Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\sqrt{\frac{a}{b+c}}\ge 2\cdot\sqrt{1+\frac{abc}{\prod(a+b)}}\Leftrightarrow\sum_{cyc}\frac{a}{b+c}+2\sum_{cyc}\sqrt{\frac{ab}{(b+c)(c+a)}}\ge 4+\frac{4abc}{\prod(a+b)}$$

$$\sum_{cyc}a(a+b)(a+c)+2\sum_{cyc}(a+b)\sqrt{ab(b+c)(c+a)}\ge 4\prod_{cyc}(a+b)+4abc\Leftrightarrow$$

$$\sum_{cyc}a^3+2\sum_{cyc}(a+b)\sqrt{(b^2+bc)(a^2+ac)}\ge 3\sum_{cyc}a^2b+9abc$$

By Schur's inequality: $\sum a^3\ge\sum a^2b-3abc$

By BCS inequality: $2\sum(a+b)\sqrt{(b^2+bc)(a^2+ac)}\ge 2\sum(a+b)\left(ab+c\sqrt{ab}\right)=$

$$=2\sum a^2b+2\sum(a+b)c\sqrt{ab}\overset{AM-GM}{\ge}2\sum a^2b+2\sum 2\sqrt{ab}\cdot c\sqrt{ab}=$$

$$=2\sum a^2b+12abc\to$$

$$\sum_{cyc}a^3+2\sum_{cyc}(a+b)\sqrt{(b^2+bc)(a^2+ac)}\ge 3\sum_{cyc}a^2b+9abc\to 1(true).$$

$$\sum_{cyc}\sqrt{\frac{a}{b+c}}\overset{BCS}{\le}\sqrt{3\sum_{cyc}\frac{a}{b+c}}\overset{(*)}{\le}\sqrt{\frac{3(R+4r)}{2r}}\Leftrightarrow$$

$$2r\sum_{cyc}a(a+b)(a+c)\le(R+4r)\prod_{cyc}(a+b)\Leftrightarrow$$

$$2r\left[\left(\sum_{cyc}a^2\right)\cdot\left(\sum_{cyc}a\right)+3abc\right]\le(R+4r)\left[\left(\sum_{cyc}a\right)\cdot\left(\sum_{cyc}ab\right)-abc\right]\Leftrightarrow$$

$$8rs(s^2-r^2-4Rr+3Rr)\le 2s(R+4r)(s^2+r^2+4Rr-2Rr)\Leftrightarrow$$

$$4r(s^2-r^2-Rr)\le(R+4r)(s^2+r^2+2Rr)\Leftrightarrow$$

$-4r(r^2+Rr)\le Rs^2+(R+4r)(r^2+2Rr)$**, which is true because**

$$-4r(r^2+Rr)<0<Rs^2+(R+4r)(r^2+2Rr)\to(*)(true).$$

Therefore,

$$\sqrt{\frac{3(R+4r)}{2r}}\ge\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\ge 2\cdot\sqrt{1+\frac{abc}{(a+b)(b+c)(c+a)}}$$

2166. Prove that for any c_a, c_b, c_c −cevians in ΔABC holds:

$$\frac{\sqrt{c_a}+\sqrt{c_b}+\sqrt{c_c}}{c_a+c_b+c_c} \le \sqrt{\frac{3}{2F\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}}$$

Proposed by Radu Diaconu-Romania

Solution by Adrian Popa-Romania

$$ah_a = 2F \to a = \frac{2F}{h_a} \to \frac{1}{a} = \frac{h_a}{2F}$$

$$2F\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 2F\left(\frac{h_a}{2F}+\frac{h_b}{2F}+\frac{h_c}{2F}\right) = h_a+h_b+h_c \le c_a+c_b+c_c \to$$

$$\sqrt{\frac{3}{2F\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}} \ge \sqrt{\frac{3}{c_a+c_b+c_c}}$$

We must to prove that:

$$\frac{\sqrt{c_a}+\sqrt{c_b}+\sqrt{c_c}}{c_a+c_b+c_c} \le \sqrt{\frac{3}{c_a+c_b+c_c}} \to \left(\frac{\sqrt{c_a}+\sqrt{c_b}+\sqrt{c_c}}{c_a+c_b+c_c}\right)^2 \le \frac{3}{c_a+c_b+c_c} \to$$

$\left(\sqrt{c_a}+\sqrt{c_b}+\sqrt{c_c}\right)^2 \le 3(c_a+c_b+c_c)$ true from BCS inequality.

2167. In ΔABC the following relationship holds:

$$\frac{12r^3}{R^2} \le \frac{ah_b+bh_c+ch_a}{a+b+c} \le \frac{s^2}{4R+r}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\Omega = \frac{ah_b+bh_c+ch_a}{a+b+c} = \frac{2F}{2s}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) = r\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \overset{(1)}{\le} \frac{s^2}{4R+r}$$

$$(1) \leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \le \frac{s^2}{4R+r} = \frac{s^2}{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)}$$

$$Let\ x = s-a; y = s-b; z = s-c \to x+y+z = s; a = y+z; b = z+x; c = x+y$$

$$\leftrightarrow \frac{y+z}{x+z}+\frac{x+z}{x+y}+\frac{x+y}{y+z}\le\frac{(x+y+z)^2}{xy+yz+zx}$$

$$\leftrightarrow (xy+yz+zx)[(y+z)^2(x+y)+(x+z)^2(y+z)+(x+y)^2(x+z)]\le$$

$$\le (x+y)(y+z)(z+x)(x+y+z)^2$$

$\leftrightarrow x^3z^2+x^2y^3+y^2z^3-xyz(xy+yz+zx)\ge 0$; which is true because

$$x^3z^2+x^2y^3+y^2z^3=\frac{(xy)^2}{\frac{1}{x}}+\frac{(xy)^2}{\frac{1}{y}}+\frac{(yz)^2}{\frac{1}{z}}\overset{BCS}{\ge}\frac{(xy+yz+zx)^2}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}=$$

$$=\frac{xyz(xy+yz+zx)^2}{xy+yz+zx}=xyz(xy+yz+zx)\to(1) is\ true.$$

$$\Omega=\frac{ah_b+bh_c+ch_a}{a+b+c}=r\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\overset{AM-GM}{\ge}3r\overset{(2)}{\ge}\frac{12r^3}{R^2}$$

$$(2)\leftrightarrow R^2\ge 4r^2\leftrightarrow R\ge 2r(Euler).$$

2168. In ΔABC the following relationship holds:

$$1\le\frac{2}{\sqrt{\frac{7}{4}+\sum\cos^2\frac{A}{2}}}\le\left(\frac{R}{2r}\right)^{\frac{3}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

For the left side we must show that: $\sqrt{\frac{7}{4}+\sum\cos^2\frac{A}{2}}\le 2\leftrightarrow\frac{7}{4}+\sum\cos^2\frac{A}{2}\le 4\leftrightarrow$

$$\sum_{cyc}\cos^2\frac{A}{2}\le\frac{9}{4};(1)$$

$$\sum_{cyc}\cos^2\frac{A}{2}=2+\frac{r}{2R};(2)$$

From (1),(2) we must show that: $2+\frac{r}{2R}\le\frac{9}{4}\leftrightarrow\frac{r}{2R}\le\frac{1}{4}\leftrightarrow 2r\le R(Euler)$.

