x^5+x^{12}}; (2)$$

$$1+x^7+x^8 \ge 3\sqrt[3]{x^{15}} = 3x^5 \to \frac{1}{1+x^7+x^8} \le \frac{1}{3x^5}; (3)$$

$$1+x^3+x^{12} = \frac{1}{2}+\frac{1}{4}+\frac{1}{4}+x^3+x^{12} \ge 5\sqrt[5]{\frac{1}{12}x^{15}} \to \frac{x^3}{1+x^3+x^{12}} \le \frac{2}{5}; (4)$$

$$\int_0^1 \frac{x^3}{1+x^3+x^{12}}dx + \int_0^1 \frac{x^5}{1+x^5+x^{10}}dx + \int_0^1 \frac{x^7}{1+x^7+x^8}dx \le$$

$$\le \int_0^1 \frac{x^3+x^5+x^7}{1+x^7+x^8}dx \le \int_0^1 \frac{x^3}{1+x^3+x^{12}}dx + \int_0^1 \frac{x^5+x^7}{3x^5}dx =$$

$$= \int_0^1 \frac{x^3}{1+x^3+x^{12}}dx + \frac{4}{9} \le \frac{2}{5} + \frac{4}{9} < 1$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\because \frac{x^3}{1+x^3+x^{12}} < \frac{1}{3} \Leftrightarrow x^{12} - x^3 - (x^3-1) \ge 0, \forall x \in [0,1]$$

$$(x^3-1)(x^3(x^6+x^3+1)-1) \ge 0 \text{ true } \forall x \in [0,1]$$

$$\because \frac{x^5}{1+x^5+x^{10}} < \frac{1}{3} \Leftrightarrow x^{10} - x^5 - (x^5-1) \ge 0$$

$$(x^5-1)^2 \ge 0 \text{ true } \forall x \in [0,1]$$

$$\because \frac{x^7}{1+x^7+x^8} < \frac{1}{3} \Leftrightarrow x^8 - x^7 - (x^7-1) \ge 0$$

$$(x-1)(x^7-x^6-x^5-x^4-x^3-x^2-x-1)\ge 0 \text{ true } \forall x\in[0,1]$$

Hence,

$$\int_0^1 \frac{x^3}{1+x^3+x^{12}}dx+\int_0^1 \frac{x^5}{1+x^5+x^{10}}dx+\int_0^1 \frac{x^7}{1+x^7+x^8}dx$$

$$<\int_0^1 \frac{1}{3}dx+\int_0^1 \frac{1}{3}dx+\int_0^1 \frac{1}{3}dx=1$$

Solution 4 by Ravi Prakash-New Delhi-India

$$\text{For } 0\le x\le 1\to 1+x^3+x^{12},1+x^5+x^{10},1+x^7+x^8\ge 1$$

$$\int_0^1 \frac{x^3}{1+x^3+x^{12}}dx+\int_0^1 \frac{x^5}{1+x^5+x^{10}}dx+\int_0^1 \frac{x^7}{1+x^7+x^8}dx\le$$

$$\le\int_0^1 x^3dx+\int_0^1 x^5dx+\int_0^1 x^7dx=\frac{1}{4}+\frac{1}{6}+\frac{1}{8}=\frac{13}{24}<1$$

1329. If $0<a\le b$ then:

$$\int_a^b\int_a^b\int_a^b \frac{x^2+y^2+z^2}{x+y+z}dxdydz\le \log\left(\frac{b}{a}\right)^{\frac{(b-a)(b^3-a^3)}{3}}$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

We must to prove that:

$$\frac{x^2+y^2+z^2}{x+y+z}\le\frac{\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}}{3}\Leftrightarrow\frac{x^2+y^2+z^2}{x+y+z}\le\frac{x^3z+y^3x+z^3y}{3xyz}$$

$$3x^3yz+3xy^3z+3xyz^3\le x^4z+y^3x^2+z^3xy+x^3yz+y^4x+x^3z^2+y^3xz+z^4y$$

$$f(x,y,z)=x^4z+y^4x+z^4y+x^3z^2+y^3x^2+z^3y^2-2x^3yz-2y^3zx-2z^3xy\ge 0$$

$$f(x,y,z)\ge 0 \text{ if }\begin{cases} f(x,1,1)\ge 0,\forall x\in\mathbb{R}\\ f(0,y,z)\ge 0,\forall y,z\in\mathbb{R}\end{cases}$$

$$f(x,1,1)=(x-1)^2(x^2+x+2)\ge 0,\forall x\in\mathbb{R}$$

$$f(0,y,z)=z^4y+z^3y^2>0 \text{ because } y,z>0. \text{ Thus,}$$

$$3\frac{x^2+y^2+z^2}{x+y+z}\le\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\Leftrightarrow$$

$$3\int_a^b\int_a^b\int_a^b\frac{x^2+y^2+z^2}{x+y+z}dxdydz\le\int_a^b\int_a^b\int_a^b\frac{x^2}{y}dxdydz+$$

$$+\int_a^b\int_a^b\int_a^b\frac{y^2}{z}dxdydz+\int_a^b\int_a^b\int_a^b\frac{z^2}{x}dxdydz=I_1+I_2+I_3$$

$$I_1=\int_a^b\int_a^b\int_a^b\frac{x^2}{y}dxdydz=\int_a^b x^2dx\int_a^b\frac{1}{y}dy\int_a^b dz=\frac{x^3}{3}\cdot logy\cdot z\Big|_a^b=$$

$$=\frac{(b-a)(b^3-a^3)}{3}log\left(\frac{b}{a}\right)=log\left(\frac{b}{a}\right)^{\frac{(b-a)(b^3-a^3)}{3}}$$

Similarly,

$$I_2=\int_a^b\int_a^b\int_a^b\frac{y^2}{z}dxdydz=log\left(\frac{b}{a}\right)^{\frac{(b-a)(b^3-a^3)}{3}}$$

$$I_3=\int_a^b\int_a^b\int_a^b\frac{z^2}{x}dxdydz=log\left(\frac{b}{a}\right)^{\frac{(b-a)(b^3-a^3)}{3}}$$

$$3\int_a^b\int_a^b\int_a^b\frac{x^2+y^2+z^2}{x+y+z}dxdydz\le 3log\left(\frac{b}{a}\right)^{\frac{(b-a)(b^3-a^3)}{3}}$$

Therefore,

$$\int_a^b\int_a^b\int_a^b\frac{x^2+y^2+z^2}{x+y+z}dxdydz\le log\left(\frac{b}{a}\right)^{\frac{(b-a)(b^3-a^3)}{3}}$$

1330. If $0<a\le b<\frac{\pi}{2}$ then:

$$\frac{3}{2}\int_a^b\frac{sinx}{x}dx\le b-a+cos(\sqrt{ab})sin\left(\frac{b-a}{2}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\begin{cases}\sqrt{ab}\le\dfrac{a+b}{2}\\ cosx\downarrow, x\in\left(0,\dfrac{\pi}{2}\right)\end{cases}\to cos(\sqrt{ab})\ge cos\left(\frac{a+b}{2}\right)$$

$$\cos(\sqrt{ab})\sin\left(\frac{b-a}{2}\right) \geq \cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) = \frac{\sin b - \sin a}{2} = \frac{1}{2}\int_a^b \cos x\, dx$$

We must show that:

$$\frac{3}{2}\int_a^b \frac{\sin x}{x}dx \leq \int_a^b dx + \frac{1}{2}\int_a^b \cos x\, dx$$

$$3\int_a^b \frac{\sin x}{x}dx \leq \int_a^b (2x + \cos x)\, dx \Leftrightarrow \frac{3\sin x}{x} \leq 2 + \cos x \Leftrightarrow$$

$$3\sin x \leq 2x + x\cos x$$

$$f(x) = 2x + x\cos x, x \in \left(0, \frac{\pi}{2}\right), f'(x) = 2 - 2\cos x - x\sin x$$

$$f''(x) = \sin x - x\cos x > 0; \left(\sin x \geq x\cos x \Leftrightarrow \tan x \geq x, \forall x \in \left(0, \frac{\pi}{2}\right)\right) \to$$

$$3\sin x \leq 2x + x\cos x, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Solution 2 by Remus Florin Stanca-Romania

Let's prove that:

$$\frac{3}{2}\int_a^b \frac{\sin x}{x}dx \leq b - a + \cos\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) \Leftrightarrow$$

$$3\int_a^b \frac{\sin x}{x}dx \leq 2(b-a) + 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) \Leftrightarrow$$

$$3\int_a^b \frac{\sin x}{x}dx \leq 2\int_a^b dx + \sin b - \sin a \Leftrightarrow$$

$$3\int_a^b \frac{\sin x}{x}dx \leq 2\int_a^b dx + \int_a^b \cos x\, dx \Leftrightarrow \ 3\int_a^b \frac{\sin x}{x}dx \leq \int_a^b (2 + \cos x)dx$$

We need to prove that:

$$\frac{3\sin x}{x} \leq 2 + \cos x \Leftrightarrow 3\sin x \leq 2x + x\cos x \Leftrightarrow 2x + x\cos x - 3\sin x \geq 0 \Leftrightarrow$$

Let $f: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, f(x) = 2x + x\cos x - 3\sin x \to f'(x) = 2 - 2\cos x - x\sin x$

$$f''(x) = \sin x - x\cos x, f'''(x) = x\sin x \geq 0 \to f''(x) > 0 \to f'(x) > 0 \to f \nearrow \left[0, \frac{\pi}{2}\right]$$

$$f(x) \geq f(0) = 0 \to 2x + x\cos x - 3\sin x \geq 0 \to$$

$$\frac{3}{2}\int_a^b \frac{sinx}{x}dxx \le b-a+cos\left(\frac{a+b}{2}\right)cos\left(\frac{a-b}{2}\right)\text{ is true.}$$

$x \to cosx$ is decreasing on $\left[0,\frac{\pi}{2}\right], \frac{a+b}{2} \ge \sqrt{ab} \to cos\left(\frac{a+b}{2}\right) \le cos(\sqrt{ab}) \to$

$$b-a+cos\left(\frac{a+b}{2}\right)sin\left(\frac{b-a}{2}\right) \le b-acos(\sqrt{ab})sin\left(\frac{b-a}{2}\right);(2)$$

From (1),(2) it follows that:

$$\frac{3}{2}\int_a^b \frac{sinx}{x}dx \le b-a+cos(\sqrt{ab})sin\left(\frac{b-a}{2}\right)$$

Solution 3 by Samar Das-India

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$p=\frac{3}{2}\int_a^b \frac{sinx}{x}dx \le \frac{3}{2}\cdot\frac{sinb}{b}(b-a)=(b-a)\frac{sinb}{b}+\frac{1}{2}(b-a)\frac{sinb}{b}$$

$$\le (b-a)+\frac{b-a}{2}\cdot\frac{sinb}{b};(1)$$

$$f(x)=sinx-x \to f'(x)=cosx-1, x\in\left[0,\frac{\pi}{2}\right] \to sinx \le x \to \frac{sinb}{b} \le 1$$

$$cos(\sqrt{ab})sin\left(\frac{a+b}{2}\right) \ge cos\left(\frac{a+b}{2}\right)sin\left(\frac{b-a}{2}\right)$$

$$\because \frac{a+b}{2} \ge \sqrt{ab} \to cos\left(\frac{a+b}{2}\right) \le cos(\sqrt{ab})$$

$$=\frac{1}{2}(sinb-sina);(2)$$

$$q=(b-a)\frac{sinb}{b}-(sinb-sina)=(b-a)\left(\frac{sinb}{b}-\frac{sinb-sina}{b-a}\right);\left(0<a\le b<\frac{\pi}{2}\right)$$

$$=\frac{1}{b}(bsinb-asinb-bsinb+bsina)=\frac{1}{b}(bsina-asinb)=$$

$$=a\left(\frac{sina}{a}-\frac{sinb}{b}\right)<0$$

$$\because (b-a)f(a) \le \int_a^b \frac{sinx}{x}dx \le (b-a)f(b) \to (b-a)\frac{sina}{a} \le (b-a)\frac{sinb}{b} \to$$

$$\frac{sina}{a} \le \frac{sinb}{b}$$

$$q \le 0 \to \frac{1}{2}(b-a)\frac{sinb}{b} - \frac{1}{2}(sinb - sina) < 0 \to$$

$$\frac{1}{2}(b-a)\frac{sinb}{b} < \frac{1}{2}(sinb - sina); (3)$$

$$p = \frac{3}{2}\int_a^b \frac{sinx}{x}dx \overset{(1)}{\le} (b-a) + \frac{1}{2}(b-a)\frac{sinb}{b} \overset{(3)}{\le} (b-a) + \frac{1}{2}(sinb - sina) \overset{(2)}{\le}$$

$$\le (b-a) + cos(\sqrt{ab})sin\left(\frac{b-a}{2}\right)$$

1331. If $0 < a \le b, f: [a,b] \to [0,\infty), f$ –continuous function, then:

$$2(b-a)\int_a^b f(x)dx \ge \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy + \left(\int_a^b \sqrt{f(x)}dx\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\int_a^b \sqrt{f(x)}dx\right)^2 + \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy =$$

$$= \int_a^b\int_a^b \sqrt{f(x)}\sqrt{f(y)}dxdy + \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy =$$

$$= \int_a^b\int_a^b \left[\sqrt{f(x)f(y)} + \sqrt{\frac{1}{2}\left(f^2(x)+f^2(y)\right)}\right]dxdy \overset{CBS}{\le}$$

$$\le \int_a^b\int_a^b \sqrt{2\left[f(x)f(y) + \frac{1}{2}\left(f^2(x)+f^2(y)\right)\right]}dxdy = \int_a^b\int_a^b \left(f(x)+f(y)\right)dxdy =$$

$$2(b-a)\int_a^b f(x)dx$$

Solution 2 by Remus Florin Stanca-Romania

$$2(b-a)\int_a^b f(x)dx \ge \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy + \left(\int_a^b \sqrt{f(x)}dx\right)^2 \Leftrightarrow$$

$$2(b-a)\int_a^b f(x)dx \ge \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{\frac{f^2(x)+f^2(y)}{2}}dxdy + \left(\int_a^b \sqrt{f(x)}dx\right)^2 \Leftrightarrow$$

$$\int_a^b\int_a^b (f(x)+f(y))dxdy \ge \int_a^b\int_a^b \sqrt{\frac{f^2(x)+f^2(y)}{2}}dxdy + \int_a^b\int_a^b \sqrt{f(x)}\sqrt{f(y)}dxdy$$

We need to prove that: $a+b \ge \sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab} \Leftrightarrow \frac{a+b}{2} \ge \frac{1}{2}\sqrt{\frac{a^2+b^2}{2}}+\frac{1}{2}\sqrt{ab};(1)$

Let $f:\mathbb{R}_+\to\mathbb{R}$ such that: $f(x)=\sqrt{x}, f'(x)=\frac{1}{2}x^{-\frac{1}{2}}, f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}\le 0 \Rightarrow$

f –concave$\Rightarrow \forall t_1,t_2\in\left(0,\frac{\pi}{2}\right)$ such that $t_1+t_2=1$ and $\forall x_1,x_2\in I$:

$$t_1f(x_1)+t_2f(x_2)\le f(t_1x_2+t_2x_2);(JENSEN)$$

$$\frac{1}{2}\sqrt{\frac{a^2+b^2}{2}}+\frac{1}{2}\sqrt{ab}\le\sqrt{\frac{a^2+b^2+2ab}{4}}=\frac{a+b}{2}\Rightarrow (1) \text{ is true.}$$

1332. If $e\le a\le b$ then:

$$\int_a^b (logx)^{logx}dx\cdot\int_a^b (logx)^{-logx}dx \le log\left(\sqrt{\frac{b}{a}}\right)^{b^2-a^2}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

Let be $x,y\ge e$. WLOG $x\le y\to 1\le logx\le logy\to$

$$(logx)^{logx}\le(logy)^{logy},\frac{1}{y}\le\frac{1}{x}\to\frac{1}{y}(logx)^{logx}\le\frac{1}{x}(logy)^{logy}\to$$

$$\to(logx)^{logx}\cdot(logy)^{-logy}\le\frac{x}{y}\to$$

$$\to\int_a^b\int_a^b(logx)^{logx}\cdot(logy)^{-logy}dxdy\le\int_a^b\int_a^b\frac{x}{y}dxdy\to$$

$$\to\int_a^b(logx)^{logx}dx\cdot\int_a^b(logy)^{-logy}dx\le\int_a^b xdx\cdot\int_a^b\frac{1}{y}dy=$$

$$=\frac{b^2-a^2}{2}\cdot(logb-loga)=log\left(\sqrt{\frac{b}{a}}\right)^{b^2-a^2}$$

$$\int_a^b (\log x)^{\log x}dx \cdot \int_a^b (\log x)^{-\log x}dx \le \log\left(\sqrt{\frac{b}{a}}\right)^{b^2-a^2}$$

1333. Prove without any software:

$$\int_{\frac{1}{16}}^{\frac{1}{8}} \log\left(1+4x+8x^2+\frac{32x^4}{3}\right)dx < \frac{3}{128}$$

Proposed by Nikos Ntorvas-Greece

Solution by proposer

Let be the function $f(x) = 36e^x - 6x^3 - 18x^2, x \in \mathbb{R}; f''(x) = 36(e^x - x - 1) \ge 0$

Due to the well-known inequality $e^x \ge x+1, \forall x \in \mathbb{R}$ we have that f –is convex on $\mathbb{R}$.