For the right side, we must show that:

$$\frac{\sqrt{\frac{7}{4}+\sum cos^2\frac{A}{2}}}{2} \geq \sqrt{\left(\frac{2r}{R}\right)^3} \leftrightarrow \frac{7}{4}+\sum_{cyc} cos^2\frac{A}{2} \geq 4\left(\frac{2r}{R}\right)^3 \leftrightarrow$$

$$\frac{7}{4}+2+\frac{r}{2R} \geq 4\left(\frac{2r}{R}\right)^3 \leftrightarrow 4\left(\frac{2r}{R}\right)^3 - \frac{1}{4}\cdot\frac{2r}{R} - \frac{15}{4} \leq 0; (3)$$

Let $x = \frac{2r}{R} \leq 1 (Euler); (4)$

From (3),(4) we must show that: $4x^3 - \frac{x}{4} - \frac{15}{4} \leq 0 \leftrightarrow 16x^3 - x - 15 \leq 0 \leftrightarrow$

$(x-1)(16x^2+16x+15) \leq 0$, which is true because $x \leq 1$ and $16x^2+16x+15 > 0$.

2169. In ΔABC the following relationship holds:

$$m_a^3 m_b + m_b^3 m_c + m_c^3 m_a \geq 81r^3(2R-r)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that: $\sum x^3 y \geq xyz \sum x, \forall x,y,z > 0$

$$\sum x^3 y = \sum \left(\frac{4}{7}x^3 y + \frac{1}{7}y^3 z + \frac{2}{7}z^3 x\right) \overset{AM-GM}{\geq} \sum x^2 yz$$

$$\sum_{cyc} m_a^3 m_b \geq m_a m_b m_c \sum_{cyc} m_a$$

We know that: $m_a \geq \sqrt{s(s-a)} (and\ analogs)$

$$\prod_{cyc} m_a \geq s^2 r \geq \frac{27Rr}{2}\cdot r = \frac{27}{2}Rr^2; (i)$$

We know that: $m_a \geq \frac{b^2+c^2}{4R} (Tereshin) \to$

$$\sum_{cyc} m_a \geq \frac{\sum a^2}{2R} = \frac{s^2-r^2-4Rr}{R} \overset{Gerretsen}{\geq} \frac{(16Rr-5r^2)-r^2-4Rr}{R} \to$$

$$\sum_{cyc} m_a \geq \frac{6r(2R-r)}{R}; (ii)$$

From $(i),(ii)$ it follows that:

$$m_a^3m_b+m_b^3m_c+m_c^3m_a\ge 81r^3(2R-r)$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$m_a^3m_b+m_b^3m_c+m_c^3m_a\overset{AM-GM}{\ge}3\sqrt[3]{(m_am_bm_c)^4}\overset{m_am_bm_c\ge r_ar_br_c=s^2r}{\ge}3\sqrt[3]{(s^2r)^4}$$

We need to prove: $\sqrt[3]{(s^2r)^4}\ge 27r^3(2R-r)\leftrightarrow(s^2r)^4\ge 27^3r^9(2R-r)^3\leftrightarrow$

$$(s^2)^4\ge 27^3r^5(2R-r)^3;(1)$$

But $s^2\ge 16Rr-5r^2(Gerretsen)\to(s^2)^4\ge(16Rr-5r^2)^4\overset{(2)}{\ge}27^3r^5(2R-r)^3$

$$(2)\leftrightarrow(16t-5)^4\ge 27^3(2t-1)^3;\left(t=\frac{R}{r}\ge 2\right)\leftrightarrow$$

$2(t-2)[t^2(32768t-54156)+28986t-5077]\ge 0$ true, because

$t\ge 2\to 2(t-2)\ge 0$ and $t^2(32768t-54156)+28986t-5077>0$

$\to(2)\to(1)$ is true.

2170. In ΔABC the following relationship holds:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge\frac{3s(s^2-3r^2-6Rr)}{s^2-r^2-4Rr}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge\frac{3s(s^2-3r^2-6Rr)}{s^2-r^2-4Rr}=\frac{3\sum a^3}{\sum a^2};(*)$$

It is suffices to prove that: $\sum\frac{a^2}{b}\ge\frac{3\sum a^3}{\sum a^2};(1)$

$$(1)\leftrightarrow\sum\frac{a^2}{b}-2\sum a+2\sum b\ge\frac{3\sum a^3}{\sum a^2}-\sum a=\frac{\sum\left(a^3+b^3+c^3-a(a^2+b^2+c^2)\right)}{\sum a^2}$$

$$\sum\frac{a^2-2ab+b^2}{b}\ge\frac{\sum\left(c^2(c-a)-b^2(a-b)\right)}{\sum a^2}\leftrightarrow$$

$$\left(\sum a^2\right)\left(\sum\frac{(a-b)^2}{b}\right)\ge\sum a^2(a-b)-\sum b^2(a-b)=\sum(a+b)(a-b)^2\leftrightarrow$$

$$\sum\left(\frac{a^2+b^2+c^2}{b}-(a+b)\right)(a-b)^2\ge 0\leftrightarrow$$

$$\sum (a^2-ab+c^2)ac(a-b)^2 \geq 0 \leftrightarrow \sum s_c(a-b)^2 \geq 0$$

$$\because s_c = ac(a^2-ab+c^2), s_a = ba(b^2-bc+a^2), s_b = cb(c^2-ca+b^2)$$

$$\sum s_a = \sum ab(b^2-bc+a^2) = \sum a^3b - \sum a^2bc \geq \sum a^2bc \geq 0 \to \sum s_a \geq 0.$$

$$\sum s_as_b = \sum a^2bc(b^2-bc+a^2)(a^2-ab+c^2) =$$

$$= abc\left(\sum a^5 - \sum a^4b + \sum a^3b^2 - 2\sum a^3bc + 2\sum a^2b^2c\right)$$

By AM-GM: $a^5 + a^3b^2 \geq 2a^4b$ and $a^{4b} + a^2bc^2 \geq 2a^3bc \to$

$$\sum a^5 - \sum a^4b + \sum a^3b^2 - 2\sum a^3bc + 2\sum a^2b^2c \geq \sum a^2b^2c \geq 0 \to$$

$$\sum s_as_b \geq 0 \to \sum s_c(a-b)^2 \geq 0 \to$$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{3\sum a^3}{\sum a^2} = \frac{3s(s^2-3r^2-6Rr)}{s^2-r^2-4Rr}$$

2171. In $\Delta ABC, g_a$ –Gergonne cevian, the following relationship holds:

$$\sum_{cyc} \sqrt{m_a + h_a + g_a + \sqrt{3}a} \leq 2 \cdot \sqrt{12R+3r}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc} \sqrt{m_a + h_a + g_a + \sqrt{3}a} \leq 2 \cdot \sqrt{12R+3r}; (*)$$

We know that: $max\{h_a, g_a, w_a, s_a\} \leq m_a (and\ analogs)$

$$LHS \leq \sum_{cyc} \sqrt{3m_a + \sqrt{3}a} \overset{BCS}{\leq} \sqrt{3\sum_{cyc}(3m_a + \sqrt{3}a)} = \sqrt{9\sum_{cyc} m_a + 6\sqrt{3}s}$$

We know that: $\sum m_a \leq 4R + r$ and $\sqrt{3}s \leq 4R + r \leftrightarrow 3\sum r_ar_b \leq (\sum r_a)^2$ is true.

$$\to 9\sum_{cyc} m_a + 6\sqrt{3}s \leq 9(4R+r) + 6(4R+r) = 3(20R+5r)$$

Therefore,

$$\sum_{cyc}\sqrt{m_a+h_a+g_a+\sqrt{3}a}\le 2\cdot\sqrt{12R+3r}$$

2172. **In ΔABC the following relationship holds:**

$$h_a^3h_b+h_b^3h_c+h_c^3h_a\ge\frac{54r^4(5R-r)}{R}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that: $\sum x^3y\ge xyz\sum x,\forall x,y,z>0$

$$\because\sum x^3y=\sum\left(\frac{4}{7}x^3y+\frac{1}{7}y^3z+\frac{2}{7}z^3x\right)\overset{AM-GM}{\ge}\sum x^2yz\to$$

$$\sum_{cyc}h_a^3h_b\ge h_ah_bh_c\left(\sum_{cyc}h_a\right)$$

$$\prod_{cyc}h_a=\prod_{cyc}\frac{2sr}{a}=\frac{2s^2r^2}{R}\ge\frac{27Rr}{2}\cdot\frac{2r^2}{R}=27r^3;(i)$$

$$\sum_{cyc}h_a=2sr\sum_{cyc}\frac{1}{a}=\frac{1}{2R}\sum_{cyc}ab=\frac{s^2+r^2+4Rr}{2R}$$

$$\to\sum_{cyc}h_a\overset{Gerretsen}{\ge}\frac{16Rr-5r^2+r^2+4Rr}{2R}=\frac{2r(5R-r)}{R};(ii)$$

From $(i),(ii)$ it follows that:

$$h_a^3h_b+h_b^3h_c+h_c^3h_a\ge\frac{54r^4(5R-r)}{R}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc}h_a^3h_b=\sum_{cyc}\frac{h_a^3}{\frac{1}{h_b}}\overset{Holder}{\ge}\frac{(\sum h_a)^3}{3\sum\frac{1}{h_a}}=\frac{(\sum h_a)^3}{3\cdot\frac{1}{r}}=\frac{\left(\sum\frac{2S}{a}\right)^3}{3\cdot\frac{1}{r}}=\frac{r}{3}\left(\frac{s^2+r^2+4Rr}{2R}\right)^3\ge$$

$$\overset{Gerretsen}{\ge}\frac{r}{3}\cdot\left(\frac{20Rr-4R^2}{2R}\right)^3=\frac{r^4}{3}\left(\frac{10R-2r}{R}\right)^3\overset{(1)}{\ge}\frac{54r^4(5R-r)}{R}$$

$$(1) \leftrightarrow (10R-2r)^3 \geq 810R^2 - 162rR^2; \left(\because t = \frac{R}{r} \geq 2\right)$$

$$\leftrightarrow (10t-2)^3 \geq 810R^3 - 162t^2 \leftrightarrow 190t^3 - 438t^2 + 120t - 8 \geq 0 \leftrightarrow$$

$2(t-2)(5t-1)(19t-2) \geq 0$, which is true by:

$t \geq 2 \to 2(t-2) \geq 0; 5t-1 \geq 9 > 0; 19t-2 \geq 36 > 0 \to (1)$ is true.

2173. In ΔABC the following relationship holds:

$$\left(\frac{2r}{R}\right)^{\frac{3}{2}} \leq \frac{3}{\sqrt{8+\sum tan^2\frac{A}{2}}} \leq 1$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

For RHS, we must show that: $\sqrt{8+\sum tan^2\frac{A}{2}} \geq 3 \Leftrightarrow \sum tan^2\frac{A}{2} \geq 1; (1)$

But $\sum tan^2\frac{A}{2} = \frac{(4R+r)^2}{s^2} - 2; (2)$. Form (1),(2) we must show:

$$\frac{(4R+r)^2}{s^2} - 2 \geq 1 \Leftrightarrow (4R+r)^2 \geq s^2 (Doucet)$$

For LHS, we must show that:

$$\frac{\sqrt{8+\sum tan^2\frac{A}{2}}}{3} \leq \sqrt{\left(\frac{R}{2r}\right)^3} \Leftrightarrow 8+\sum tan^2\frac{A}{2} \leq 9\left(\frac{R}{2r}\right)^3 \Leftrightarrow 6+\frac{(4R+r)^2}{s^2} \leq 9\left(\frac{R}{2r}\right)^3; (3)$$

$$s^2 \geq 3r(4R+r)(Doucet); (4).$$

From (3),(4) we must show that:

$$6+\frac{(4R+r)^2}{s^2} \leq 9\left(\frac{R}{2r}\right)^3 \Leftrightarrow 6+\frac{4R}{3r}+\frac{1}{3} \leq 9\left(\frac{R}{2r}\right)^3 \Leftrightarrow$$

$$9\left(\frac{R}{2r}\right)^3 - \frac{8}{3}\left(\frac{r}{2r}\right) - \frac{19}{3} \geq 0. Let \frac{R}{2r} = x \geq 1 (Euler)$$

We must show that: $9x^3 - \frac{8}{3}x - \frac{19}{3} \geq 0 \Leftrightarrow 27x^3 - 8x - 19 \geq 0 \Leftrightarrow$

$$(x-1)(27x^2+27x+19) \geq 0 \; true \; \forall x \geq 1.$$

2174. **If in $\Delta ABC, \Omega -$ first Brocard point then:**

$$\Omega A^2 + \Omega B^2 + \Omega C^2 \geq \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$\Delta A\Omega C: \frac{A\Omega}{\sin\omega} = \frac{AC}{\sin(A\Omega C)}$$

$$\angle A\Omega C = 180^o - \omega - (A - \omega) =$$

$$= 180^o - A \Rightarrow \frac{A\Omega}{\sin\omega} = \frac{b}{\sin A}$$

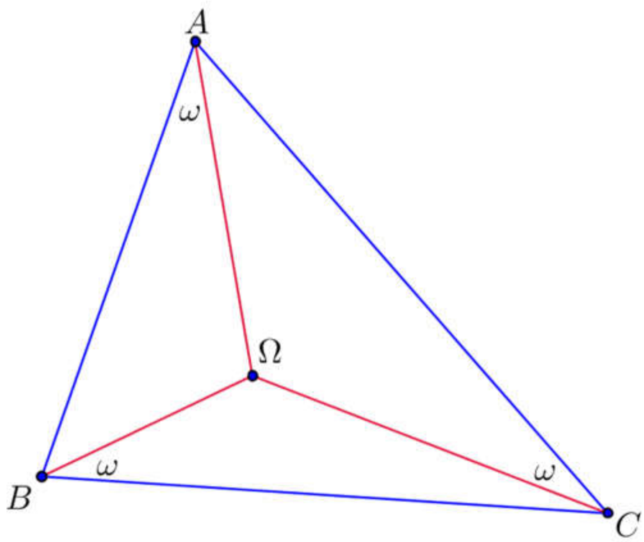
$$A\Omega = \frac{b sim\omega}{\sin A} = \frac{b}{a} 2R\sin\omega$$

$$B\Omega = \frac{c}{b} 2R\sin\omega; C\Omega = \frac{a}{c} 2R\sin\omega$$

Hence, $\Omega A^2 + \Omega B^2 + \Omega C^2 =$

$$4R^2\sin^2\omega\left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right)$$

$$\sin^2\omega = \frac{4F^2}{a^2b^2 + b^2c^2 + c^2a^2}$$



We must to prove that: $4R^2 \cdot \frac{4F^2}{a^2b^2+b^2c^2+c^2a^2} \cdot \frac{b^4c^2+c^4a^2+a^4b^2}{a^2b^2c^2} \geq \frac{a^2b^2+b^2c^2+c^2a^2}{a^2+b^2+c^2}$

$$\frac{b^4c^2 + c^4a^2 + a^4b^2}{a^2b^2 + b^2c^2 + c^2a^2} \geq \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2} \Leftrightarrow$$

$$(b^4c^2 + c^4a^2 + a^4b^2)(a^2 + b^2 + c^2) \geq (a^2b^2 + b^2c^2 + c^2a^2)^2 (BCS)$$