The tangent for $A\big(0, f(0)\big)$ is the $y = 36x + 36$. Because f –is convex we have that

$$f(x) \ge y, \forall x \in \mathbb{R}; (1)$$

So, we have that (1) holds for $x \coloneqq 4x > 0$

$$f(4x) \ge y \Rightarrow 36e^{4x} - 6(4x)^3 - 18(4x)^2 \ge 36(4x) + 36 \Leftrightarrow$$

$$4x \ge \log\left(1+4x+8x^2+\frac{32x^4}{3}\right)$$

Therefore,

$$\int_{\frac{1}{16}}^{\frac{1}{8}} \log\left(1+4x+8x^2+\frac{32x^4}{3}\right)dx < \frac{3}{128}$$

1334. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^2} \cdot \left(\sum_{i+j+k+l=n} ijkl\right)\left(\sum_{i+j+k=n} ijk\right)^{-1}$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$$1) \because \binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

$$LHS=\binom{r}{r}+\binom{r+1}{r}+\dots+\binom{n}{r}=\binom{r+1}{r}+\binom{r+1}{r}+\dots+\binom{n}{r}=$$

$$=\binom{r+2}{r}+\binom{r+2}{r}+\dots+\binom{n}{r}=\dots=\binom{n+1}{r+1}$$

$$2)\ \forall n\geq 2, A_n=\sum_{i+j=n} ij=\binom{n+1}{3}$$

$$LHS_{A_n}=\sum_{i=1}^{n-1} i(n-i)=\sum_{i=1}^{n-1} i[(n+1)-(i+1)]=(n+1)\sum_{n=1}^{n-1} i-2\sum_{i=1}^{n-1}\binom{i+1}{2}=$$

$$=\frac{(n+1)n(n-1)}{2}-2\binom{n+1}{3}=3\binom{n+1}{3}-2\binom{n+1}{3}=\binom{n+1}{3}$$

$$\because B_n=\sum_{i+j+k=n} ijk=\sum_{k=1}^{n-2} k\sum_{i+j=n-k} ij=\sum_{k=1}^{n-2} kA_{n-k}=$$

$$=1\cdot A_{n-1}+2\cdot A_{n-2}+3A_{n-3}+\dots+(n-2)A_2=\sum_{r=2}^{n-1}(n-r)A_r=$$

$$=\sum_{r=2}^{n-1}(n-2)\binom{r+1}{3}=\sum_{r=2}^{n-1}[(n+2)-(r+2)]\binom{r+1}{3}=$$

$$=(n+2)\sum_{r=2}^{n-1}\binom{r+1}{3}-4\sum_{r=2}^{n-1}\binom{r+2}{4}=(n+2)\binom{n+1}{4}-4\binom{n+2}{5}=\binom{n+2}{5}$$

$$\because C_n=\sum_{i+j+k+l=n} ijkl=\sum_{l=1}^{n-3} l\sum_{i+j+k=n-l} ijk=$$

$$=1\cdot B_{n-1}+2\cdot B_{n-2}+\dots+(n-3)\cdot B_3=\sum_{r=3}^{n-1}(n-r)S_r=$$

$$=\sum_{r=3}^{n-1}(n-r)\binom{r+2}{5}=\sum_{r=3}^{n-1}[n+3-(r+3)]\binom{r+2}{5}=$$

$$=\sum_{r=3}^{n-1}(n+3)\binom{r+3}{5}-\sum_{r=3}^{n-1}(r+3)\binom{r+2}{5}=$$

$$= (n+3)\left(\binom{5}{5}+\binom{6}{5}+\cdots+\binom{n+1}{5}\right)-6\sum_{r=3}^{n-1}\binom{r+3}{6}=$$

$$=7\binom{n+3}{7}-6\binom{n+3}{7}=\binom{n+3}{7}$$

Now,

$$\Omega=\lim_{n\to\infty}\frac{C_n}{n^2B_n}=\lim_{n\to\infty}\frac{\binom{n+3}{7}}{n^2\binom{n+2}{5}}=\frac{1}{42}\lim_{n\to\infty}\frac{(n+3)!\,(n-3)!}{(n-4)!\,(n+2)!\,n^2}=$$

$$=\frac{1}{42}\lim_{n\to\infty}\frac{(n+3)(n-3)}{n^2}=\frac{1}{42}$$

1335. Let $(x_n)_{n\geq 1},(y_n)_{n\geq 1}$ be sequences of positive real numbers such that:

$$x_1=1, nx_{n+1}=(n-1)x_n+x_n^{1-n}\ ;\ y_1>0, y_{n+1}=\frac{(n+x_n)n^ny_n}{y_n^n+n^n(n-x_n)}$$

Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{y_n}\sum_{k=1}^{n}\left(cos^2\frac{4\pi x_ky_k}{2y_n+x_n}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania

$$nx_{n+1}=(n-1)x_n+x_n^{1-n}\Leftrightarrow x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}$$

$$n=1\Rightarrow x_2=\frac{1+0\cdot x_1^1}{1\cdot x_1^0}=1$$

$$n=2\Rightarrow x_3=\frac{1+x_2^2}{2\cdot x_2^1}=1$$

Suppose that $x_n=1,\forall n\in\mathbb{N}$, then $x_{n+1}=\frac{1+(n-1)\cdot 1^n}{n\cdot 1^{n-1}}=1$.

So, $x_n=1,\forall n\in\mathbb{N}$. Now,

$$n=1\Rightarrow y_2=\frac{2\cdot 1^1\cdot y_1}{y_1^1+1^1\cdot 0}=\frac{2y_1}{y_1}=2$$

$$n=2\Rightarrow y_3=\frac{3\cdot 2^2\cdot 2}{2^2+2^2\cdot 1}=\frac{24}{8}=3$$

Suppose that $y_n=n,\forall n\in\mathbb{N}$, then we have:

$$y_{n+1}=\frac{(n+1)\cdot n^n\cdot n}{n^n+n^n(n-1)}=\frac{(n+1)n^{n+1}}{n^n(1+n-1)}=n+1.$$

So, $y_n=n,\forall n\in\mathbb{N}$. Thus,

$$\Omega=\lim_{n\to\infty}\frac{1}{y_n}\sum_{k=1}^{n}\left(\cos^2\frac{4\pi x_k y_k}{2y_n+x_n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\cos^2\frac{2k\pi}{2n+1}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos^2\frac{4\pi\frac{k}{n}}{2+\frac{1}{n}}=\int_0^1\cos^2(2\pi x)\,dx=\int_0^1\frac{1+\cos(4\pi x)}{2}dx=\frac{1}{2}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$nx_{n+1}=(n-1)x_n+x_n^{1-n}\Leftrightarrow x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}$$

$$n=1\Rightarrow x_2=\frac{1+0\cdot x_1^1}{1\cdot x_1^0}=1$$

$$n=2\Rightarrow x_3=\frac{1+x_2^2}{2\cdot x_2^1}=1$$

Suppose that $x_n=1,\forall n\in\mathbb{N}$, then $x_{n+1}=\frac{1+(n-1)\cdot 1^n}{n\cdot 1^{n-1}}=1$.

So, $x_n=1,\forall n\in\mathbb{N}$. Now,

$$n=1\Rightarrow y_2=\frac{2\cdot 1^1\cdot y_1}{y_1^1+1^1\cdot 0}=\frac{2y_1}{y_1}=2$$

$$n=2\Rightarrow y_3=\frac{3\cdot 2^2\cdot 2}{2^2+2^2\cdot 1}=\frac{24}{8}=3$$

Suppose that $y_n=n,\forall n\in\mathbb{N}$, then we have:

$$y_{n+1}=\frac{(n+1)\cdot n^n\cdot n}{n^n+n^n(n-1)}=\frac{(n+1)n^{n+1}}{n^n(1+n-1)}=n+1.$$

So, $y_n=n,\forall n\in\mathbb{N}$. Thus,

$$\Omega=\lim_{n\to\infty}\frac{1}{y_n}\sum_{k=1}^{n}\left(\cos^2\frac{4\pi x_k y_k}{2y_n+x_n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\cos^2\frac{2k\pi}{2n+1}\right)$$

$$a_n=\sum_{k=1}^{n}\left(\cos^2\frac{2k\pi}{2n+1}\right)\to 2a_n=\sum_{k=1}^{n}\left(1+\cos\frac{4k\pi}{2n+1}\right)=n+\sum_{k=1}^{n}\cos\frac{4k\pi}{2n+1}=$$

$$=n-\frac{1}{2}+\frac{\sin\left(\frac{2n+1}{2}\cdot\frac{4\pi}{2n+1}\right)}{2\sin\frac{2\pi}{2n+1}}$$

$$\frac{2a_n}{n}=1-\frac{1}{2n}+\frac{\frac{2\pi}{2n+1}}{\sin\frac{2\pi}{2n+1}}\cdot\frac{2n+1}{2\pi}\cdot\frac{1}{2n}\cdot\sin 2\pi=1-\frac{1}{2n}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{y_n}\sum_{k=1}^{n}\left(\cos^2\frac{4\pi x_k y_k}{2y_n+x_n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\cos^2\frac{2k\pi}{2n+1}\right)=\lim_{n\to\infty}\frac{a_n}{n}=\frac{1}{2}$$

1336. For $x\in[-1,1]$ define:

$$P(x)=\frac{\cos(n\cdot\cos^{-1}x)}{n}\cdot\sum_{k=1}^{n}\cos^{n-1}\frac{k\pi}{n}\cos\frac{(n-1)k\pi}{n}$$

Prove that $P(x)$ is a polynomial of degree n with rational coefficients and coefficient to x^n is 1.

Proposed by Florică Anastase-Romania

Solution by Adrian Popa-Romania

$$(\cos t+i\sin t)^n=\cos(nt)+i\sin(nt);(1)$$

$$(\cos t+i\sin t)^n=\binom{n}{0}\cos^n t+i\binom{n}{1}\cos^{n-1}t\sin t-\binom{n}{2}\cos^{n-2}t\sin^2 t-$$

$$-i\binom{n}{3}\cos^{n-3}\sin^3 t+\cdots+\binom{n}{n}(i\sin t)^n;(2)$$

From (1),(2) it follows that:

$$\cos(nt)=\binom{n}{0}\cos^n t-\binom{n}{2}\cos^{n-2}t\sin^2 t+\binom{n}{4}\cos^{n-4}t\sin^4 t+\cdots=$$

$$=\binom{n}{0}\cos^n t-\binom{n}{2}\cos^{n-2}t(1-\cos^2 t)+\binom{n}{4}\cos^{n-4}t(1-\cos^2 t)^2+\cdots\overset{t=\cos^{-1}x}{\Longrightarrow}$$

$$\cos(n\cos^{-1}x)=\binom{n}{0}x^n-\binom{n}{2}(x^{n-2}-x^n)+\binom{n}{4}(x^{n-4}-2x^{n-2}+x^n)-\cdots=$$

$$=x^n\left(\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots\right)-x^{n-2}\left(\binom{n}{2}+2\binom{n}{4}+3\binom{n}{6}+\cdots\right)+\cdots=$$

$$=2^{n-1}x^n-px^{n-2}+qx^{n-4}+\cdots,p,q\in\mathbb{Q}.\text{ Next,}$$

$$\frac{1}{n}\sum_{k=1}^{n}cos^{n-1}\frac{k\pi}{n}cos\frac{(n-1)k\pi}{n}=\int_0^1 cos^{n-1}(\pi x)\cdot cos(n-1)\pi x\,dx=I_{n-1}$$

$$I_{n-1}=\int_0^1 cos^{n-1}(\pi x)\cdot cos(n-1)\pi x\,dx=$$

$$=\int_0^1 cos^{n-1}(\pi x)cos(\pi x)(cos(\pi x)cos(n-2)\pi x-sin(\pi x)sin(n-2)\pi x)\,dx=$$

$$=\int_0^1\left(cos^{n-2}(\pi x)cos(n-2)\pi x cos^2(\pi x)-cos^{n-2}(\pi x)cos(\pi x)sin(\pi x)sin(n-2)\pi x\right)dx=$$

$$=\int_0^1\left(cos^{n-2}(\pi x)cos(n-2)\pi x\frac{1+cos(2\pi x)}{2}-cos^{n-2}(\pi x)sin(n-2)\pi x\frac{sin(2\pi x)}{2}\right)dx=$$

$$=\frac{1}{2}I_{n-2}+\frac{1}{2}\int_0^1 cos^{n-2}(\pi x)\big(cos(n-2)\pi x cos(2\pi x)-sin(n-2)\pi x sin(2\pi x)\big)dx=$$

$$=\frac{1}{2}I_{n-2}+\frac{1}{2}\int_0^1 cos^{n-2}(\pi x)cos(\pi x)\,dx$$

Similarly,

$$I_{n-2}=\frac{1}{2}I_{n-3}+\frac{1}{2}\int_0^1 cos^{n-3}(\pi x)cos(n+1)\pi x\,dx$$

$$I_{n-3}=\frac{1}{2}I_{n-4}+\frac{1}{2}\int_0^1 cos^{n-4}(\pi x)cos(n+2)\pi x\,dx$$

$$\vdots$$

$$i_1=\frac{1}{2}I_0+\frac{1}{2}\int_0^1 cos^0 x\cdot cos(2n-2)\pi x\,dx=\frac{1}{2}I_0=\frac{1}{2}\to I_{n-1}=\frac{1}{2^{n-1}}$$

Therefore, $P(x)=x^n-\frac{p}{2^{n-1}}x^{n-2}+\frac{q}{2^{n-1}}x^{n-4}-\cdots,p,q\in\mathbb{Q}$

1337. $x_0=\frac{1}{2},x_1=1,15^{x_{n+2}}=12^{x_{n+1}}+9^{x_n}$

Prove that the sequence $(x_n)_{n\geq 1}$is increasing, bounded and find:

$$\Omega=\lim_{n\to\infty}x_n$$

Proposed by 1 by Marian Ursărescu-Romania

Solution 1 by proposer

$$15^{x_2}=12^{x_1}+9^{x_0}=1+3=15\to x_2=1$$

$x_1 \geq x_0$. Suppose that: $x_0 \leq x_1 \leq \cdots \leq x_k$ and we prove that $x_k \leq x_{k+1}, \forall k \in N$.

$$15^{x_{k+1}} - 15^{x_k} = 12^{x_k} + 9^{x_{k-1}} - 12^{x_k} - 12^{x_{k-1}} = (12^{x_k} - 12^{x_{k-1}}) + (9^{x_k} - 9^{x_{k-1}}) \geq$$

$0 \to (x_n)_{n\geq 1}$ −increasing→ $x_n \geq x_0$ inferior bounded.

We must to prove that $x_n < 2, \forall n \in N$.