2175. **In ΔABC**

$$\sum \cos^3\frac{A}{2}\cos\frac{B}{2} \geq \frac{p}{4R}\left(\frac{3\sqrt{3}}{4} + \frac{p}{2R}\right)$$

Proposed by Marin Chirciu – Romania

Solution by Tran Hong-DongThap-Vietnam

Lemma: In any triangle ABC :

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\geq\frac{3\sqrt{3}}{4}+\frac{p}{2R};\ (1)$$

Proof:

Let $\varphi(x)=\sin x,(0<x<\pi)\rightarrow\varphi''(x)=-\sin x<0,\forall x\in(0;\pi)$

By Popoviciu's inequality:

$$\varphi(A)+\varphi(B)+\varphi(C)+3\varphi\left(\frac{A+B+C}{3}\right)\leq 2\varphi\left(\frac{A+B}{2}\right)+2\varphi\left(\frac{B+C}{2}\right)+2\varphi\left(\frac{B+C}{2}\right);$$

$$\leftrightarrow\sin A+\sin B+\sin C+3\sin\frac{\pi}{3}\leq 2\left(\sin\left(\frac{A+B}{2}\right)+\sin\left(\frac{B+C}{2}\right)+\sin\left(\frac{C+A}{2}\right)\right);$$

$$\leftrightarrow\frac{p}{R}+\frac{3\sqrt{3}}{2}\leq 2\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right);$$

$$\leftrightarrow\frac{p}{2R}+\frac{3\sqrt{3}}{4}\leq\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2};$$

$\rightarrow(1)$ is true. Now,

$$\sum\cos^3\frac{A}{2}\cos\frac{B}{2}=\sum\frac{\left(\cos\frac{A}{2}\right)^2}{\frac{1}{\cos\frac{A}{2}\cos\frac{B}{2}}}\overset{C-S}{\geq}\frac{\left(\sum\cos\frac{A}{2}\right)^2}{\frac{1}{\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)}\sum\cos\frac{A}{2}}$$

$$=\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\sum\cos\frac{A}{2}=\frac{p}{4R}\sum\cos\frac{A}{2}\overset{\text{By }(1)}{\geq}\frac{p}{4R}\left(\frac{3\sqrt{3}}{4}+\frac{p}{2R}\right).\text{ Proved.}$$

2176. If H −ortocenter in acute ΔABC then:

$$M=max\left(1+\frac{s}{AH},1+\frac{s}{BH},1+\frac{s}{CH}\right)\geq\frac{12r(2r+s)}{R(7R-2r)}$$

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

ΔABC is acute: $cos\ A, cos\ B, cos\ C>0\Rightarrow AH=2R\cdot cos\ A(and\ analogs)$

$$\prod_{cyc}cos\ A\overset{AM-GM}{\leq}\left(\frac{1}{3}\sum_{cyc}cos\ A\right)^3=\left(\frac{1}{3}\left(1+\frac{r}{R}\right)\right)^3\overset{Euler}{\leq}\frac{1}{8}\Rightarrow\prod_{cyc}AH\leq R^3;(1)$$

$$M \ge \sqrt[3]{\prod_{cyc}\left(1+\frac{s}{AH}\right)} \overset{Holder}{\ge} 1+\frac{s}{\sqrt[3]{AH}} \overset{(1)}{\ge} 1+\frac{s}{R} \overset{(*)}{\ge} \frac{12r(2r+s)}{R(7R-2r)}$$

$$(*) \Leftrightarrow 7R^2-2Rr-24r^2+7s(R-2r)\ge 0 \Leftrightarrow (R-2r)(7R+12r+7s)\ge 0,$$

which is true from $R \ge 2r\ (Euler)$.

Therefore,

$$M = max\left(1+\frac{s}{AH},1+\frac{s}{BH},1+\frac{s}{CH}\right) \ge \frac{12r(2r+s)}{R(7R-2r)}$$

2177. In ΔABC the following relationship holds:

$$\sum_{cyc} sec^3\frac{A}{2}sec\frac{B}{2} \ge \frac{8R\sqrt{3}}{s} \ge \frac{16}{3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc} x^3y\right)\left(\sum_{cyc} xyz^2\right) \overset{BCS}{\ge} \left(\sum_{cyc} x^2yz\right), \forall x,y,z>0$$

$$\Rightarrow \sum_{cyc} x^3y \ge xyz\left(\sum_{cyc} x\right), \forall x,y,z>0$$

$$\Rightarrow \sum_{cyc} sec^3\frac{A}{2}sec\frac{B}{2} \ge \prod_{cyc} sec\frac{A}{2}\sum_{cyc} sec\frac{A}{2}$$

$$\prod_{cyc} sec\frac{A}{2} = \prod_{cyc}\sqrt{\frac{bc}{s(s-a)}} = \frac{4Rrs}{s^2r} = \frac{4R}{s}$$

$x \to secx$ is convexe on $\left(0,\frac{\pi}{2}\right)$, then:

$$\sum_{cyc} sec\frac{A}{2} \overset{Jensen}{\ge} 3sec\frac{\pi}{3} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\Rightarrow \sum_{cyc} sec^3\frac{A}{2}sec\frac{B}{2} \ge \frac{8R\sqrt{3}}{s} \overset{Mitrinovic}{\ge} \frac{8\sqrt{3}\cdot 2}{3\sqrt{3}} = \frac{16}{3}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

❖ $\sum\left(\sec^3\frac{A}{2}\sec\frac{B}{2}\right)=\sum\frac{1}{\cos^3\frac{A}{2}\cos\frac{B}{2}}=\sum\frac{\left(\frac{1}{\cos\frac{A}{2}}\right)^3}{\cos\frac{B}{2}}\overset{Holder}{\geq}\frac{\left(\sum\frac{1}{\cos\frac{A}{2}}\right)^3}{3\sum\cos\frac{A}{2}}$

$$\overset{Am-Gm}{\geq}\frac{\left(3.\sqrt[3]{\frac{1}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}}\right)^3}{3\sum\cos\frac{A}{2}}=$$

$$=\frac{9}{\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\sum\cos\frac{A}{2}}\overset{\sum\cos\frac{A}{2}\leq\frac{3\sqrt{3}}{2}}{\geq}\frac{9}{\frac{p}{4R}\cdot\frac{3\sqrt{3}}{2}}=\frac{8\sqrt{3}R}{p};$$

❖ $\frac{8\sqrt{3}R}{p}\geq\frac{16}{3}\leftrightarrow 3\sqrt{3}R\geq 2p\leftrightarrow p\leq\frac{3\sqrt{3}}{2}R$ ($\therefore$ true by Mitrinovic)

❖

2178. In ΔABC the following relationship holds:

$$\sum_{cyc}w_a^3w_b\geq 243r^4$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc}w_a^3w_b\overset{w_a\geq h_a}{\geq}\sum_{cyc}h_a^3h_b\overset{AM-GM}{\geq}3\sqrt[3]{(h_ah_bh_c)^4}=3\sqrt[3]{\left(\frac{2s^2r^2}{R}\right)^4}$$

We need to prove: $3\sqrt[3]{\left(\frac{2s^2r^2}{R}\right)^4}\geq 243r^4\Leftrightarrow\left(\frac{2s^2r^2}{R}\right)^4\geq 81^3(r^4)^3\Leftrightarrow$

$$\left(\frac{2s^2r^2}{R}\right)^4\geq(3^3\cdot r^3)^4\Leftrightarrow 2s^2\geq 27Rr;(1)$$

Other, $s^2\geq 16Rr-5r^2(Gerretsen)\rightarrow 2s^2\geq 2(16Rr-5r^2)\overset{(2)}{\geq}27Rr$

$(2)\Leftrightarrow 5rr\geq 10r^2\Leftrightarrow R\geq 2r(Euler)\rightarrow(2)\rightarrow(1)$ is true.