Suppose $x_0, x_1, \dots, x_k < 2$, then $x_k < 2 \to 12^{x_k} < 12^2$ and $x_{k-1} < 2 \to 9^{x_{k-1}} < 9^2 \to$

$12^{x_k} + 9^{x_{k-1}} \leq 144 + 81 \to 15^{x_{k+1}} < 15^2 \to x_{k+1} < 2 \to (x_n)_{n\geq 1}$ −is convergent.

$$\lim_{n\to\infty} x_n = l, l \in \mathbb{R} \to \lim_{n\to\infty} 15^{x_{n+1}} = \lim_{n\to\infty}(12^{x_n} + 9^{x_{n-1}}) \to 15^l = 12^l + 9^l \to$$

$l = 2$ unique solution.Therefore,

$$\Omega = \lim_{n\to\infty} x_n = 2$$

Solution 2 by Iulian Cristi-Romania

By induction $x_n < 2, \forall n \in N$. $x_0 = \frac{1}{2} < 2$. Suppose $x_i < 2, \forall i \in \overline{0, k+1}$

$15^{x_{k+3}} = 12^{x_{k+2}} + 9^{x_{k+1}} < 12^2 + 9^2 = 15^2 \to x_{k+3} < 2$. Hence, $x_n < 2, \forall n \in N$

$x_0 < x_1$.Suppose $x_k \leq x_{k+1}, \forall k \in \overline{0, n-1} \to x_n \leq x_{n+1} \leq x_{n+2} \to$

$9^{x_n} \leq 9^{x_{n+1}}$ and $12^{x_{n+1}} \leq 12^{x_{n+2}} \to 15^{x_{n+2}} \leq 15^{x_{n+3}} \to (x_n)_{n\in N}$ −is increasing

$\to (x_n)_{n\in N}$ −is convergent. Therefore,

$$\Omega = \lim_{n\to\infty} x_n = 2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$x_0 = \frac{1}{2}, x_1 = 1, 15^{x_2} = 12^{x_1} + 9^{x_0} = 15 \to x_2 = 1.$$

We assume that $x_n \leq 2, \forall n \geq 0$. We have that $x_0, x_1, x_2 \leq 2$.

Assume that $x_k \leq 2, \forall 0 \leq k \leq n+1$, we have:

$15^{x_{n+2}} = 12^{x_{n+1}} + 9^{x_n} \leq 12^2 + 9^2 = 15^2 \to x_{n+2} \leq 2$. Thus, by the principle of mathematical induction, $x_n \leq 2, \forall n \geq 0$.

Assume that $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \to 12^{x_n} + 9^{x_{n-1}} \leq 12^{x_{n+1}} + 9^{x_n} \to$

$15^{x_{n+1}} \leq 15^{x_{n+2}} \to x_{n+1} \leq x_{n+2}$. Thus, $(x_n)_{n\geq 1}$ −increasing sequence and bounded, then $(x_n)_{n\geq 1}$ −converges. Then

$$\lim_{n\to\infty} 15^{x_{n+2}} = \lim_{n\to\infty}(12^{x_{n+1}} + 9^{x_n}) \to$$

$$15^l = 12^l + 9^l \to l = 2.$$

Therefore,

$$\Omega = \lim_{n\to\infty} x_n = 2$$

1338. Find:

$$\Omega = \lim_{n\to\infty} n\left(\int_0^1 \frac{1}{x} log(1+x+x^2+x^3)\,dx - \sum_{k=0}^{n} \frac{1}{(2k+1)^2}\right)$$

Proposed by Mohammed Bouras-Fes-Morocco

Solution 1 by Naren Bhandari-Bajura-Nepal

$$\int_0^1 \frac{1}{x} log(1+x+x^2+x^3)\,dx = \int_0^1 \left(\frac{log(1-x^4)}{x} - \frac{log(1-x)}{x}\right)dx =$$

$$= \left[-\frac{1}{4}Li_2(x^4) + Li_2(x)\right]_0^1 = \frac{\pi^2}{8}$$

$$\frac{\pi^2}{8} - \sum_{k=0}^{n} \frac{1}{(2k+1)^2} = \sum_{k=0}^{n} \frac{1}{(2k+3)^2} = \sum_{k=0}^{n} \frac{1}{(2k+2n+3)^2} = \frac{1}{4}\psi_1\left(n+\frac{3}{2}\right)$$

$$\Omega = \lim_{n\to\infty} n\left(\frac{\pi^2}{8} - \sum_{k=0}^{n} \frac{1}{(2k+1)^2}\right) = \lim_{n\to\infty} \frac{n}{4}\psi_1\left(n+\frac{3}{2}\right) = \lim_{n\to\infty} \frac{n}{4}\left(\frac{1}{n} - \frac{1}{n^2} + o(n^{-3})\right) = \frac{1}{4}$$

Solution 2 by Probal Chakraborty-India

$$\Omega = \lim_{n\to\infty} n\left(\int_0^1 \frac{1}{x} log(1+x+x^2+x^3)\,dx - \sum_{k=0}^{n} \frac{1}{(2k+1)^2}\right) =$$

$$= \lim_{n\to\infty} n\left[\int_0^1 \frac{log[(1+x)(1+x^2)]}{x}dx - \sum_{k=0}^{n} \frac{1}{(2k+1)^2}\right] =$$

$$= \lim_{n\to\infty} n\left[\int_0^1 \left(\frac{log(1+x)}{x} + \frac{log(1+x^2)}{x}\right)dx - \sum_{k=0}^{n} \frac{1}{(2k+1)^2}\right] =$$

$$= \lim_{n\to\infty} n\left[\frac{\pi^2}{12} + \frac{\pi^2}{24} - \frac{\pi^2}{8} + \frac{1}{4}\psi_1\left(n+\frac{1}{2}\right)\right]; (1)$$

$$-\psi_1\left(n+\frac{1}{2}\right) = \frac{\pi^2}{2} - 4\sum_{k=0}^{n} \frac{1}{(2k+1)^2} \to \sum_{k=0}^{n} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} - \frac{1}{4}\psi_1\left(n+\frac{3}{2}\right); (2)$$

$$\Omega = \lim_{n\to\infty} n\left(\int_0^1 \frac{1}{x}\log(1+x+x^2+x^3)\,dx - \sum_{k=0}^{n}\frac{1}{(2k+1)^2}\right) =$$

$$= \lim_{n\to\infty} n\left[\frac{\pi^2}{8}-\frac{\pi^2}{8}+\frac{1}{4}\psi_1\left(n+\frac{3}{2}\right)\right] = \lim_{n\to\infty}\frac{n}{4}\psi_1\left(n+\frac{3}{2}\right) = \frac{1}{4}$$

1339. Prove that:

$$\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k^2+k}\right) = \frac{\pi}{4}$$

Proposed by Mohammed Bouras-Fes-Morocco

Solution 1 by Samar Das-India

$$\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k^2+k}\right) = \lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k(k+1)}\right) =$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{\frac{1}{n}}{\frac{k+1}{n}\cdot\frac{k}{n}+1}\right) = \lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{\frac{k+1}{n}-\frac{k}{n}}{1+\frac{k+1}{n}\cdot\frac{k}{n}}\right) =$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\left(tan^{-1}\left(\frac{k+1}{n}\right) - tan^{-1}\left(\frac{k}{n}\right)\right) = \lim_{n\to\infty} tan^{-1}\left(1+\frac{1}{n}\right) = \frac{\pi}{4}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

If $(\alpha,\beta)\in\mathbb{R}_+^*\times\mathbb{R}$ and $\gamma=\alpha^2+\beta^2+\beta$ then:

$$tan^{-1}\left(\frac{x+\beta+1}{\alpha}\right) + tan^{-1}\left(\frac{x+\beta}{\alpha}\right) = tan^{-1}\left(\frac{\alpha}{x^2+(2\beta+1)x+\gamma}\right)$$

For $\beta=0, x=k$ then $\gamma=k^2>0$. So, suppose $k=\alpha, \alpha=n, \beta=0$, we have:

$$tan^{-1}\left(\frac{n}{k^2+k+n^2}\right) = tan^{-1}\left(\frac{k+1}{n}\right) - tan^{-1}\left(\frac{k}{n}\right)$$

Thus,

$$\lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k^2+k}\right) = \lim_{n\to\infty}\sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k(k+1)}\right) =$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\tan^{-1}\left(\frac{\frac{1}{n}}{\frac{k+1}{n}\cdot\frac{k}{n}+1}\right)=\lim_{n\to\infty}\sum_{k=0}^{n}\tan^{-1}\left(\frac{\frac{k+1}{n}-\frac{k}{n}}{1+\frac{k+1}{n}\cdot\frac{k}{n}}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\left(\tan^{-1}\left(\frac{k+1}{n}\right)-\tan^{-1}\left(\frac{k}{n}\right)\right)=\lim_{n\to\infty}\tan^{-1}\left(1+\frac{1}{n}\right)=\frac{\pi}{4}$$

Solution 3 by Mohammad Rostami-Kabul-Afghanistan

$$\tan^{-1}\left(\frac{n}{n^2+k^2+k}\right)=\alpha+\beta\to\tan(\alpha+\beta)=\frac{n}{n^2+k^2+k}\to$$

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}=\frac{n}{n^2+k^2+k}\to\frac{n^2(\tan\alpha+\tan\beta)}{n^2-n^2\tan\alpha\cdot\tan\beta}=\frac{n}{n^2+k^2+k}$$

$$\begin{cases}n^2(\tan\alpha+\tan\beta)=n\\-n^2\tan\alpha\cdot\tan\beta=k^2+k\end{cases}\to\begin{cases}S=\tan\alpha+\tan\beta=\frac{1}{n}\\P\backslash\tan\alpha\cdot\tan\beta=\frac{k^2+k}{n^2}\end{cases}$$

$$Z^2-SZ+P=0\to Z^2-\frac{1}{n}Z-\frac{k^2+k}{n^2}=0\to$$

$$Z=\frac{1}{2}\left(\frac{1}{n}\pm\sqrt{\left(\frac{2k+1}{n}\right)^2}\right)=\frac{1}{2}\left(\frac{1}{n}\pm\frac{2k+1}{n}\right)\to\begin{cases}Z=\frac{2k+2}{2n}=\frac{k+1}{n}=\tan\alpha\\Z=-\frac{2k}{2n}=-\frac{k}{n}=\tan\beta\end{cases}\to$$

$$\alpha=\tan^{-1}\left(\frac{k+1}{n}\right),\beta=\tan^{-1}\left(-\frac{k}{n}\right)$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}\tan^{-1}\left(\frac{n}{n^2+k^2+k}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}(\alpha+\beta)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\left(\tan^{-1}\left(\frac{k+1}{n}\right)-\tan^{-1}\left(\frac{k}{n}\right)\right)=\lim_{n\to\infty}\tan^{-1}\left(1+\frac{1}{n}\right)=\frac{\pi}{4}$$

1340. Prove that:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}-\frac{3n+1}{2\sqrt[3]{n}}\right)=\zeta\left(\frac{1}{3}\right)$$

-where $\zeta(z)$ – is Riemann zeta function.

Proposed by Naren Bhandari-Bajura-Nepal

Solution and generalization by Serlea Kabay-Liberia

$$Let\ \omega(a)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k^a}-\frac{n}{(1-a)n^a}-\frac{1}{2n^a}\right),\forall a\in\mathbb{R}_+$$

$$\omega(a)=\lim_{n\to\infty}\left(H_n^{(a)}-\frac{n}{(1-a)n^a}-\frac{1}{2n^a}\right)=$$

$$=\lim_{n\to\infty}\left(\zeta(a)-\zeta(a,n+1)-\frac{n}{(1-a)n^a}-\frac{1}{2n^a}\right)=$$

$$=\lim_{n\to\infty}\left(\zeta(a)-\frac{1}{n^a}(-\frac{n}{1-a}+\frac{n}{1-a}-\frac{1}{2n^a}+\frac{1}{2n^a}+\frac{a}{12n}+o\left(\frac{1}{n}\right)\right)=$$

$$=\lim_{n\to\infty}\left(\zeta(a)-\frac{a}{12n^{a+1}}+o\left(\frac{1}{n}\right)^a\right)$$

$$\therefore\ \omega(a)=\zeta(a),\forall a\in\mathbb{R}_+$$

$$\text{For}=\frac{1}{3},\omega\left(\frac{1}{3}\right)=\zeta\left(\frac{1}{3}\right)$$

1341. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n}k^k\cdot e^{\sum_{k=1}^{n}ke^{\frac{k}{n^2}}}}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by Ali Jaffal-Lebanon

$$Let\ \Omega_n=\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n}k^k\cdot e^{\sum_{k=1}^{n}ke^{\frac{k}{n^2}}}}$$

$$log\Omega_n=-\frac{1}{2}logn+\frac{1}{n^2}\sum_{k=1}^{n}klogk+\frac{1}{n^2}\sum_{k=1}^{n}ke^{\frac{k}{n^2}}=$$

$$=-\frac{1}{2}logn+\frac{1}{n^2}\sum_{k=1}^{n}klog\left(\frac{k}{n}\right)+\frac{n(n+1)}{2n^2}logn+\frac{1}{n^2}\sum_{k=1}^{n}ke^{\frac{k}{n^2}}=$$

$$=\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}log\left(\frac{k}{n}\right)+\frac{1}{n}logn+\frac{1}{n^2}\sum_{k=1}^{n}ke^{\frac{k}{n^2}}$$

$$\lim_{n\to\infty}\frac{1}{n}logn = 0$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}log\left(\frac{k}{n}\right) = \int_0^1 xlogx\,dx = \left[\frac{x^2}{2}logx\right]_0^1 - \int_0^1\frac{x^2}{2}dx = -\frac{1}{4}$$

Next, $1 + x \le e^x \le 1 + x + (e-2)x^2, \forall x \in [0,1]$. Let $1 \le k \le n$, then

$$1 + \frac{k}{n^2} \le e^{\frac{k}{n^2}} \le 1 + \frac{k}{n^2} + (e-2)\frac{k^2}{n^4}$$

$$1 + \frac{k^2}{n^2} \le ke^{\frac{k}{n^2}} \le k + \frac{k^2}{n^2} + (e-2)\frac{k^3}{n^4}$$

$$\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6n^2} \le \sum_{k=1}^{n} ke^{\frac{k}{n^2}} \le \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6n^2} + (e-2)\frac{n^2(n+1)^2}{4n^4}$$

$$\lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \frac{1}{2}, \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^4} = 0, \lim_{n\to\infty}(e-2)\frac{n^2(n+1)^2}{4n^6} = 0$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} ke^{\frac{k}{n^2}} = \frac{1}{2}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n} k^k\cdot e^{\sum_{k=1}^{n} ke^{\frac{k}{n^2}}}}\right) = \sqrt[4]{e}$$

Solution 2 by proposer

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \to \forall\varepsilon > 0, \exists\delta_\varepsilon > 0 \text{ such that } \left|\frac{e^x-1}{x} - 1\right| < \varepsilon \to$$

$$(1-\varepsilon)\frac{k}{n^2} + 1 < e^{\frac{k}{n^2}} < (1+\varepsilon)\frac{k}{n^2} + 1 \leftrightarrow$$

$$\frac{1}{n^2}\sum_{k=1}^{n}\left[(1-\varepsilon)\frac{k^2}{n^2} + k\right] \le \frac{1}{n^2}\sum_{k=1}^{n} ke^{\frac{k}{n^2}} \le 1/n^2\sum_{k=1}^{n}\left[(1+\varepsilon)\frac{k^2}{n^2} + k\right]$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\left[(1-\varepsilon)\frac{k^2}{n^2} + k\right] = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\left[(1+\varepsilon)\frac{k^2}{n^2} + k\right] = \frac{1}{2} \to$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}ke^{\frac{k}{n^2}}=\frac{1}{2}\to\lim_{n\to\infty}\left(\sqrt[n^2]{e^{\sum_{k=1}^{n}ke^{\frac{k}{n^2}}}}\right)=\sqrt{e};(1)$$

Now,

$$\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n}k^k}=\sqrt[n^2]{\prod_{k=1}^{n}\left(\frac{k}{n}\right)^k}\cdot\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{n^{\frac{n(n+1)}{2}}}=\sqrt[n^2]{\prod_{k=1}^{n}\left(\frac{k}{n}\right)^k}\cdot n^{\frac{1}{2n}}=a_n\cdot n^{\frac{1}{2n}};n^{\frac{1}{2n}}\to 1$$

$$a_n=\sqrt[n^2]{\prod_{k=1}^{n}\left(\frac{k}{n}\right)^k}\to loga_n=\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}log\left(\frac{k}{n}\right)=\sigma_\Delta(f,\xi);$$

$$\Delta=\left(0,\frac{1}{n},\frac{2}{n},\dots,1\right),\xi\in\left\{\frac{1}{n},\frac{2}{n},\dots,1\right\},f:[0,1]\to\mathbb{R},f(x)=\begin{cases}xlogx,x>0\\0,x=0\end{cases}$$

$$F(x)=\begin{cases}\frac{x^2}{2}logx-\frac{x^2}{4},x>0\\0,x=0\end{cases}\to\lim_{n\to\infty}loga_n=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}log\left(\frac{k}{n}\right)=\int_0^1 f(x)dx=-\frac{1}{4}$$

$$\to\lim_{n\to\infty}a_n=\frac{1}{\sqrt[4]{e}};(2)$$

From (1),(2) it follows that:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n}k^k\cdot e^{\sum_{k=1}^{n}ke^{\frac{k}{n^2}}}}\right)=\sqrt[4]{e}$$

1342.

$$\Omega_n=sin\left(\pi\cdot\sqrt[p]{\prod_{k=1}^{n}(n+k)}\right),n\in\mathbb{N}$$

Find $p\in\mathbb{N},p\geq 2$ such that the sequence $(\Omega_n)_{n\geq1}$ is a convergent one.

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\lim_{n\to\infty}\Omega_n=\lim_{n\to\infty}sin\left(\pi\cdot\sqrt[p]{\prod_{k=1}^{n}(n+k)}\right)$$

$$=\lim_{n\to\infty}(-1)^n\cdot sin\left[\pi\sqrt[p]{(n+1)(n+2)\dots(n+p)}-n\right]=$$

$$\lim_{n\to\infty}(-1)^n\cdot sin\left(\frac{\pi((1+2+\dots+p)n^{p-1}+\dots+1}{\sqrt[p]{(n+1)\dots(n+p)}^{p-1}+\dots+n^{p-1}}\right)=$$

$$=\lim_{n\to\infty}(-1)^n sin\left(\frac{\pi\cdot p(p+1)}{2p}\right)=\lim_{n\to\infty}(-1)^n sin\left(\frac{\pi(p+1)}{2}\right)$$

If p –is even number, then $\lim_{n\to\infty}(-1)^n sin\left(\frac{\pi}{2}+k\pi\right)=\pm(-1)^n$ is divergent.

If p –is odd number, then $\lim_{n\to\infty}(-1)^n sin(k+1)\pi=0$.

$$|(-1)^n sin(k+1)\pi|=sin(k+1)\pi=0\to p\text{ –is odd.}$$

1343. Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Remus Florin Stanca-Romania

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\left(\frac{1}{3n+1}-\frac{1}{3n+2}\right)=$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)(3n+1)}-\frac{1}{(n+1)(3n+2)}\right)=$$

$$=3\left(\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{3n+1}-\frac{1}{3n+3}\right)-\sum_{n=0}^{\infty}\left(\frac{1}{3n+2}-\frac{1}{3n+3}\right)\right)=$$

$$=\frac{3}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{3}}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}-\sum_{n=0}^{\infty}\frac{1}{n+\frac{2}{3}}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}$$

$$\because \psi(z+1)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)$$

$$s=\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n}\right)\right)+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n}+\sum_{n=0}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\frac{2}{3}}\right)-\sum_{n=0}^{\infty}\frac{1}{n}+\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{n+1}-\frac{1}{n}\right)=$$

$$=\frac{1}{2}\left(-\gamma-\psi\left(\frac{4}{3}\right)\right)+\left(\psi\left(\frac{5}{3}\right)+\gamma\right)+\frac{1}{2}(-\gamma-\psi(1))=\psi\left(\frac{5}{3}\right)-\frac{1}{2}\psi\left(\frac{4}{3}\right)-\frac{\psi(1)}{2}$$

$$\psi(1-x)-\psi(x)=\pi ctg(\pi x),\psi(x+1)=\psi(x)+\frac{1}{x}$$

$$s=\psi\left(1+\frac{2}{3}\right)-\frac{1}{2}\psi\left(1+\frac{1}{3}\right)-\frac{\psi(1)}{2}=\psi\left(\frac{2}{3}\right)+\frac{3}{2}-\frac{1}{2}\left(\psi\left(\frac{1}{3}\right)+3\right)-\frac{\psi(1)}{2}=$$

$$=\psi\left(\frac{2}{3}\right)-\frac{\psi\left(\frac{1}{3}\right)}{2}-\frac{\psi(1)}{2}$$

$$\psi\left(1-\frac{1}{3}\right)-\psi\left(\frac{1}{3}\right)=\pi ctg\left(\frac{\pi}{3}\right)=\frac{\pi}{\sqrt{3}}\Rightarrow$$

$$s=\frac{\pi}{\sqrt{3}}+\frac{1}{2}\left(-\frac{\pi}{2\sqrt{3}}-\frac{3log3}{2}-\gamma\right)+\frac{\gamma}{2}=\frac{\pi}{\sqrt{3}}-\frac{\pi}{4\sqrt{3}}-\frac{3log3}{4}=\frac{\pi\sqrt{3}-3log3}{4}$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\frac{\pi\sqrt{3}-3log3}{4}$$

Solution 2 by Probal Chakraborty-India

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\Omega+\frac{1}{2}=S=$$

$$=\sum_{n=0}^{\infty}\left[\frac{3}{2(3n+1)}-\frac{3}{3n+2}+1/2(n+1)\right]$$

$$\sum_{n=0}^{\infty}F(t)=\int_0^{\infty}\frac{f(t)}{e^t-1}dt=\Omega$$

$$f(t)=\frac{1}{2}L^{-1}\left(\frac{1}{s+\frac{1}{3}}\right)-L^{-1}\left(\frac{1}{s+\frac{2}{3}}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right)=\frac{1}{2}e^{-\frac{1}{3}t}-e^{-\frac{2}{3}t}+\frac{1}{2}e^{-t}$$

$$\Omega=\frac{1}{2}\int_0^{\infty}\frac{e^{-\frac{1}{3}t}}{e^t-1}dt-\int_0^{\infty}\frac{e^{-\frac{3}{2}t}}{e^t-1}dt+\frac{1}{2}\int_0^{\infty}\frac{e^{-t}}{e^t-1}dt=\frac{1}{2}\int_0^{\infty}\frac{e^{-\frac{t}{3}}-2e^{-\frac{2}{3t}}+e^{-t}}{e^t-1}dt=$$

$$\overset{e^{-t}}{=}\frac{1}{2}\int_0^1\frac{z^{\frac{1}{3}}-2z^{\frac{2}{3}}+z}{1-z}dz=\frac{3}{2}\int_0^1\frac{t-2t^2+t^3}{1-t^3}t^2dt\overset{z=t^3}{=}$$

$$=\frac{3}{2}\int_0^1\frac{t^3-2t^4+t^5}{1-t^3}dt=\frac{1}{4}\left(-2+\sqrt{3}\pi-3\log 3\right)=-\frac{1}{2}+\frac{\sqrt{3}\pi}{4}-\frac{3\log 3}{4}$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\frac{\pi\sqrt{3}-3\log 3}{4}$$

Solution 3 by Probal Chakraborty-India

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\sum_{(n=0)}^{\infty}\left[\frac{3}{2(3n+1)}-\frac{3}{3n+2}+\frac{1}{2(n+1)}\right]=$$

$$=\lim_{m\to\infty}\left[\sum_{n=0}^{m}\frac{3}{2(3n+1)}-3\sum_{n=0}^{m}\frac{1}{3n+2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\right]=$$

$$=\lim_{m\to\infty}\left[\frac{1}{2}\psi\left(m+\frac{4}{3}\right)-\psi\left(m+\frac{5}{3}\right)+\frac{1}{2}\psi(m+2)\right]+\left[-\frac{1}{2}\psi\left(\frac{1}{3}\right)+\psi\left(\frac{2}{3}\right)+\frac{\gamma}{3}\right]=$$

$$=0+\frac{\gamma}{2}+\frac{\pi}{4\sqrt{3}}+\frac{3\log 3}{4}$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\frac{\pi\sqrt{3}-3\log 3}{4}$$

Solution 4 by Syed Shahabudeen-India

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\sum_{n=0}^{\infty}\left(\frac{1}{2(n+1)}+\frac{1}{2\left(n+\frac{1}{3}\right)}-\frac{1}{n+\frac{2}{3}}\right)=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}+\frac{1}{n+\frac{1}{3}}-\frac{n}{n+\frac{2}{3}}\right)=$$

$$=\frac{1}{2}\left(2\psi\left(\frac{2}{3}\right)+2\gamma-\left(\psi\left(\frac{1}{3}\right)+\gamma\right)\right)=\frac{1}{2}\left(2\psi\left(\frac{2}{3}\right)-\psi\left(\frac{1}{3}\right)+\gamma\right)=\frac{\pi\sqrt{3}-3log3}{4}$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\frac{\pi\sqrt{3}-3log3}{4}$$

Solution 5 by Ravi Prakash-New Delhi-India

Let $a_n=\frac{1}{(n+1)(3n+1)(3n+2)}$

$$a_n=\frac{3}{(3n+1)(3n+1)(3n+3)}=3\left[\frac{1}{2}\cdot\frac{1}{3n+1}-\frac{1}{3n+2}+\frac{1}{2}\cdot\frac{1}{3n+3}\right]=$$

$$=\frac{3}{2}\left(\frac{1}{3n+1}-\frac{1}{3n+2}\right)-\frac{3}{2}\left(\frac{1}{3n+2}-\frac{1}{3n+3}\right)$$

$$\sum_{k=0}^{\infty}a_k=\frac{3}{2}\sum_{k=0}^{\infty}\left(\frac{1}{3k+1}-\frac{1}{3k+2}\right)-\frac{3}{2}\sum_{k=0}^{\infty}\left(\frac{1}{3k+2}-\frac{1}{3k+3}\right)=$$

$$=\frac{3}{2}\sum_{k=0}^{\infty}\int_0^1 x^{3k}(1-x)\,dx-\frac{3}{2}\sum_{k=0}^{\infty}\int_0^1 x^{3k+1}(1-x)\,dx$$

$$\frac{2}{3}\sum_{k=0}^{\infty}a_k=\int_0^1\left((1-x)\sum_{k=0}^{\infty}x^{3k}\right)dx-\int_0^1\left((1-x)\sum_{k=0}^{\infty}x^{3k+1}\right)dx=$$

$$=\int_0^1\left[(1-x)\cdot\frac{1}{1-x^3}-\frac{(1-x)x}{1-x^3}\right]dx=$$

$$=\int_0^1\left[\frac{1}{1+x+x^2}-\frac{x}{1+x+x^2}\right]dx=\int_0^1\frac{1-x}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx\overset{x+\frac{1}{2}=t}{=}$$

$$=\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1-\left(t-\frac{1}{2}\right)}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt=\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\left[tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}-\frac{1}{2}\left[log\left(t^2+\left(\frac{2t}{\sqrt{3}}\right)\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}=$$

$$=\sqrt{3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)-\frac{1}{2}log3$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+1)(3n+2)}=\frac{\pi\sqrt{3}-3\log 3}{4}$$

1344. Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Akerele Olofin-Nigeria

$$\Omega=\sum_{n=0}^{\infty}\left(\frac{4}{3(4n+1)}+\frac{4}{4n+3}-\frac{1}{3(n+1)}-\frac{2}{2n+1}\right)=$$

$$=\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{4}}+\sum_{n=0}^{\infty}\frac{1}{n+\frac{3}{4}}-\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{n+1}-\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}$$

$$\because \psi^{m}(z)=(-1)^{m+1}(m!)\sum_{n=0}^{\infty}\frac{1}{(k+z)^{m+1}}\Rightarrow\sum_{n=0}^{\infty}\frac{1}{(k+z)^{m+1}}=\frac{\psi(z)}{(-1)^{m+1}(m!)}$$

$$\Omega=-\frac{\psi^{(0)}\left(\frac{1}{4}\right)}{3}-\psi^{(0)}\left(\frac{3}{4}\right)+\frac{\psi^{(0)}(1)}{3}+\psi^{(0)}\left(\frac{1}{2}\right)=$$

$$=-\frac{1}{3}\left(-\gamma-\frac{\pi}{2}-3\log 2\right)-\gamma+\frac{\pi}{2}-3\log 2+\frac{1}{3}(-\gamma)+(-\gamma-2\log 2)=$$

$$=\frac{\pi}{6}-\frac{\pi}{2}+2\log 2=2\log 2-\frac{\pi}{3}$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=2\log 2-\frac{\pi}{3}$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=$$

$$=\sum_{n=0}^{\infty}\left(\frac{4}{3(4n+1)}+\frac{4}{4n+3}-\frac{1}{3(n+1)}-\frac{2}{2n+1}\right)=$$

$$=\frac{1}{3}\psi(1)+\psi\left(\frac{1}{2}\right)-\frac{1}{3}\psi\left(\frac{1}{4}\right)-\psi\left(\frac{3}{4}\right)=$$

$$=-\frac{\gamma}{3}-\gamma-2log2+\frac{\gamma}{3}+\frac{\pi}{6}+log2+\gamma-\frac{\pi}{2}+3log2$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=2log2-\frac{\pi}{3}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$Let\ a_n=\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=\frac{8}{(4n+1)(4n+2)(4n+3)(4n+4)}=$$

$$=8\left[\frac{1}{6}\cdot\frac{1}{4n+1}-\frac{1}{2}\cdot\frac{1}{4n+2}+\frac{1}{2}\cdot\frac{1}{4n+3}-\frac{1}{6}\cdot\frac{1}{4n+4}\right]=$$

$$=\frac{4}{3}\left[\int_0^1 x^{4n}\,dx-\int_0^1 x^{4n+3}dx\right]-4\left[\int_0^1 x^{4n+1}dx-\int_0^1 x^{4n+2}dx\right]=$$

$$=\frac{4}{3}\int_0^1 x^{4n}(1-x^3)dx-4\int_0^1 x^{4n+1}(1-x)dx$$

$$\sum_{n=0}^{\infty}a_n=\frac{4}{3}\sum_{n=0}^{\infty}\int_0^1 x^{4n}(1-x^3)dx-4\sum_{n=0}^{\infty}x^{4n+1}(1-x)dx=$$

$$=\frac{4}{3}\int_0^1\frac{1-x^3}{1-x^4}dx-4\int_0^1\frac{(1-x)x}{1-x^4}dx=$$

$$=\frac{4}{3}\int_0^1\frac{1+x+x^2}{(1+x)(1+x^2)}dx-4\int_0^1\frac{x}{(1+x)(1+x^2)}dx=$$

$$=\frac{4}{3}\int_0^1\frac{(1-x)^2}{(1+x)(1+x^2)}dx=\frac{4}{3}\int_0^1\left(\frac{2}{1+x}-\frac{x}{1+x^2}-\frac{1}{1+x^2}\right)dx=$$

$$=\frac{4}{3}\left[2log(1+x)-\frac{1}{2}log(1+x^2)-tan^{-1}x\right]_0^1=2log2-\frac{\pi}{3}$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=2log2-\frac{\pi}{3}$$