2179. In ΔABC the following relationship holds:

$$\sum_{cyc} r_a^3 r_b \geq 3r^2(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc} r_a^3 r_b\right)\left(\sum_{cyc} r_a r_b r_c^2\right) \overset{BCS}{\geq} \left(\sum_{cyc} r_a^2 r_b r_c\right)^2$$

$$\sum_{cyc} r_a^3 r_b \geq r_a r_b r_c \sum_{cyc} r_a = s^2 r(4R+r)$$

$$\sum_{cyc} r_a^3 r_b \overset{Doucet}{\geq} 3r^2(4R+r)^2$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc} r_a^3 r_b = \sum_{cyc} \frac{r_a^3}{\frac{1}{r_b}} \overset{Holder}{\geq} \frac{(\sum r_a)3}{3\sum \frac{1}{r_a}} = \frac{(4R+r)^3}{3\cdot \frac{1}{r}} = \frac{r(4R+r)^3}{3} \overset{(1)}{\geq} 3r^2(4R+r)^2$$

$$(1) \Leftrightarrow 4R+r \geq 9r \Leftrightarrow 4R \geq 8r \Leftrightarrow R \geq 2r (Euler).$$

2180. In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a}{b^2+c^2-a^2} \geq \frac{1}{2s}\left(\frac{2R^2}{r^2} + \frac{R}{r} - 1\right)$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma: $\sum \frac{1}{cosA} = \frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}$; $\sum cosA = 1 + \frac{r}{R}$

$$\sum_{cyc} \frac{a}{b^2+c^2-a^2} = \sum_{cyc} \frac{a}{2bc\cdot cosA} = \frac{1}{2abc}\sum_{cyc}\frac{a^2}{cosA} = \frac{1}{8sRr}\sum_{cyc}\frac{(2RsinA)^2}{cosA} =$$

$$= \frac{R}{2sr}\sum_{cyc}\frac{1-cos^2A}{cosA} = \frac{R}{2sr}\sum_{cyc}\frac{1}{cosA} - \frac{R}{2sr}\sum_{cyc} cosA =$$

$$=\frac{1}{2sr}\left(\frac{s^2R+Rr^2-4R^3}{s^2-(2R+r)^2}-(R+r)\right)=\frac{1}{2s}\cdot\frac{8R^2+6Rr+r^2-s^2}{s^2-4R^2-4Rr-r^2}\overset{Gerretsen}{\geq}$$

$$\geq\frac{1}{2s}\cdot\frac{8R^2+6Rr+r^2-(4R^2+4Rr+3r^2)}{(4R^2+4Rr+3r^2)-4R^2-4Rr-r^2}=\frac{1}{2s}\left(\frac{2R^2}{r^2}+\frac{R}{r}-1\right)$$

Therefore,

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}\geq\frac{1}{2s}\left(\frac{2R^2}{r^2}+\frac{R}{r}-1\right)$$

2181. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}\geq\frac{9R}{4F}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\Delta ABC\text{ -acute}\Rightarrow b^2+c^2-a^2>0(and\ analogs)$$

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}=\sum_{cyc}\frac{a^3}{a^2b^2+a^2c^2-a^4}\overset{Holder}{\geq}\frac{(\sum a)^3}{3\sum(a^2b^2+a^2c^2-a^4)}=$$

$$=\frac{(2s)^3}{3(2\sum a^2b^2-\sum a^4)}=\frac{8s^3}{3\cdot 16F^2}=\frac{s^2\cdot s}{6rsF}\overset{s^2\geq\frac{27Rr}{2}}{\geq}\frac{\frac{27Rr}{2}}{6rF}=\frac{9R}{4F}$$

Therefore,

$$\sum_{cyc}\frac{a}{b^2+c^2-a^2}\geq\frac{9R}{4F}$$

2182. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{cosB+cosC}{a}\leq\frac{s}{9r}\sum_{cyc}\frac{sinB+sinC}{a}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$\sum_{cyc}\frac{sinB+sinC}{a}=\frac{1}{2R}\sum_{cyc}\frac{b+c}{a}=\frac{1}{2R}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\geq\frac{1}{2R}\cdot 6=\frac{3}{R}$$

We must show that:

$$\sum_{cyc}\frac{cosB+cosC}{a}\le\frac{s}{3Rr};(1)$$

$$\sum_{cyc}\frac{cosB+cosC}{a}=\sum_{cyc}\frac{cosA+cosB+cosC-cosA}{a}=$$

$$=(cosA+cosB+cosC)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\sum_{cyc}\frac{cosA}{a}=$$

$$=(cosA+cosB+cosC)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\sum_{cyc}\frac{b^2+c^2-a^2}{2abc}=$$

$$=(cosA+cosB+cosC)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{1}{2abc}(a^2+b^2+c^2);(2)$$

But $cosA+cosB+cosC=\frac{R+r}{R}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{s^2+r^2+4Rr}{4Rrs}$ and

$$a^2+b^2+c^2=2(s^2-r^2-4Rr);(3)$$

From (2),(3) we get:

$$\sum_{cyc}\frac{cosB+cosC}{a}=\frac{8R^2+6Rr+r^2+s^2}{4R^2s};(4)$$

From (3),(4) we must to show:

$$\frac{8R^2+6Rr+r^2+s^2}{4R^2s}\le\frac{s}{3Rr}\Leftrightarrow 3r(8R^2+6Rr+r^2+s^2)\le 4s^2R;(5)$$

But $s^2\ge\frac{27Rr}{2}(Cosnita-Turtoiu);(6)$

From (5),(6) we must to show:

$$3r(8R^2+6Rr+s^2+r^2)\le 4R\cdot\frac{27Rr}{2}\Leftrightarrow$$

$$8R^2+6Rr+r^2+s^2\le 18R^2\Leftrightarrow s^2\le 10R^2-6Rr-r^2;(7)$$

$$s^2\le 4R^2+4Rr+3r^2(Gerretsen);(8)$$

From (7),(8) we must to show:

$$4R^2+4Rr+3r^2\le 10R^2-6Rr-r^2\Leftrightarrow 10Rr+4r^2\le 6R^2\Leftrightarrow$$

$5Rr+2r^2\le 3R^2$, true because $5Rr+2r^2\le\frac{5R^2}{2}+\frac{R^2}{2}=3R^2$.