Solution 4 by Probal Chakraborty-India

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=$$

$$=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=I+\frac{1}{6}$$

$$I=\sum_{n=1}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=$$

$$=\frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{n+1}+4\sum_{n=1}^{\infty}\frac{1}{4n+3}-\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n+1}-2\sum_{n=1}^{\infty}\frac{1}{2n+1}$$

$$\sum_{n=1}^{\infty}F(n)=\int_0^{\infty}\frac{f(t)}{e^t-1}dt,\qquad f(t)=L^{-1}\big(F(n)\big)$$

$$=\frac{1}{3}\int_0^{\infty}\frac{e^{-\frac{1}{4}t}}{e^t-1}dt+\int_0^{\infty}\frac{e^{-\frac{3t}{4}}}{e^t-1}dt-\frac{1}{3}\int_0^{\infty}\frac{e^{-t}}{e^t-1}dt-\int_0^{\infty}\frac{e^{-\frac{1}{2}t}}{e^t-1}dt\overset{e^{-t}=z}{=}$$

$$=\frac{4}{3}\int_0^1\frac{z^4}{1-z^4}dz+4\int_0^1\frac{z^6}{1-z^4}dz-\frac{4}{3}\int_0^1\frac{z^7}{1-z^4}dz-4\int_0^1\frac{z^5}{1-z^4}dz=$$

$$=\frac{4}{3}\int_0^1\frac{z^4+3z^6-z^7-3z^5}{1-z^4}dz=$$

$$=\frac{4}{3}\left(\int z^3\,dz-3\int z^2dz+3\int zdz-\frac{1}{2}\int\frac{2z}{z^2+1}dz-\int dz+2\int\frac{dz}{z+1}-\int\frac{dz}{z^2+1}\right)=$$

$$=\frac{4}{3}\left[\frac{z^4}{4}-3\frac{z^3}{3}+\frac{3}{2}z^2-\frac{1}{2}log|z^2+1|-z-2log|z+1|-tan^{-1}z\right]_0^1=$$

$$=2log2-\frac{\pi}{3}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}=2log2-\frac{\pi}{3}$$

1345. Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\varepsilon}^{1}\frac{\sqrt{x}logx}{x^3+x\sqrt{x}+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\varepsilon}^{1}\frac{\sqrt{x}logx}{x^3+x\sqrt{x}+1}dx\overset{x=z^{\frac{2}{3}}}{=}\frac{4}{9}\int_0^1\frac{logz}{z^2+z+1}dz=\frac{4}{9}\int_0^1\frac{(1-z)logz}{1-z^3}dz$$

$$\Omega=\frac{4}{9}\sum_{n=0}^{\infty}\int_0^1 (z^{3n}-z^{3n+1})logz\,dz \overset{z=e^{-t}}{=} \frac{4}{9}\sum_{n=0}^{\infty}\int_0^{\infty} t\left(e^{-(3n+2)t}-e^{-3(3n+1)t}\right)dt$$

$$\Omega=\frac{4}{9}\sum_{n=0}^{\infty}\left(\frac{1}{(3n+2)^2}-\frac{1}{(3n+1)^2}\right)=\frac{4}{81}\left(\psi^{(1)}\left(\frac{2}{3}\right)-\psi^{(1)}\left(\frac{1}{3}\right)\right)$$

Therefore,

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\varepsilon}^1 \frac{\sqrt{x}logx}{x^3+x\sqrt{x}+1}dx=\frac{4}{81}\left(\psi^{(1)}\left(\frac{2}{3}\right)-\psi^{(1)}\left(\frac{1}{3}\right)\right)$$

1346. If $m>0$, then prove:

$$\lim_{x\to 0}\frac{1}{x}\left(\psi\left(\frac{m+x}{2x}\right)-\psi\left(\frac{m}{2x}\right)\right)=\frac{1}{m}$$

Proposed by Angad Singh-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\int_0^1 \frac{t^{k-1}}{1+t}dt=\frac{1}{2}\left(\psi\left(\frac{k}{2}+\frac{1}{2}\right)-\psi\left(\frac{k}{2}\right)\right)\overset{k=\frac{m}{x}}{=}$$

$$=\frac{1}{x}\left(\psi\left(\frac{m}{2x}+\frac{1}{2}\right)-\psi\left(\frac{m}{2x}\right)\right)=\frac{2}{x}\int_0^1\frac{t^{\frac{m}{x}-1}}{1+t}dt\overset{z=\frac{m}{x}}{=}\frac{2}{m}\int_0^1\frac{zt^{z-1}}{1+z}dt\overset{IBP}{=}$$

$$=\frac{2}{m}\left[\frac{t^z}{(1+t)^2}\right]_0^1+\frac{2}{m}\int_0^1\frac{t^z}{(1+t)^2}dt$$

$$\lim_{x\to 0}\frac{1}{x}\left(\psi\left(\frac{m+x}{2x}\right)-\psi\left(\frac{m}{2x}\right)\right)=\frac{1}{m}+\frac{2}{m}\lim_{z\to\infty}\int_0^1\frac{t^z}{(1+t)^2}dt\overset{t=\frac{1}{t}}{=}$$

$$=\frac{1}{m}+\frac{2}{m}\lim_{z\to\infty}\int_0^{\infty}\frac{dt}{t^z(1+t)^2}=\frac{1}{m}$$

Therefore,

$$\lim_{x\to 0}\frac{1}{x}\left(\psi\left(\frac{m+x}{2x}\right)-\psi\left(\frac{m}{2x}\right)\right)=\frac{1}{m}$$

Solution 2 by Probal Chakraborty-India

$$\because\ \psi(z)=-\int_0^1\frac{t^{z-1}}{1-t}dt$$

$$\psi\left(\frac{m}{2x}+\frac{1}{2}\right)-\psi\left(\frac{m}{2x}\right)=\int_0^1\frac{t^{\frac{m}{2x}-1}-t^{\frac{m}{2x}+\frac{1}{2}-1}}{1-t}dt=\int_0^1\frac{t^{\frac{m}{2x}-1}}{1+t^{\frac{1}{2}}}dt\overset{t=k^2}{=}$$

$$=2\int_0^1\frac{x^{\frac{m}{k}}}{1+k}dk=2\int_0^1 k^{\frac{m}{x}}(1-k+k^2-\cdots)dk=$$

$$=2\int_0^1 k^{\frac{m}{x}}\sum_{k=0}^{\infty}(-1)^n k^n dk=2\sum_{n=0}^{\infty}(-1)^n\int_0^1 k^{\frac{m+nx}{x}}dx=2\sum_{n=0}^{\infty}(-1)^n\frac{x}{m+(n+1)x}$$

$$\Omega=2\lim_{x\to 0}\frac{1}{x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x}{m+(n+1)x}\right)=2\lim_{x\to 0}\sum_{n=0}^{\infty}(-1)^m\frac{1}{m}=$$

$$=2\cdot\frac{1}{m}\cdot\sum_{n=0}^{\infty}(-1)^n=\frac{2}{m}\cdot\frac{1}{1-(-1)}=\frac{1}{m}$$

1347. Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}$$

Proposed by George Florin Şerban-Romania

Solution 1 by proposer

$$k^4+4=(k^2+2)^2-(2k)^2=(k^2+2k+2)(k^2-2k+2)$$

$$=[(k+1)^2+1][(k-1)^2+1]$$

$$\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{4(k^2+2k+2)-(k^2-2k+2)}{2^k(k^2+2k+2)(k^2-2k+2)}=$$

$$=\frac{1}{2^{k-2}[(k-1)^2+1]}-\frac{1}{2^k[(k+1)^2+1]}$$

$$\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=$$

$$=\frac{1}{2(2^2+1)}-\frac{1}{2^3(4^2+1)}+\frac{1}{2^2(3^2+1)}-\frac{1}{2^4(5^2+1)}+\frac{1}{2^3(4^2+1)}-\frac{1}{2^5(6^2+1)}+\cdots$$

$$+\frac{1}{2^{n-4}[(n-3)^2+1]}-\frac{1}{2^{n-2}[(n-1)^2+1]}+\frac{1}{2^{n-3}[(n-2)^2+1]}-\frac{1}{2^{n-1}(n^2+1)}+$$

$$+\frac{1}{2^{n-2}[(n-1)^2+1]}-\frac{1}{2^n[(n+1)^2+1]}$$

$$\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{1}{2(2^2+1)}+\frac{1}{2^2(3^2+1)}-\frac{1}{2^{n-1}(n^2+1)}-\frac{1}{2^n[(n+1)^2+1]}$$

$$\lim_{n\to\infty}\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{1}{10}+\frac{1}{40}=\frac{1}{8}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$k^4+4=k^4-(2i)^2=(k^2-2i)(k^2+2i)=(k^2-(1+i)^2)(k^2-i^2(1+i)^2)=$$

$$=\big(k-(i+1)\big)\big(k+(i+1)\big)\big(k-i(i+1)\big)\big(k+i(i+1)\big)$$

$$\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{A}{k-(i+1)}+\frac{B}{k+(i+1)}+\frac{C}{k-(i-1)}+\frac{D}{k+(i-1)}$$

$$A=\lim_{k\to i+1}\frac{3k^2+10k+6}{\big(k+(i+1)\big)\big(k-(i-1)\big)\big(k+(i-1)\big)}=-2i$$

$$B=\lim_{k\to-(i+1)}\frac{3k^2+10k+6}{\big(k-(i+1)\big)\big(k-(i-1)\big)\big(k+(i-1)\big)}=-\frac{1}{2}i$$

$$C=\lim_{k\to i-1}\frac{3k^2+10k+6}{\big(k-(i+1)\big)\big(k+(i+1)\big)\big(k+(i-1)\big)}=\frac{1}{2}i$$

$$D=\lim_{k\to 1-i}\frac{3k^2+10k+6}{\big(k-(i+1)\big)\big(k+(i+1)\big)\big(k-(i-1)\big)}=2i$$

$$\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\sum_{k=3}^{\infty}\left(-\frac{2i}{2^k(k-1-i)}-\frac{-\frac{1}{2}i}{2^k(k+1+i)}+\frac{\frac{1}{2}i}{2^k(k+1-i)}\right)=$$

$$=\sum_{k=3}^{\infty}\left(-\frac{i}{2^{k-1}(k-1-i)}-\frac{i}{2^{k+1}(k+1+i)}+\frac{i}{2^{k+1}(k+1-i)}+\frac{i}{2^{k-1}(k-1+i)}\right)=$$

$$=\sum_{k=3}^{\infty}i\left(-\frac{1}{2^{k-1}(k-1-i)}-\frac{1}{2^{k+1}(k+1+i)}+\frac{1}{2^{k+1}(k+1-i)}+\frac{1}{2^{k-1}(k-1+i)}\right)=$$

$$=\sum_{k=3}^{\infty}i\left[\frac{1}{2^{k-1}}\left(\frac{1}{k+i-1}-\frac{1}{k-i-1}\right)-\frac{1}{2^{k+1}}\left(\frac{1}{k+1+i}-\frac{1}{k+1-i}\right)\right]=$$

$$=\sum_{k=3}^{\infty}\left[\frac{1}{2^{k-1}}\cdot\frac{2}{(k-1)^2+1}-\frac{1}{2^{k+1}}\cdot\frac{2}{(k+1)^2+1}\right]=\frac{1}{10}+\frac{1}{40}+\frac{1}{2^{n-2}((n-1)^2+1)}$$

$$\lim_{n\to\infty}\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{S}{40}=\frac{1}{8}$$

Solution 3 by Remus Florin Stanca-Romania

$$\frac{3k^2+10k+6}{k^4+4}=\frac{3k^2+10k+6}{(k^2-2k+2)(k^2+2k+2)}=$$

$$=\frac{A}{k^2-2k+2}+\frac{B}{k^2+2k+2}=\frac{k^2(A+B)+2k(A-B)+2A+2B}{(k^2-2k+2)(k^2+2k+2)}$$

$$\begin{cases}A+B=3\\A-B=5\end{cases}\Rightarrow 2A=8\Rightarrow A=4, B=-1\Rightarrow$$

$$\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{1}{2^k}\left(\frac{4}{k^2-2k+2}-\frac{1}{k^2+2k+2}\right)=$$

$$=\frac{1}{2^{k-2}(k^2-2k+2)}-\frac{1}{2^k(k^2+2k+2)}$$

Let $a_k=\frac{1}{2^{k-2}(k^2-2k+2)}\to a_{k+2}=\frac{1}{2^k(k^2+2k+2)}\to$

$$\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\sum_{k=3}^{\infty}(a_k-a_{k+2})=a_3+a_4-a_{n+1}-a_{n+2}=$$

$$=\frac{1}{10}+\frac{1}{40}-\frac{1}{2^{n-1}((n+1)^2-2(n+1)+2)}-\frac{1}{2^n((n+2)^2-2(n+2)+2)}\to$$

$$\lim_{n\to\infty}\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{1}{10}+\frac{1}{40}=\frac{1}{8}$$

Solution 4 by Bedri Hajrizi-Mitrovica-Kosovo

$$\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{1}{2^k}\cdot\frac{3k^2+10k+6}{(k^2-2k+2)(k^2+2k+2)}=$$

$$=\frac{1}{2^k}\cdot\frac{4(k^2+2k+2)-(k^2-2k+2)}{(k^2-2k+2)(k^2+2k+2)}=\frac{1}{2^k}\left(\frac{4}{(k-1)^2+1}-\frac{1}{(k+1)^2+1}\right)=$$

$$=\frac{1}{2^{k-1}((k-1)^2+1)}-\frac{1}{2^k((k+1)^2+1)}$$

$$\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\sum_{k=3}^{n}\left(\frac{1}{2^{k-1}((k-1)^2+1)}-\frac{1}{2^k((k+1)^2+1)}\right)=$$

$$=\frac{1}{2^1(2^2+1)}-\frac{1}{2^3(4^2+1)}+\frac{1}{2^2(3^2+1)}-\frac{1}{2^4(5^2+1)}+\cdots+\frac{1}{2^{n-2}((n-1)^2+1)}$$

$$-\frac{1}{2^n((n+1)^2+1)}=\frac{1}{10}+\frac{1}{40}-\frac{1}{2^{n-3}((n-2)^2+1)}$$

$$\lim_{n\to\infty}\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}=\frac{1}{10}+\frac{1}{40}=\frac{1}{8}$$

1348. Compute the sum in a closed form:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)^3}{(1+2+3+4+\cdots+n)^3}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Izumi Ainsworth-Lima-Peru

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)^3}{(1+2+3+4+\cdots+n)^3}=8\sum_{n=1}^{\infty}\frac{H_n^3}{n^3(n+1)^3}=$$

$$=8\sum_{n=1}^{\infty}h_n^3\left(-\frac{1}{(n+1)^3}-\frac{3}{(n+1)^2}-\frac{6}{n+1}+\frac{1}{n^3}-\frac{3}{n^2}+\frac{6}{n}\right)=$$

$$=8\left[-\sum_{n=1}^{\infty}\frac{\left(H_{n+1}-\frac{1}{n+1}\right)^3}{(n+1)^3}-3\sum_{n=1}^{\infty}\frac{\left(H_{n+1}-\frac{1}{n+1}\right)^3}{(n+1)^3}-6\sum_{n=1}^{\infty}\frac{\left(H_{n+1}-\frac{1}{n+1}\right)^3}{n+1}+\sum_{n=2}^{\infty}\frac{H_n^3}{n^3}-3\sum_{n=2}^{\infty}\frac{H_n^2}{n^2}+6\sum_{n=2}^{\infty}\frac{H_n^3}{n}\right]=$$

$$=8\left[-\left(\sum_{n=2}^{\infty}\frac{H_n^3}{n^3}-3\sum_{n=2}^{\infty}\frac{H_n^2}{n^4}+3\sum_{n=2}^{\infty}\frac{H_n}{n^5}-\sum_{n=2}^{\infty}\frac{1}{n^6}\right)\right.$$

$$-3\left(\sum_{n=2}^{\infty}\frac{H_n^3}{n^2}-3\sum_{n=2}^{\infty}\frac{H_n^2}{n^3}+3\sum_{n=2}^{\infty}\frac{H_n}{n^4}-\sum_{n=2}^{\infty}\frac{1}{n^5}\right)$$

$$-6\left(\sum_{n=2}^{\infty}\frac{H_n^3}{n}-3\sum_{n=2}^{\infty}\frac{H_n^2}{n^2}+3\sum_{n=2}^{\infty}\frac{H_n}{n^3}-\sum_{n=2}^{\infty}\frac{1}{n^4}\right)+\sum_{n=2}^{\infty}\frac{H_n^3}{n^3}-3\sum_{n=2}^{\infty}\frac{H_n^2}{n^2}$$

$$\left.+6\sum_{n=2}^{\infty}\frac{H_n^3}{n}\right]=$$

$$= 8\left[1 + 3 - 6\sum_{n=1}^{\infty}\frac{H_n^3}{n^2} + 6 + 3\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} - 1\right) + 9\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n^3} - 1\right) + 18\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n^2} - 1\right)\right.$$

$$- 3\left(\sum_{n=1}^{\infty}\frac{H_n}{n^5} - 1\right) - 9\left(\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 1\right) - 18\left(\sum_{n=1}^{\infty}\frac{H_n}{n^3} - 1\right) + (\zeta(6) - 1)$$

$$\left. + 3(\zeta(5) - 1) + 6(\zeta(4) - 1)\right] =$$

$$= 8\left[10 - 6\big(10\zeta(5) + \zeta(2)\zeta(3)\big) + 3\left(\frac{97}{24}\zeta(6) - 2\zeta^2(3) - 1\right)\right.$$

$$+ 9\left(\frac{7}{2}\zeta(5) - \zeta(2)\zeta(3) - 1\right) + 18\left(\frac{17}{4}\zeta(4) - 1\right)$$

$$- 3\left(\frac{7}{4}\zeta(6) - \frac{1}{2}\zeta^2(3) - 1\right) - 9(3\zeta(5) - \zeta(2)\zeta(3) - 1) - 18\left(\frac{5}{4}\zeta(4) - 1\right)$$

$$\left. + \zeta(6) + 3\zeta(5) + 6\zeta(4) - 10\right] =$$

$$= 3\left(21\zeta(6) - 140\zeta(5) + 160\zeta(4) - 12\zeta^2(3) - 16\zeta(2)\zeta(3)\right)$$

1349. Prove that:

$$\lim_{x\to 0}\frac{1}{x^2}\left(\psi(x) + \omega\psi(\omega x) + \omega^2\psi(\omega^2 x) + \frac{3}{x}\right) = -3\zeta(3)$$

-where $\omega^2 + \omega + 1 = 0$ and $\psi(x)$ is the digamma function.