2183. In ΔABC the following relationship holds:

$$\sum_{cyc} tan^3\frac{A}{2}tan\frac{B}{2} \ge \frac{r(4R+r)}{s^2} \ge \frac{2r}{3R}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc} x^3y\right)\left(\sum_{cyc} xyz^2\right) \overset{BCS}{\ge} \left(\sum_{cyc} x^2yz\right)^2, \forall x,y,z \ge 0$$

$$\sum_{cyc} x^3y \ge xyz\sum_{cyc} x \to \sum_{cyc} tan^3\frac{A}{2}tan\frac{B}{2} \ge \prod_{cyc} tan\frac{A}{2}\left(\sum_{cyc} tan\frac{A}{2}\right)$$

$$\sum_{cyc} tan\frac{A}{2} = \sum_{cyc}\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = r\sum_{cyc}\frac{1}{s-a} = \frac{4R+r}{s}$$

$$\prod_{cyc} tan\frac{A}{2} = \prod_{cyc}\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{r}{s}$$

$$\sum_{cyc} tan^3\frac{A}{2}tan\frac{B}{2} \ge \frac{r(4R+r)}{s^2} \overset{Doucet}{\ge} \frac{3r(4R+r)}{(4R+r)^2} = \frac{3r}{4R+r} \overset{Euler}{\ge} \frac{3r}{4R+\frac{R}{2}}$$

Therefore,

$$\sum_{cyc} tan^3\frac{A}{2}tan\frac{B}{2} \ge \frac{r(4R+r)}{s^2} \ge \frac{2r}{3R}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

❖ $$\sum\left(\tan^3\frac{A}{2}\tan\frac{B}{2}\right) = \sum\frac{\tan^3\frac{A}{2}}{\frac{1}{\tan\frac{B}{2}}} \overset{Holder}{\ge} \frac{\left(\sum\tan\frac{A}{2}\right)^3}{3\sum\frac{1}{\tan\frac{A}{2}}} = \frac{\left(\frac{4R+r}{p}\right)^3}{3.\frac{p}{r}} = \frac{r(4R+r)^3}{3p^4} \overset{(1)}{\ge} \frac{r(4R+r)}{p^2};$$

$$(1) \leftrightarrow (4R+r)^2 \ge 3p^2 \leftrightarrow (r_a+r_b+r_c)^2 \ge 3(r_ar_b+r_br_c+r_cr_a);$$

$$\leftrightarrow r_a^2+r_b^2+r_c^2 \ge r_ar_b+r_br_c+r_cr_a; (\therefore \text{true})$$

$\to (1)$ is true.

❖ $\frac{r(4R+r)}{p^2} \overset{(2)}{\geq} \frac{2r}{3R}$;

$\leftrightarrow 3R(4R+r) \geq 2p^2 \leftrightarrow 12R^2 + 3Rr \geq 2p^2$; But:

$$p^2 \leq 4R^2 + 4Rr + 3r^2 \leftrightarrow 2p^2 \leq 8R^2 + 8Rr + 6r^2 \overset{(3)}{\leq} 12R^2 + 3Rr;$$

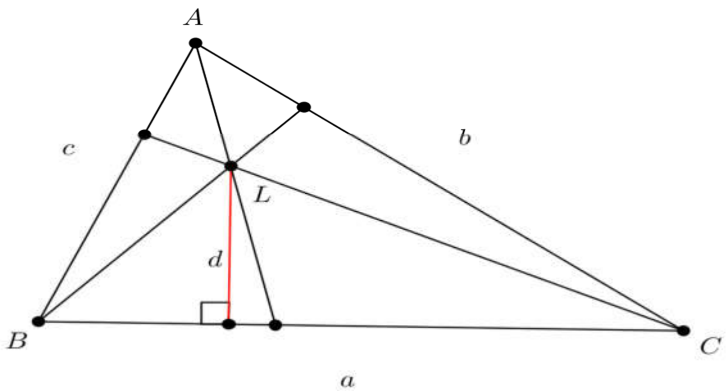
$(3) \leftrightarrow 4R^2 - 5Rr + 6r^2 \geq 0 \leftrightarrow (R-2r)(4R-3r) \geq 0$;

Which is true by: $R \geq 2r \leftrightarrow R - 2r \geq 0;\ 4R - 3r \geq 5r > 0$.

$\rightarrow (3) \rightarrow (2)$ is true.

2184. L −Lemoine's point of ΔABC; d −distance of L from BC.

Prove that: $a \geq 2\sqrt{3} \cdot d$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$BC = Bx, BA = By, B(0,0), C(a,0), A(0,c), L(l_1, l_2)$$

L −is the barycentre of $(A, a^2), (B, b^2), (C, c^2) \rightarrow$

$$l_2 = \frac{a^2c}{a^2+b^2+c^2} \rightarrow d = l_2 \sin B = \frac{1}{2R} \cdot \frac{a^2bc}{a^2+b^2+c^2} \rightarrow$$

$$\frac{a}{d} = 2R \cdot \frac{a^2+b^2+c^2}{abc} \overset{AM-GM}{\geq} \frac{3\sqrt[3]{(abc)^2}}{2sr} = \frac{3\sqrt[3]{(4Rrs)^2}}{2sr}$$

$$\because R \geq 2r(Euler), R \geq \frac{2\sqrt{3}s}{9}(Mitrinovic)$$

$$\frac{a}{d} \geq \frac{3\sqrt[3]{16s^2r^2 \cdot 2r \cdot \frac{2\sqrt{3}}{9}s}}{2sr} = 2\sqrt{3}$$

Solution 2 by proposer

$$BC = Bx, BA = By, B(0,0), L(l_1, l_2), l_2 = \frac{a^2c}{a^2+b^2+c^2}$$

$$d = l_2 \cdot sinB = \frac{a \cdot a \cdot c \cdot sinB}{a^2+b^2+c^2} = \frac{2aF}{a^2+b^2+c^2} \to$$

$$a^2+b^2+c^2 = \frac{2aF}{d}; (1)$$

$$a^2+b^2+c^2 \ge 4\sqrt{3}F \overset{(1)}{\to} \frac{2aF}{d} \ge 4\sqrt{3}F \to a \ge 2\sqrt{3} \cdot d$$

2185. In acute ΔABC the following relationship holds:

$$\sum_{cyc} sec^3AsecB \ge 48$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

In acute ΔABC: $secA, secB, secC > 0 \to$

$$\sum_{cyc} sec^3AsecB = secAsecBsecC \sum_{cyc} \frac{sec^2A}{secC} \overset{Bergstrom}{\ge}$$

$$\ge secAsecBsecC \cdot \frac{(secA+secB+secC)^2}{secA+secB+secC} \to$$

$$\sum_{cyc} sec^2AsecB \ge secAsecBsecC(secA+secB+secC)$$

We must show that:

$$\frac{1}{cosAcosBcosC}\left(\frac{1}{cosA}+\frac{1}{cosB}+\frac{1}{cosC}\right) \ge 48 \Leftrightarrow$$

$$\frac{1}{cosA}+\frac{1}{cosB}+\frac{1}{cosC} \ge 48cosAcosBcosC; (1)$$

But $cosAcosBcosC \le \frac{1}{8}; (2)$

From (1),(2) we must show that:

$$\frac{1}{cosA}+\frac{1}{cosB}+\frac{1}{cosC} \ge 6; (3)$$

$$\frac{1}{cosA}+\frac{1}{cosB}+\frac{1}{cosC}\overset{Cauchy}{\geq}\frac{9}{cosA+cosB+cosC};(4)$$