Proposed by Angad Singh-India

Solution by Probal Chakraborty-India

$$\lim_{x\to 0}\frac{1}{x^2}\left(\psi(x) + \omega\psi(\omega x) + \omega^2\psi(\omega^2 x) + \frac{3}{x}\right) =$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left(-\gamma + \int_0^1\frac{1 - t^x}{1 - t}dt - \gamma\omega + \int_0^1\frac{1 - t^{\omega x}}{1 - t}dt - \omega^2\gamma + \int_0^1\frac{1 - t^{\omega^2 x}}{1 - t}dt + \frac{3}{x}\right) =$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left(-\gamma(1 + \omega + \omega^2) + \int_0^1\frac{3 - t^x - t^{\omega x} - t^{\omega^2 x}}{1 - t}dt + \frac{3}{x}\right) =$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left(\int_0^1\frac{3 - \left(t^x + t^{\omega x} + t^{\omega^2 x}\right)}{1 - t}dt + \frac{3}{x}\right) =$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left(\sum_{n=0}^{\infty}\int_0^1 (3t^n - t^{x+n} - t^{\omega x+n} - t^{\omega^2 x+n})dt + \frac{3}{x}\right) =$$

$$= \lim_{x\to 0}\left(\frac{1}{x^2}\left(\sum_{n=0}^{\infty}\left(\frac{3}{n+1} + \frac{3}{x} - \frac{1}{x+n+1} - \frac{1}{\omega x+n+1} - \frac{1}{\omega^2 x+n+1}\right)\right)\right) =$$

$$= \lim_{x\to 0}\left(\frac{1}{x^2}\left(\sum_{n=0}^{\infty}\left(\frac{3x+3n+3}{(n+1)x} - \frac{1}{x+n+1} - \frac{1}{\omega x+n+1} - \frac{1}{\omega^2 x+n+1}\right)\right)\right) =$$

$$= \lim_{x\to 0}\left(\frac{1}{x^2}\left(\sum_{n=0}^{\infty}\frac{3(x+n+1)}{(n+1)x} - \frac{(x+n+1)(\omega x+n+1) + (\omega^2 x+n+1)(x+n+1) + (\omega x+n+1)(\omega^2 x+n+1)}{(x+n+1)(\omega x+n+1)(\omega^2 x+n+1)}\right)\right)$$

$$= \lim_{x\to 0}\left(\frac{3}{x^2}\left(\sum_{n=0}^{\infty}\frac{n+x+1}{(n+1)x} - \frac{(n+1)^2}{(\omega x^2 + \omega x(n+1) + (n+1)^2)(\omega^2 x+n+1)}\right)\right) =$$

$$= \lim_{x\to 0}\left(3\left(\sum_{n=0}^{\infty}\frac{n+x+1}{(n+1)x^3} - \frac{(n+1)^2}{(x^3 + (1-\omega x)(n+1)^2 + \omega x^2(n+1) + (n+1)^3)x^2}\right)\right) =$$

$$= \lim_{x\to 0} 3\left[\sum_{n=0}^{\infty}\frac{n+x+1}{(n+1)x^3} - \frac{(n+1)^2}{(x^3 + (2-\omega x+n)(n+1)^2 + \omega x^2(n+1))x^2}\right] =$$

$$= -3\sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = -3\zeta(3)$$

1350.

$$x_1 = p+1, p, n \in \mathbb{N}, p \geq 2, x_{n+1} = x_n^2 - (2p-1)x_n + p^2, n \geq 1.$$

Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{x_k - p + 1}$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$P(n): x_n \geq n + p, \forall n \geq 1$$

$$P(1): x_1 \geq p + 1 \text{ true.}$$

Suppose that $P(k): x_k \geq p + k$ is true, then we have:

$$x_{k+1} = x_k(x_k - 2p + 1) + p^2 > (p + k)(k - p + 1) + p^2 =$$

$= k^2 - p^2 + p + k + p^2 = k^2 + k + p > p + k + 1 \Leftrightarrow k^2 \geq 1$ **true. So,** $x_n \geq n + p$ **and**

$$\lim_{n\to\infty} x_n = +\infty.$$

$$\frac{1}{x_{k+1} - p} = \frac{1}{x_k^2 - 2px_k + x_k + p^2 - p} = \frac{1}{(x_k - p)^2 + x_k - p} =$$

$$= \frac{1}{(x_k - p)(x_k - p + 1)} = \frac{1}{x_k - p} - \frac{1}{x_k - p + 1} \Rightarrow$$

$$\frac{1}{x_k - p + 1} = \frac{1}{x_k - p} - \frac{1}{x_k - p + 1} \Rightarrow$$

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{x_k - p + 1} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{x_k - p} - \frac{1}{x_k - p + 1}\right) = \lim_{n\to\infty} \left(\frac{1}{x_1 - p} - \frac{1}{x_{n+1} - p}\right) =$$

$$= \frac{1}{x_1 - p} = 1$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$x_{n+1} = x_n^2 - 2px_n + x_n + p^2$$

$$x_{n+1} - x_n = x_n^2 - 2px_n + p^2 = (x_n - p)^2 > 0 \to x_{n+1} > x_n, \forall n \in N$$

$$\begin{cases} x_1 = p + 1 \\ x_2 = x_1 + (x_1 - p)^2 = p + 2 \\ x_3 = x_2 + (x_2 - p)^2 = p + 6 \\ x_4 = x_3 + (x_3 - p)^2 = p + 42 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

$$(1)\begin{cases} x_1 - p + 1 = p + 1 - p + 1 = 2 \\ x_2 - p + 1 = p + 2 - p + 1 = 3 \\ x_3 - p + 1 = p + 6 - p + 1 = 7 \\ x_4 - p + 1 = p + 42 - p + 1 = 43 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

Let $s_n = x_n - p + 1$. From (1) it follows that s_n is Sylvester's sequence

$$\left(s_n + \prod_{i=1}^{n-1} s_i = s_{n-1}(s_{n-1} - 1) + 1, s_1 = 2\right)$$

$$s_i = s_{i-1}(s_{i-1} - 1) + 1 \Leftrightarrow s_i - 1 = s_{i-1}(s_{i-1} - 1) \Leftrightarrow$$

$$\frac{1}{s_i-1}=\frac{1}{s_{i-1}(s_{i-1}-1)}\Leftrightarrow$$

$$\begin{cases}\dfrac{1}{s_i-1}=\dfrac{1}{s_{i-1}-1}-\dfrac{1}{s_i-1}\to\dfrac{1}{s_{i-1}}=\dfrac{1}{s_{i-1}-1}-\dfrac{1}{s_i-1}\Rightarrow\dfrac{1}{s_i}=\dfrac{1}{s_i-1}-\dfrac{1}{s_{i+1}-1}\\ i\to i+1\end{cases}$$

$$\begin{cases}\dfrac{1}{s_1}=\dfrac{1}{s_1-1}-\dfrac{1}{s_2-1}\\ \dfrac{1}{s_2}=\dfrac{1}{s_2-1}-\dfrac{1}{s_3-1}\\ \dfrac{1}{s_3}=\dfrac{1}{s_3-1}-\dfrac{1}{s_4-1}\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \dfrac{1}{s_{n-1}}=\dfrac{1}{s_{n-1}-1}-\dfrac{1}{s_n-1}\end{cases}\to\sum_{k=1}^{n-1}\frac{1}{s_k}=\frac{1}{s_1-1}-\frac{1}{s_n-1}\Big|+\frac{1}{s_n}$$

$$\Leftrightarrow\sum_{k=1}^{n}\frac{1}{s_k}=\frac{1}{s_1-1}-\frac{1}{s_n-1}+\frac{1}{s_n}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{x_k-p+1}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{s_k}=\lim_{n\to\infty}\left(\frac{1}{s_1-1}-\frac{1}{s_n-1}+\frac{1}{s_n}\right)=\frac{1}{s_1-1}=1$$

1351. Prove the summation:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n}\right)^2}{(1+2+3+4+\dots+n)^2}=4\big(7\zeta(4)-6\zeta(3)\big)$$

Proposed by Srinvasa Raghava-AIRMC-India

Solution 1 by Izumi Ainsworth-Lima-Peru

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n}\right)^2}{(1+2+3+4+\dots+n)^2}=4\sum_{n=1}^{\infty}\frac{H_n^2}{n^2(n+1)^2}=$$

$$=4\sum_{n=1}^{\infty}H_n^2\left(\frac{1}{(n+1)^2}+\frac{1}{n^2}+\frac{2}{n+1}-\frac{2}{n}\right)=$$

$$= 4\left\{\sum_{n=1}^{\infty}\frac{\left(H_{n+1}-\frac{1}{n+1}\right)^2}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}+\left[\sum_{n=1}^{\infty}\frac{\left(H_{n+1}-\frac{1}{n+1}\right)^2}{n+1}-\sum_{n=1}^{\infty}\frac{H_n^2}{n}\right]\right\}=$$

$$= 4\left\{\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}+\sum_{n=1}^{\infty}\frac{1}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}+2\left[\sum_{n=2}^{\infty}\frac{H_n^2}{n}-2\sum_{n=2}^{\infty}\frac{H_n}{n^2}+\sum_{n=2}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{H_n^2}{n}\right]\right\}=$$

$$= 4\left\{2(\frac{17}{4}\zeta(4)-1-2\left(\frac{5}{4}\zeta(4)-1\right)+\zeta(4)-1+2[-1-2(2\zeta(3)-1)+\zeta(3)-1]\right\}$$

$$= 4\big(7\zeta(4)-6\zeta(3)\big)$$

Solution 2 by Probal Chakraborty-India

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)^2}{(1+2+3+4+\cdots+n)^2}=4\sum_{n=1}^{\infty}\frac{H_n^2}{n^2(n+1)^2}=$$

$$= 4\lim_{M\to\infty}\left(\sum_{n=1}^{M}\left(\frac{H_n}{n}\right)^2+2\sum_{n=1}^{M}\frac{H_n^2}{n+1}+\sum_{n=1}^{M}\frac{H_n^2}{(n+1)^2}-2\sum_{n=1}^{M}\frac{H_n^2}{n}\right)$$

$$= 4(L_1+2L_2+L_3-2L_4)$$

$$L_1=\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\sum_{n=1}^{\infty}\left[\int_0^1 x^{n-1}\log(1-x)dx\int_0^1 y^{n-1}\log(1-y)dy\right]=$$

$$= \int_0^1\int_0^1\frac{\log(1-x)\log(1-y)}{1-xy}dxdy=\int_0^1\log(1-y)\int_0^1\frac{\log(1-x)}{1-xy}dxdy\overset{xy=t}{=}$$

$$= \int_0^1\frac{\log(1-y)}{y}\left(\int_0^y\frac{\log\left(1-\frac{t}{y}\right)}{1-t}dt\right)dy=$$

$$= \int_0^1\frac{\log(1-y)}{y}dy\left(-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{ky^k}\int_0^y t^{k+n}dt\right)=$$

$$= \int_0^1\frac{\log(1-y)}{y}dy\left(-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{y^{k+n+1}}{ky^k(n+k+1)}\right)=$$

$$= \int_0^1\frac{\log(1-y)}{y}dy\left(-\sum_{n=0}^{\infty}\frac{y^{n+1}}{n+1}\sum_{k=1}^{\infty}\frac{n+1}{k(n+k+1)}\right)=$$

$$=\int_0^1\frac{\log(1-y)}{y}dy\left(-\sum_{n=0}^{\infty}\frac{y^{n+1}H_{n+1}}{n+1}\right)=\int_0^1\frac{\log(1-y)}{y}\left(-Li_2(y)-\frac{1}{2}\log^2(1-y)\right)=$$

$$=-\int_0^1\frac{Li_2(y)\log(1-y)}{y}dy-\frac{1}{2}\int_0^1\frac{\log^3(1-y)}{y}dy=$$

$$=\sum_{n=1}^{\infty}\frac{H_n}{n^3}+3\zeta(4)=\frac{5}{4}\zeta(4)+3\zeta(4)=\frac{17}{4}\zeta(4)$$

$$S=4\sum_{n=1}^{\infty}\left(\frac{H_n^2}{n^2}+2\frac{H_n^2}{n+1}+\frac{H_n^2}{(n+1)^2}-2\frac{H_n^2}{n}\right)=4(L_1+2L_2+L_3-2L_4)$$

$$L_1=\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\frac{17}{4}\zeta(4)$$

$$L_2=\sum_{n=1}^{\infty}\frac{H_n^2}{n+1}=\sum_{n=1}^{\infty}\left(\frac{H_{n+1}^2}{n+1}-\frac{2H_{n+1}}{(n+1)^2}+\frac{1}{(n+1)^3}\right)\overset{H_{n+1}=H_n+\frac{1}{n+1}}{=}$$

$$=\sum_{n=1}^{\infty}\frac{H_n^2}{n}-2\sum_{n=1}^{\infty}\frac{H_n}{n^2}+\zeta(3)=\sum_{n=1}^{\infty}\frac{H_n^2}{n}-3\zeta(3)=L_4-3\zeta(3)$$

$$L_3=\sum_{n=1}^{\infty}\frac{H_n^2}{n+1}=\frac{11}{4}\zeta(4),\qquad L_4=\sum_{n=1}^{\infty}\frac{H_n^2}{n}$$

$$S=4\left(\frac{17}{4}\zeta(4)+2L_4-6\zeta(3)+\frac{11}{4}\zeta(4)-2L_4\right)=4\left(\frac{28}{4}\zeta(4)-6\zeta(3)\right)$$

$$=4\big(7\zeta(4)-6\zeta(3)\big)$$

1352. Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $a_n=\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=\frac{(-1)^n\cdot 4}{(2n+1)(2n+2)(2n+3)(2n+4)}=$

$$=4(-1)^n\left[\frac{1}{6(2n+1)}-\frac{1}{2(2n+2)}+\frac{1}{2(2n+3)}-\frac{1}{6(2n+4)}\right]=$$

$$=\frac{4}{6}(-1)^n\left[\frac{1}{2n+1}-\frac{1}{2n+4}\right]+2(-1)^{n+1}\left[\frac{1}{2n+2}-\frac{1}{2n+3}\right]=$$

$$=\frac{2}{3}(-1)^n\int_0^1 x^{2n}(1-x^3)dx+2(-1)^n\int_0^1 x^{2n+1}(1-x)dx\Rightarrow$$

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=$$

$$=\frac{2}{3}\sum_{n=0}^{\infty}\int_0^1 x^{2n}(1-x^3)dx-2\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}(1-x)dx=$$

$$=\frac{2}{3}\int_0^1\sum_{n=0}^{\infty}(x^{-2})^n(1-x^3)dx-2\int_0^1\sum_{n=0}^{\infty}(-x^2)^n x(1-x)dx=$$

$$=\frac{2}{3}\int_0^1\frac{1-x^3}{1+x^2}dx-2\int_0^1\frac{x(1-x)}{1+x^2}dx=\frac{2}{3}\int_0^1\frac{1-x^3-3x+3x^2}{1+x^2}dx=$$

$$=\frac{2}{3}\int_0^1\left(3-x-\frac{2(x+1)}{x^2+1}\right)dx=\frac{2}{3}\left[3x-\frac{1}{2}x^2-log(x^2+1)-2tan^{-1}x\right]_0^1=$$

$$=\frac{5}{3}-\frac{2}{3}log2-\frac{\pi}{3}$$

Solution 2 by Probal Chakraborty-India

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=\frac{1}{6}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}$$

$$\Omega=\frac{1}{6}-\left[\sum_{n=1}^{\infty}\left\{-\frac{(-1)^{n-1}}{3(n+2)}+\frac{2(-1)^{n-1}}{3(2n+1)}+\frac{2(-1)^{n-1}}{2n+1}-\frac{1}{n+1}\right\}\right]=\frac{1}{6}-I$$

$$\because\sum_{m=1}^{\infty}(-1)^{m-1}F(m)=\int_0^{\infty}\frac{f(t)}{e^t+1}dt\,,f(t)=L^{-1}\big(F(m)\big)$$

$$I=\sum_{n=1}^{\infty}(-1)^{n-1}\left\{-\frac{1}{3(n+2)}+\frac{1}{3\left(n+\frac{1}{2}\right)}+\frac{1}{n+\frac{3}{2}}-\frac{1}{n+1}\right\}=$$

$$=\frac{1}{3}\int_0^{\infty}\frac{-e^{-2t}+e^{-\frac{1}{2}t}}{e^t+1}dt+\int_0^1\frac{e^{-\frac{3}{2}t}}{e^t+1}dt-\int_0^{\infty}\frac{e^{-t}}{e^t+1}dt=$$

$$=\frac{1}{3}\int_0^\infty \frac{e^{-\frac{1}{2}t}-e^{-2t}+3e^{-\frac{3}{2}t}-3e^{-t}}{e^t+1}dt \overset{e^{-t}=z}{=} \frac{1}{3}\int_0^1 \frac{z^{\frac{1}{2}}-z^2+3z^{\frac{3}{2}}-3z}{z+1}dz \overset{z=t^2}{=}$$

$$=\frac{2}{3}\int_0^1 \frac{t^2-t^5+3t^\$-3t^3}{1+t^2}dt=\frac{2}{3}\int_0^1\left(\frac{2t}{t^2+1}+\frac{2}{t^2+1}-t^3-2t-2+3t^2-t^3\right)dt=$$

$$=\frac{2}{3}\left[log(t^2+1)-\frac{1}{4}t(t^3-4t^2+4t+8)+tan^{-1}t\right]_0^1=\frac{5}{3}-\frac{2}{3}log2-\frac{\pi}{3}$$

Solution 3 by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(n+1)(n+2)(2n-1)(2n+3)}=$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{3(2n-1)}+\frac{2}{2n+1}-\frac{1}{3(n+1)}-\frac{1}{n}\right)$$

$$\because \sum_{n=1}^{\infty}(-1)^{n-1}f(n)=\int_0^\infty \frac{L^{-1}\{f(s)\}}{e^t+1}dt$$

$$\Omega=\int_0^\infty \frac{L^{-1}\left\{\frac{2}{3(2s-1)}+\frac{2}{2s+1}-\frac{1}{3(s+1)}-\frac{1}{s}\right\}}{e^t+1}dt=$$

$$=-\frac{2}{3}\left[\frac{1}{2}x^2+log(x^2+1)-3x+2tan^{-1}x\right]_0^1= \ -\frac{2}{3}\left(\frac{1}{2}+log2-3+\frac{\pi}{2}\right)$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=\frac{1}{3}(5-\pi-log4)$$

Solution 4 by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=$$

$$=\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3(2n+1)}+\frac{2}{2n+3}-\frac{1}{n+1}-\frac{1}{3(n+2)}\right)=$$

$$=\sum_{n=0}^{\infty}\left(\frac{2}{3(4n+1)}+\frac{2}{4n+3}-\frac{1}{2n+1}-\frac{1}{3(2n+2)}\right)-\left(\frac{2}{3(4n+3)}+\frac{2}{4n+5}-\frac{1}{2n+2}-1/3(2n+3)\right)$$

$$=\sum_{n=0}^{\infty}\left(\frac{2}{3(4n+1)}+\frac{4}{3(4n+3)}-\frac{2}{4n+5}+\frac{1}{3(n+1)}-\frac{1}{2n+1}+\frac{1}{3(2n+3)}\right)=$$

$$=-\frac{1}{6}\psi\left(\frac{1}{4}\right)-\frac{1}{3}\psi\left(\frac{3}{4}\right)+\frac{1}{2}\psi\left(\frac{5}{4}\right)-\frac{1}{3}\psi(1)+\frac{1}{2}\psi\left(\frac{1}{2}\right)-\frac{1}{6}\psi\left(\frac{3}{2}\right)=$$

$$=\gamma\left(\frac{1}{6}+\frac{1}{3}-\frac{1}{2}+\frac{1}{3}-\frac{1}{2}+\frac{1}{6}\right)+\left(2-\frac{1}{3}\right)-\pi\left(-\frac{1}{12}+\frac{1}{6}+\frac{1}{4}\right)$$

$$-\log 2\left(-\frac{1}{2}-1+\frac{3}{2}+1-\frac{1}{3}\right)$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=\frac{1}{3}(5-\pi-\log 4)$$

Solution 5 by Serlea Kabay-Liberia

$$\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=-\frac{1}{n+1}-\frac{1}{3(n+2)}+\frac{2}{2n+3}+\frac{2}{3(2n+1)}$$

$$\Omega=\sum_{n=0}^{\infty}\left(-\frac{(-1)^n}{n+1}-\frac{(-1)^n}{3(n+2)}+\frac{2(-1)^n}{2n+3}+\frac{2(-1)^n}{3(2n+1)}\right)$$

$$A=-\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=\sum_{n=0}^{\infty}\frac{1}{2n+2}-\sum_{n=0}^{\infty}\frac{1}{2n+1}=-\frac{1}{2}\left(\psi(1)-\psi\left(\frac{1}{2}\right)\right)$$

$$A=-\frac{1}{2}\left(-\gamma-(\gamma-2\log 2)\right)=-\log 2$$

$$B=-\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}=\frac{1}{3}\left(\sum_{n=0}^{\infty}\frac{1}{2n+3}-\sum_{n=0}^{\infty}\frac{1}{2n+2}\right)=-\frac{1}{6}\left(\psi\left(\frac{3}{2}\right)-\psi(1)\right)$$

$$B=-\frac{1}{6}(2-\gamma-2\log 2+\gamma)=-\frac{1}{3}(1-\log 2)$$

$$C=\sum_{n=0}^{\infty}\frac{2(-1)^n}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{2}}=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{4}}-\sum_{n=0}^{\infty}\frac{1}{n+\frac{3}{4}}\right)=-\frac{1}{2}\left(\psi\left(\frac{1}{4}\right)-\psi\left(\frac{3}{4}\right)\right)$$

$$C=-\frac{1}{2}\left(-\gamma-\frac{\pi}{2}-3\log 2-\left(-\gamma+\frac{\pi}{2}-3\log 2\right)\right)=-\frac{1}{2}(-\pi)=\frac{\pi}{2}$$

$$D=\sum_{n=0}^{\infty}\frac{2(-1)^n}{3(2n+3)}=\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{3}{2}}=\frac{1}{3}\left(\sum_{n=0}^{\infty}\frac{1}{2n+\frac{3}{2}}-\sum_{n=0}^{\infty}\frac{1}{2n+\frac{5}{2}}\right)$$

$$D=-\frac{1}{6}\left(\psi\left(\frac{3}{4}\right)-\psi\left(\frac{5}{4}\right)\right)=\frac{4}{6}-\frac{\pi}{6}$$

Therefore,

$$\Omega=A+B+C+D=\frac{1}{3}(5-\pi-log4)$$

Solution 6 by Mohammad Rostami-Kabul-Aghanistan

$$B(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt$$

$$\eta(s)=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x+1}dx$$

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=$$

$$=\sum_{n=0}^{\infty}(-1)^n\left(-\frac{1}{n+1}+\frac{-\frac{1}{3}}{n+2}+\frac{\frac{2}{3}}{2n+1}+\frac{2}{2n+3}\right)=$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}-\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}+\frac{2}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{3}{2}}=$$

$$=-\eta(1)-\frac{1}{3}\Phi(-1,1,2)+\frac{2}{3}B(1)+\Phi\left(-1,1,\frac{3}{2}\right)=$$

$$=-\int_0^{\infty}\frac{1}{1+e^x}dx-\frac{1}{3}\int_0^{\infty}\frac{e^{-2x}}{1+e^{-x}}dx+\frac{2}{3}\cdot\frac{\pi}{4}+\int_0^{\infty}\frac{e^{-\frac{3}{2}x}}{1+e^{-x}}dx\overset{e^x=t}{=}$$

$$=-\int_1^{\infty}\frac{1}{u(u+1)}du-\frac{1}{3}\int_0^1\frac{t}{1+t}dt+\frac{\pi}{6}+\int_0^1\frac{\sqrt{t}}{1+t}dt=$$

$$=-\int_0^1\left(\frac{1}{u}-\frac{1}{1+u}\right)du-\frac{1}{3}\int_0^1\left(1-\frac{1}{1+t}\right)dt+\frac{\pi}{6}+2\int_0^1\frac{y^2}{1+y^2}dy=$$

$$=-log\left(\frac{u}{u+1}\right)_0^1-\frac{1}{3}[u-log(1+t)]_0^1+\frac{\pi}{6}+2\int_0^1\left(1-\frac{1}{1+y^2}\right)dy=$$

$$=\frac{1}{3}(5-\pi-\log 4)$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=\frac{1}{3}(5-\pi-\log 4)$$

Solution 7 by Syed Shahabudeen-India

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)(2n+1)(2n+3)}=$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}-\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}+\frac{2}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{3}{2}}=A+\frac{B}{3}+\frac{2}{3}C+2D;(1)$$

$$A=-\log 2;\left(\because -\log(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n+1}\right)$$

$$B=\log 2-1, C=\tan^{-1}(1)=\frac{\pi}{4}\left(\because \tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\right)$$

$$D=1-\tan^{-1}(1)=1-\frac{\pi}{4}$$

$$\Omega=-\log 2+\frac{1}{3}(\log 2-1)+\frac{2}{3}\tan^{-1}(1)-2(\tan^{-1}1-1)=\frac{1}{3}(5-\pi-\log 4)$$

1353. If $n,k\in\mathbb{N}, n\geq k$ and $f_k:\mathbb{R}\to\left[0,\frac{n}{n+k-1}\right]$ continuous function. Prove that:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{1-f(k)+\frac{1-k}{n}f(k)}\right)<\log 2$$

Proposed by Florică Anastase-Romania

Solution 1 by Nassim Nicholas Taleb-New York-USA

Determine the summand $S(n,k)=f(n,k)\left(\frac{(1-k)f(n,k)}{n}-f(n,k)+1\right)^{\frac{1}{k+n-1}}$

$S(\cdot)$ –is maximum for $f(n,k)=\frac{n}{n+k-1}-\epsilon, 0<\epsilon<\frac{n}{n+k-1}$.

Maximizing, we have $\frac{\partial S(n,k)}{\partial \epsilon} = 0$ for $\epsilon = \frac{n}{(k+n-1)(k+n)}$. Allora

$$\sum_{k=1}^{n} S^{Max}(n,k) = \sum_{k=1}^{n} \left(\frac{1}{k+n}\right)^{\frac{k+n}{-1+k+n}} \le \sum_{k=1}^{n} \left(\frac{1}{k+n}\right) = \psi^{(0)}(2n+1) - \psi^{(0)}(n+1),$$

Where ψ –is the polygamma function. Allora

$$\lim_{n\to\infty} \left(\psi^{(0)}(2n+1) - \psi^{(0)}(n+1)\right) = log2$$

Solution 2 by proposer

$$\frac{n}{n+k} = \frac{(n+k-1)f(k) + n - (n+k-1)f(k)}{n+k} \overset{AM-GM}{\ge}$$

$$\ge \sqrt[n+k]{f^{n+k-1}(k)\big(n-(n+k-1)f(k)\big)} \Leftrightarrow$$

$$f^{n+k-1}(k)\big(n-(n+k-1)f(k)\big) \le \left(\frac{n}{n+k}\right)^{n+k-1} \cdot \frac{n}{n+k} \Leftrightarrow$$

$$f(k)\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)} \le \frac{n}{n+k}$$

$$\frac{f(k)}{n} \cdot \sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)} \le \frac{1}{n} \cdot \frac{n}{n+k} \Leftrightarrow$$

$$\frac{1}{n}\sum_{k=1}^{n} f(k) \cdot \sqrt[n+k-1]{1 - f(n) + \frac{1-k}{n} f(n)} \le$$

$$\le \frac{1}{n}\sum_{k=1}^{n} f(k) \cdot \sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)} \le \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{n}{n+k} =$$

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{n}{n+k} = \int_0^1 \frac{1}{x+1} dx = log2$$

Therefore,

$$\lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} f(k) \cdot \sqrt[n+k-1]{1 - f(k) + \frac{1-k}{n} f(k)}\right) < log2$$

1354. If $(a_n)_{n\ge1}, a_1 = e, n \in \mathbb{N}, n \ge 2\ a_n = e^{\sqrt[n]{e}-1} \cdot a_{n-1}$ then find:

$$\Omega = \lim_{n\to\infty} log_{n+1}a_n$$

Proposed by Florică Anastase-Romania

Solution 1 by Mikael Bernardo-Mozambique

$$(a_n)_{n\ge1}, a_1 = e, n \in \mathbb{N}, n \ge 2\ a_n = e^{\sqrt[n]{e}-1} \cdot a_{n-1} \Rightarrow a_{n+1} > a_n \Rightarrow (a_n)_{n\ge1} \uparrow$$

$$\Omega = \lim_{n\to\infty} log_{n+1}a_n = \lim_{n\to\infty} log_{n+1}\left(e^{\sqrt[n]{e}-1} \cdot a_{n-1}\right) =$$

$$= \lim_{n\to\infty}\left(\left(\sqrt[n]{e}-1\right)log_{n+1}e + log_{n+1}a_{n-1}\right) = \lim_{n\to\infty}\frac{\sqrt[n]{e}-1}{log(n+1)} + \lim_{n\to\infty} log_{n+1}e^{loga_{n-1}} =$$

$$= \lim_{n\to\infty}\frac{loga_{n-1}}{n}\cdot\lim_{n\to\infty}\frac{n}{log(n+1)} \overset{LC-S}{=} \lim_{n\to\infty}\frac{loga_n - loga_{n-1}}{n+1-n}\cdot\lim_{n\to\infty}\frac{1}{\frac{1}{n+1}} =$$

$$= \lim_{n\to\infty}\left(log\left(\frac{a_n}{a_{n-1}}\right)\right) = \lim_{n\to\infty} log\left(\frac{e^{\sqrt[n]{e}-1}\cdot a_{n-1}}{a_{n-1}}\right)(n+1) =$$

$$= \lim_{n\to\infty} log\left(e^{\sqrt[n]{e}-1}\right)(n+1) = \lim_{n\to\infty}\frac{log\left(e^{\sqrt[n]{e}-1}\right)}{\frac{1}{n+1}} \overset{LC-S}{=}$$

$$= \lim_{n\to\infty}\frac{\frac{e^{\sqrt[n]{e}-1}e^{\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{e^{\sqrt[n]{e}-1}}}{-\frac{1}{(n+1)^2}} = \lim_{n\to\infty} e^{\frac{1}{n}} = 1$$