From (3),(4) we must show:

$$\frac{9}{cosA+cosB+cosC}\geq 6\Leftrightarrow cosA+cosB+cosC\leq\frac{3}{2}\text{ which is true.}$$

2186. In acute ΔABC the following relationship holds:

$$\sum_{cyc}cot^3AcotB\geq\frac{4}{3}\left(\frac{r^2}{R^2}+\frac{2r}{R}-1\right)$$

Proposed by Marin Chirciu-Romania

Solution by Mohammed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc}x^3y\right)\left(\sum_{cyc}xyz^2\right)\overset{BCS}{\geq}\left(\sum_{cyc}x^2yz\right),\forall x,y,z>0$$

$$\Rightarrow\sum_{cyc}x^3y\geq xyz\left(\sum_{cyc}x\right),\forall x,y,z>0$$

ABC is acute triangle, then $cotA,cotB,cotC>0$

$$\sum_{cyc}cot^3AcotB\geq\left(\prod_{cyc}cotA\right)\left(\sum_{cyc}cotA\right)$$

$$\sum_{cyc}cotA\overset{Jensen}{\geq}3cot\frac{\pi}{3}=\sqrt{3}$$

$$\prod_{cyc}cotA=\frac{s^2-(2R+r)^2}{4R^2}\cdot\frac{2R^2}{sr}=\frac{s^2-(2R+r)^2}{2sr}\overset{Walker,Euler\ Mitrinovic}{\geq}$$

$$\geq\frac{2R^2+8Rr+3r^2-(2R+r)^2}{2\cdot\frac{3\sqrt{3}R}{2}\cdot\frac{R}{2}}=\frac{4}{3\sqrt{3}}\left(\frac{r^2}{R^2}+\frac{2r}{R}-1\right)$$

Therefore,

$$\sum_{cyc}cot^3AcotB\geq\frac{4}{3}\left(\frac{r^2}{R^2}+\frac{2r}{R}-1\right)$$

2187. In acute ΔABC the following relationship holds:

$$\sum_{cyc} \cos^3 A \cos B \geq \frac{1}{2}\left(\frac{r^3}{R^3}+\frac{3r^2}{R^2}+\frac{r}{R}-1\right)$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc} x^3 y\right)\left(\sum_{cyc} xyz^2\right) \overset{BCS}{\geq} \left(\sum_{cyc} x^2 yz\right), \forall x,y,z>0$$

$$\Rightarrow \sum_{cyc} x^3 y \geq xyz\left(\sum_{cyc} x\right), \forall x,y,z>0$$

ABC is acute triangle, then $\cos A, \cos B, \cos C > 0$

$$\sum_{cyc} \cos^3 A \cos B \geq \left(\prod_{cyc} \cos A\right)\left(\sum_{cyc} \cos A\right)$$

$$\sum_{cyc} \cos A = 1+\frac{r}{R}, \prod_{cyc} \cos A = \frac{s^2-(2R+r)^2}{4R^2}$$

$$\prod_{cyc} \cos A \overset{Walker}{\geq} \frac{2R^2+8Rr+3r^2-(2R+r)^2}{4R^2} = \frac{-R^2+2Rr+r^2}{2R^2}$$

$$\sum_{cyc} \cos^3 A \cos B \geq \frac{1}{2}\left(-1+\frac{2r}{R}+\frac{r^2}{R^2}\right)\left(1+\frac{r}{R}\right) = \frac{1}{2}\left(\frac{r^3}{R^3}+\frac{3r^2}{R^2}+\frac{r}{R}-1\right)$$

$$\sum_{cyc} \cos^3 A \cos B \geq \frac{1}{2}\left(\frac{r^3}{R^3}+\frac{3r^2}{R^2}+\frac{r}{R}-1\right)$$

2188. In ΔABC the following relationship holds:

$$\frac{1}{R} \leq \frac{1}{h_a+h_b}+\frac{1}{h_b+h_c}+\frac{1}{h_c+h_a} \leq \frac{1}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$\sum_{cyc}\frac{1}{h_a+h_b}=\sum_{cyc}\frac{1}{\frac{2F}{a}+\frac{2F}{b}}=\frac{1}{2sr}\sum_{cyc}\frac{ab}{a+b}\to$$

For the right side we have:

$$\frac{1}{2sr}\sum_{cyc}\frac{ab}{a+b}\le\frac{1}{2r}\Leftrightarrow\sum_{cyc}\frac{ab}{a+b}\le s;(1)$$

But $\frac{2ab}{a+b}\le\frac{a+b}{2}\to\frac{ab}{a+b}\le\frac{a+b}{4}\to$

$$\sum_{cyc}\frac{ab}{a+b}\le\frac{1}{4}\sum_{cyc}(a+b)=\frac{a+b+c}{2}=s\to(1)is\ true.$$

For the left side: $\frac{1}{h_a+h_b}+\frac{1}{h_b+h_c}+\frac{1}{h_c+h_a}\overset{Cauchy}{\ge}\frac{9}{2(h_a+h_b+h_c)}$

We must show that: $\frac{9}{2(h_a+h_b+h_c)}\ge\frac{1}{R};(2)$

But $h_a+h_b+h_c=\frac{s^2+r^2+4Rr}{2R};(3)$**. From (2),(3) we must show that:**

$$\frac{9R}{s^2+r^2+4Rr}\ge\frac{1}{R}\Leftrightarrow 9R^2\ge s^2+r^2+4Rr;(4)$$

From Gerretsen: $s^2\le 4R^2+4Rr+3r^2;(5)$

From (4),(5) we must show:

$$9R^2\ge 4R^2+8Rr+4r^2\Leftrightarrow 9R^2\ge 4(R+r)^2\Leftrightarrow 3R\ge 2(R+r)\Leftrightarrow R\ge 2r(Euler).$$

2189. In acute ΔABC the following relationship holds:

$$\sum_{cyc}tan^3A tanB\ge\frac{s^2}{r^2}\ge\frac{16R}{r}-5$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc}x^3y\right)\left(\sum_{cyc}xyz^2\right)\overset{BCS}{\ge}\left(\sum_{cyc}x^2yz\right),\forall x,y,z>0$$

$$\Rightarrow \sum_{cyc} x^3y \geq xyz\left(\sum_{cyc} x\right), \forall x,y,z > 0$$

ABC is acute triangle, then $tanA, tanB, tanC > 0$

$$\sum_{cyc} tan^3 A tanB \geq \left(\prod_{cyc} tanA\right)\left(\sum_{cyc} tanA\right) = \left(\prod_{cyc} tanA\right)^2 =$$

$$= \left(\frac{sr}{2R^2} \cdot \frac{4R^2}{s^2 - (2R+r)^2}\right)^2 \overset{Gerretsen}{\geq} \left(\frac{2sr}{4R^2 + 4Rr + 3r^2 - (2R+r)^2}\right)^2 = \frac{s^2}{r^2}$$

Therefore,

$$\sum_{cyc} tan^3 A tanB \geq \frac{s^2}{r^2} \geq \frac{16R}{r} - 5$$

2190. In ΔABC the following relationship holds:

$$(h_a + h_b + h_c)\left(\frac{1}{h_a + h_b} + \frac{1}{h_b + h_c} + \frac{1}{h_c + h_a}\right) \geq \frac{2r}{R}\left(5 - \frac{r}{R}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} \frac{1}{h_a + h_b}\right) = \frac{1}{2}\left(\sum_{cyc}(h_a + h_b)\right)\left(\sum_{cyc} 1/(h_a + h_b)\right) \overset{BCS}{\geq} \frac{9}{2} \overset{(1)}{\geq} \frac{2r}{R}\left(5 - \frac{r}{R}\right)$$

$\Leftrightarrow 9R^2 \geq 4r(5R - r) \Leftrightarrow (R - 2r)(9R - 2r) \geq 0$, which is true because $R \geq 2r (Euler)$.