Solution 2 by Ali Jaffal-Lebanon

$$loga_n = e^{\frac{1}{n}} - 1 + loga_{n-1} \Leftrightarrow loga_n - loga_{n-1} = e^{\frac{1}{n}} - 1$$

$$loga_n - loga_1 = e^{\frac{1}{2}} + e^{\frac{1}{3}} + \dots + e^{\frac{1}{n}} - (n-1)$$

$$loga_n = e^{\frac{1}{2}} + e^{\frac{1}{3}} + \dots + e^{\frac{1}{n}} - n + 2$$

$$\because 1 + x \le e^x \le 1 + x + (e-2)x^2, \forall x \in [0,1]$$

$$n + H_n - 1 \le loga_n + n - 2 \le n + H_n - 1 + (e-2)\sum_{k=2}^{n}\frac{1}{k^2}$$

$$H_n + 1 \le loga_n \le H_n + 1 + (e-2)\sum_{k=2}^{n}\frac{1}{k^2}$$

$$\because \sum_{k=2}^{n}\frac{1}{k^2} \le \sum_{k=2}^{n}\frac{1}{k(k-1)} = \sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right) < 1-\frac{1}{n} \le 1 \Rightarrow$$

$$H_n + 1 \le loga_n \le H_n + e$$

$$\lim_{n\to\infty}\frac{H_n}{log(n+1)} = \lim_{n\to\infty}\frac{logn+\gamma+\varphi(n)}{log(n+1)} = \lim_{n\to\infty}\left(\frac{logn}{log(n+1)}+\frac{\gamma+\varphi(n)}{log(n+1)}\right) = 1$$

Therefore,

$$\Omega = \lim_{n\to\infty} log_{n+1}a_n = 1$$

Solution 3 by proposer

$$a_n = e^{\sqrt[n]{e}-1}\cdot a_{n-1} \Leftrightarrow \frac{ea_n}{a_{n-1}} = e^{\sqrt[n]{e}} \Leftrightarrow 1 + loga_n - loga_{n-1} = e^{\frac{1}{n}} \Leftrightarrow$$

$$loga_n - loga_{n-1} = e^{\frac{1}{n}} - 1$$

Summing, it follows that:

$$loga_n - loga_1 = e^{\frac{1}{2}} + e^{\frac{1}{3}} + \dots + e^{\frac{1}{n}} - (n-1) \to$$

$$loga_n = e^{\frac{1}{2}} + e^{\frac{1}{3}} + \dots + e^{\frac{1}{n}} - n + 2$$

Hence,

$$\Omega = \lim_{n\to\infty} log_{n+1}a_n = \lim_{n\to\infty}\frac{loga_n}{log(n+1)} = \lim_{n\to\infty}\frac{e^{\frac{1}{2}} + e^{\frac{1}{3}} + \dots + e^{\frac{1}{n}} - n + 2}{log(n+1)} \overset{LCS}{=}$$

$$= \lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}-1}{log(n+2)-log(n+1)} = \lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\cdot\frac{1}{log\left(1+\frac{1}{n+1}\right)^{n+1}} = 1.$$

1355. Prove that:

$$\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\binom{n}{k}^2}{(a+4)^n}\right)\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n\binom{n}{k}^2}{a^n}\right) = 1;\ a > 4$$

Proposed by Mohammed Bouras-Morocco

Solution 1 by Izumi Ainsworth-Lima-Peru

$$S=\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\binom{n}{k}^2}{(a+4)^n}\right)\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n\binom{n}{k}^2}{a^n}\right)=$$

$$=\left(\sum_{n=0}^{\infty}\frac{\Gamma(2n+1)}{(a+4)^n\Gamma^2(n+1)}\right)\left(\sum_{n=0}^{\infty}\frac{\Gamma(2n+1)}{a^n\Gamma^2(2n+1)}\right)=$$

$$=\left({}_1F_0\left(\frac{1}{2};;\frac{4}{a+4}\right)\right)\left({}_1F_0\left(\frac{1}{2};;\frac{-4}{a}\right)\right);a>4$$

$${}_1F_0(a;;z)=(1-z)^a||z|<1$$

$$\Rightarrow S=\left(1-\frac{4}{a+4}\right)^{-\frac{1}{2}}\left(1-\frac{-4}{a}\right)^{-\frac{1}{2}}=\sqrt{\frac{a+4}{a}}\cdot\sqrt{\frac{a}{a+4}}=1$$

Solution 2 by Ravi Prakash-New Delhi-India

For $|x|>1$ we have:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{x^n}\binom{n}{k}^2=\sum_{n=0}^{\infty}\frac{1}{x^n}\left(\sum_{k=0}^{n}\binom{n}{k}^2\right)=\sum_{n=0}^{\infty}\frac{1}{x^n}\binom{2n}{n}=\left(1-\frac{4}{x}\right)^{-\frac{1}{2}}$$

Thus, for $a>4$ we have:

$$\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\binom{n}{k}^2}{(a+4)^n}\right)\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n\binom{n}{k}^2}{a^n}\right)=\left(1-\frac{4}{a+4}\right)^{-\frac{1}{2}}\left(1+\frac{4}{a}\right)^{-\frac{1}{2}}=$$

$$=\left(\frac{a}{a+4}\right)^{-\frac{1}{2}}\left(\frac{a+4}{a}\right)^{-\frac{1}{2}}=1$$

1356. **Solve the equation using matrix exponential:**

$$\frac{dx}{dt}=2x+3y \text{ and } \frac{dy}{dt}=3x+3y$$

Proposed by Probal Chakraborty-India

Solution by Tobi Joshua-Nigeria

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}2&3\\3&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Let $A=\begin{pmatrix}2 & 3\\ 3 & 3\end{pmatrix}$, we find eigenvalues and eigenvectors.

$$|A-\lambda I_2|=0\Rightarrow\left|\begin{pmatrix}2 & 3\\ 3 & 3\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=0$$

Characteristic equation is $\begin{vmatrix}2-\lambda & 3\\ 3 & 3-\lambda\end{vmatrix}=0\Rightarrow(2-\lambda)(3-\lambda)-9=0\Rightarrow$

$$\lambda_1=\frac{5+\sqrt{37}}{2},\lambda_2=\frac{5-\sqrt{37}}{2}\text{(eigenvalues).}$$

For λ_1 and λ_2 we have $P_1=\begin{pmatrix}\frac{\sqrt{37}-1}{6}\\ 1\end{pmatrix}; P_2=\begin{pmatrix}1\\ \frac{-1-\sqrt{37}}{6}\end{pmatrix}$

By matrix exponential diagonalization:

$$PCe^k=C_1e^{\lambda_1 t}\begin{pmatrix}\frac{\sqrt{37}-1}{6}\\ 1\end{pmatrix}+C_2e^{\lambda_2 t}\begin{pmatrix}1\\ \frac{-1-\sqrt{37}}{6}\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}C_1e^{\lambda_1 t}\frac{\sqrt{37}-1}{6}\\ C_1e^{\lambda_1 t}\end{pmatrix}+\begin{pmatrix}C_2e^{\lambda_2 t}\\ C_2e^{\lambda_2 t}\frac{-1-\sqrt{37}}{6}\end{pmatrix}=\begin{pmatrix}C_1e^{\lambda_1 t}\frac{\sqrt{37}-1}{6}+C_2e^{\lambda_2 t}\\ C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}\frac{-1-\sqrt{37}}{6}\end{pmatrix}=$$

$$=\begin{pmatrix}C_1e^{t\frac{5+\sqrt{37}}{6}}\cdot\frac{\sqrt{37}-1}{6}+C_2e^{\frac{5-\sqrt{37}}{6}t}\\ C_1e^{t\frac{5+\sqrt{37}}{6}}+C_2e^{\frac{5-\sqrt{37}}{6}t}\cdot\frac{-1-\sqrt{37}}{6}\end{pmatrix}$$

$$x(t)=C_1e^{t\frac{5+\sqrt{37}}{6}}\cdot\frac{\sqrt{37}-1}{6}+C_2e^{\frac{5-\sqrt{37}}{6}t}$$

$$y(t)=C_1e^{t\frac{5+\sqrt{37}}{6}}+C_2e^{\frac{5-\sqrt{37}}{6}t}\cdot\frac{-1-\sqrt{37}}{6}$$

1357. Find:

$$\Omega=\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\int_\epsilon^\infty\frac{\log x}{x^4-x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Mohammad Rostami-Kabul-Afghanistan

$$\Omega=\int_0^\infty\frac{\log x}{x^4+x^2+1}dx=\int_0^\infty\frac{1}{x^4-x^2+1}\frac{d}{da}\bigg|_{a=0}x^a dx=\frac{d}{da}\bigg|_{a=0}\int_0^\infty\frac{x^a}{x^4-x^2+1}dx=$$

$$=\frac{d}{da}\bigg|_{a=0}\int_0^\infty \frac{x^a(x^2+1)}{(x^2+1)(x^4-x^2+1)}dx=\frac{d}{da}\bigg|_{a=0}\left(\int_0^\infty \frac{x^{a+2}}{x^6+1}dx+\int_0^\infty \frac{x^a}{x^6+1}dx\right)$$

$$I_1=\int_0^\infty \frac{x^{a+2}}{x^6+1}dx=\int_0^\infty \frac{t^{\frac{a+2}{6}}}{t+1}\cdot\frac{1}{6t^{\frac{5}{6}}}dt=\frac{1}{6}\int_0^\infty \frac{t^{\frac{a-3}{6}}}{t+1}dt \overset{t=x^6\to x=\sqrt[6]{t}}{=}$$

$$\left(\because\begin{cases} m-1=\dfrac{a-3}{6} \\ m+n=1\end{cases}\Rightarrow\begin{cases} m=\dfrac{a+3}{6} \\ n=\dfrac{3-a}{6}\end{cases}\right)$$

$$I_1=\frac{1}{6}B(m,n)=\frac{1}{6}B(n,m)=\frac{1}{6}B(n,1-n)=\frac{\pi}{6sin\left(\frac{\pi}{2}-\frac{\pi}{2}a\right)}=\frac{\pi}{6cos\left(\frac{\pi}{2}a\right)}$$

$$I_2=\int_0^\infty \frac{x^a}{x^6+1}dx=\int_0^\infty \frac{t^{\frac{a}{6}}}{t+1}\cdot\frac{dt}{6t^{\frac{5}{6}}}=\frac{1}{6}\int_0^\infty \frac{t^{\frac{a-5}{6}}}{t+1}dt=$$

$$\left(\because\begin{cases} m-1=\dfrac{a-5}{6} \\ m+n=1\end{cases}\Rightarrow\begin{cases} m=\dfrac{a+1}{6} \\ n=\dfrac{5-a}{6}\end{cases}\right)$$

$$\mathrm{I}_2=\frac{1}{6}\mathrm{B}(\mathrm{m},\mathrm{n})=\frac{1}{6}\mathrm{B}(\mathrm{n},\mathrm{m})=\frac{1}{6}\mathrm{B}(\mathrm{n},1-\mathrm{n})=\frac{\pi}{6\sin\left(\frac{5\pi}{6}-\frac{\pi}{6}\mathrm{a}\right)}$$

$$\Omega=\frac{d}{da}\bigg|_{a=0}(I_1+I_2)=\left[\frac{\pi}{6cos\left(\frac{\pi}{2}a\right)}+\frac{\pi}{6\sin\left(\frac{5\pi}{6}-\frac{\pi}{6}a\right)}\right]_{a=0}=$$

$$=\frac{\pi^2cos\left(\pi-\frac{\pi}{6}\right)}{36sin^2\left(\pi-\frac{\pi}{6}\right)}=-\frac{\sqrt{3}}{18}\pi^2$$

Solution 2 by Serlea Kabay-Liberia

$$\Omega=\int_0^\infty \frac{logx}{x^4-x^2+1}dx=\int_0^\infty \frac{(1+x^2)logx}{1+x^6}dx=\int_0^\infty \frac{logx}{1+x^6}dx+\int_0^\infty \frac{x^2logx}{1+x^6}dx$$

$$I_1(n):=\int_0^\infty \frac{x^n}{1+x^6}dx\overset{x^6=u}{=}\frac{1}{6}\int_0^\infty \frac{u^{\frac{n}{6}-\frac{5}{6}}}{1+u}du$$

$$\because\int_0^\infty \frac{x^n}{1+x}dx=-\pi csc(\pi n);n>-1$$

$$I_1(n)=-\frac{1}{6}\pi csc\left(\frac{\pi}{6}(n-5)\right)\Rightarrow I_1'(0)=-\frac{\pi^2}{6\sqrt{3}}$$

Also,

$$I_2(n)=\int_0^\infty\frac{x^{2+n}}{1+x^6}dx\overset{x^6=u}{=}\frac{1}{6}\int_0^\infty\frac{u^{\frac{n-3}{6}}}{1+u}du=\frac{1}{6}\pi csc\left(\frac{\pi}{6}(n-3)\right)\Rightarrow I_2'(0)=0$$

Now, $I=I_1'(0)+I_2'(0)=-\frac{\pi^2\sqrt{3}}{18}$

Solution 3 by Rana Ranino-Setif-Algerie

$$\Omega=\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\int_\epsilon^\infty\frac{logx}{x^4-x^2+1}dx=\int_0^\infty\frac{(1+x^2)logx}{x^6+1}dx\overset{t=x^6}{=}\frac{1}{36}\int_0^\infty\frac{t^{-\frac{5}{6}}-t^{-\frac{1}{2}}}{1+t}logtdt$$

$$\Omega=\frac{\pi}{36}\frac{d}{ds}\bigg|_{s=0}\int_0^\infty\frac{t^{-\left(\frac{5}{6}-s\right)}-t^{-\left(\frac{1}{2}-s\right)}}{1+t}dt$$

$$\because\int_0^\infty\frac{t^{-a}}{1+t}dt=\frac{\pi}{sin(\pi a)}$$

$$\Omega=\frac{\pi}{36}\frac{d}{ds}\bigg|_{s=0}\left[\frac{1}{sin\left(\frac{5\pi}{6}-\pi s\right)}-\frac{1}{sin\left(\frac{\pi}{2}-\pi s\right)}\right]=$$

$$=\frac{\pi}{36}\frac{d}{ds}\bigg|_{s=0}\left[\frac{1}{sin\left(\frac{\pi}{6}+s\pi\right)}-\frac{1}{cos(s\pi)}\right]=-\frac{\pi^2}{36}\cdot\frac{cos\left(\frac{\pi}{6}\right)}{sin^2\left(\frac{\pi}{6}\right)}=-\frac{\pi^2}{6\sqrt{3}}$$

Therefore,

$$\Omega=\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\int_\epsilon^\infty\frac{logx}{x^4-x^2+1}dx=-\frac{\pi^2}{6\sqrt{3}}$$

Solution 4 by Probal Chakraborty-Kolkata-India

$$\Omega=\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\int_\epsilon^\infty\frac{logx}{x^4-x^2+1}dx=\int_0^\infty\frac{logx}{x^6+1}dx=\int_0^\infty\frac{x^2logx}{x^6+1}dx+\int_0^\infty\frac{(1+x^2)logx}{x^6+1}dx$$

$$\overset{t=x^6}{=}\frac{\pi}{36}\frac{d}{ds}\bigg|_{s=0}\int_0^\infty\frac{t^{-\left(\frac{5}{6}-s\right)}-t^{-\left(\frac{1}{2}-s\right)}}{1+t}dt=$$

$$=\frac{1}{36}\left(\frac{d}{ds}\left[\Gamma\left(\frac{1}{6}+s\right)\Gamma\left(\frac{5}{6}-s\right)+\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)\right]\right)=$$