2191. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{h_b + h_c} \geq \sum_{cyc} \frac{1}{r_b + r_c}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{1}{r_b+r_c}=\frac{\sum(r_a+r_b)(r_a+r_c)}{\prod(r_a+r_b)}=\frac{(\sum r_a)^2+\sum r_ar_b}{\prod(r_a+r_b)}$$

$$\sum_{cyc}\frac{1}{r_b+r_c}=\frac{(4R+r)^2+s^2}{4Rs^2}$$

$$\sum_{cyc}\frac{1}{h_b+h_c}=\frac{1}{2sr}\sum_{cyc}\frac{bc}{b+c}=2R\sum_{cyc}\frac{1}{a(b+c)}=\frac{R}{s}\sum_{cyc}\left(\frac{1}{a}+\frac{1}{b+c}\right)=$$

$$=\frac{R}{s}\left(\sum_{cyc}\frac{1}{a}+\sum_{cyc}\frac{1}{b+c}\right)\overset{BCS}{\geq}\frac{R}{s}\left(\frac{9}{\sum a}+\frac{9}{\sum(b+c)}\right)\Rightarrow$$

$$\sum_{cyc}\frac{1}{h_b+h_c}\geq\frac{27R}{4s^2}\overset{(1)}{\geq}\frac{(4R+r)^2+s^2}{4Rs^2}=\sum_{cyc}\frac{1}{r_b+r_c}$$

$$(1)\Leftrightarrow s^2\leq 11R^2-8Rr-r^2$$

By Gerretsen: $s^2\leq 4R^2+4Rr+3r^2\leq 11R^2-8Rr-r^2\Leftrightarrow$

$7R^2-12Rr-4r^2\geq 0\Leftrightarrow(R-2r)(7R+2r)\geq 0$, which is true from $R\geq 2r(Euler)$.

2192. In ΔABC the following relationship holds:

$$\frac{4R}{r}\leq\frac{(h_a+r_a)(h_b+r_b)(h_c+r_c)}{h_ah_bh_c}\leq\frac{2R^2}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$h_a+r_a=\frac{2F}{a}+\frac{F}{s-a}=\frac{F(2s-2a+a)}{a(s-a)}=\frac{F(b+c)}{a(s-a)}$$

$$\frac{(h_a+r_a)(h_b+r_b)(h_c+r_c)}{h_ah_bh_c}=\frac{F^3(a+b)(b+c)(c+a)}{8(s-a)(s-b)(s-c)}=\frac{2s(s^2+r^2+2Rr)}{8sr^2}=$$

$$=\frac{s^2+r^2+2Rr}{4r^2}$$

We must show that: $\frac{4R}{r}\leq\frac{s^2+r^2+2Rr}{4r^2}\leq\frac{2R^2}{r^2}$

For LHS, we have: $s^2+r^2+2Rr\geq 16Rr\Leftrightarrow s^2\geq 14Rr-r^2;(1)$

By Gerretsen: $s^2 \geq 16Rr - 5r^2; (2)$

From (1),(2) we must show that: $16Rr - 5r^2 \geq 14Rr - r^2 \Leftrightarrow$

$$2Rr \geq 4r^2 \Leftrightarrow R \geq 2r(Euler).$$

For RHS, we have: $s^2 + r^2 + 2Rr \leq 8R^2 \Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2; (3)$

By Gerretsen: $s^2 \leq 4R^2 + 4Rr + 3r^2; (4)$

From (3),(4) we must show that: $4R^2 + 4Rr + 3r^2 \leq 8R^2 - 2Rr - r^2 \Leftrightarrow$

$6Rr + 4r^2 \leq 4R^2$, which is true because $6Rr + 4r^2 \leq 3R^2 + R^2 = 4R^2$.

2193. In ΔABC the following relationship holds:

$$8 \leq \frac{(h_a + r_a)(h_b + r_b)(h_c + r_c)}{r_a r_b r_c} \leq \frac{4R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$h_a + r_a = \frac{2F}{a} + \frac{F}{s-a} = \frac{F(2s - 2a + a)}{a(s-a)} = \frac{F(b+c)}{a(s-a)}$$

$$\frac{(h_a + r_a)(h_b + r_b)(h_c + r_c)}{r_a r_b r_c} = \frac{F^3(a+b)(b+c)(c+a)}{abc(s-a)(s-b)(s-c) \cdot \frac{F^3}{(s-a)(s-b)(s-c)}} =$$

$$= \frac{(a+b)(b+c)(c+a)}{abc} = \frac{2s(s^2 + r^2 + 2Rr)}{4sRr} = \frac{s^2 + r^2 + 2Rr}{2Rr}$$

We must show that: $8 \leq \frac{s^2+r^2+2Rr}{2Rr} \leq \frac{4R}{r}$

For LHS, we have: $s^2 + r^2 + 2Rr \geq 16Rr \Leftrightarrow s^2 \geq 14Rr - r^2; (1)$

By Gerretsen: $s^2 \geq 16Rr - 5r^2; (2)$

From (1),(2) we must show that: $16Rr - 5r^2 \geq 14Rr - r^2 \Leftrightarrow$

$$2Rr \geq 4r^2 \Leftrightarrow R \geq 2r(Euler).$$

For RHS, we have: $s^2 + r^2 + 2Rr \leq 8R^2 \Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2; (3)$

By Gerretsen: $s^2 \leq 4R^2 + 4Rr + 3r^2; (4)$

From (3),(4) we must show that: $4R^2 + 4Rr + 3r^2 \leq 8R^2 - 2Rr - r^2 \Leftrightarrow$

$6Rr + 4r^2 \leq 4R^2$, which is true because $6Rr + 4r^2 \leq 3R^2 + R^2 = 4R^2$.

2194. In ΔABC the following relationship holds:

$$\sum_{cyc} csc^3 A csc B \geq \frac{16}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Sohini Mondal-India

$$\sum_{cyc} csc^3 A csc B = \sum_{cyc} \frac{1}{sin^3 A sin B} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{1}{(sinAsinBsinC)^4}} \geq$$

$$\geq 3\sqrt[3]{\frac{1}{\left(\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}\right)^4}} = \frac{3\cdot 2^4\cdot R^4}{\sqrt[3]{(abc)^4}}$$

We need to show that: $\frac{3\cdot 2^4\cdot R^4}{\sqrt[3]{(abc)^4}} \geq \frac{16}{3} \Leftrightarrow 3^2 R^4 \geq \sqrt[3]{(abc)^4} \Leftrightarrow 3^6 R^{12} \geq (4FR)^4 \Leftrightarrow$

$3\sqrt{3}R^2 \geq 4rs \Leftrightarrow R^2 \geq 2r\cdot\frac{2}{3\sqrt{3}}s$ true.

$R \geq 2r(Euler)$ and $R \geq \frac{2}{3\sqrt{3}}s(Mitrinovic)$

Therefore,

$$\sum_{cyc} csc^3 A csc B \geq \frac{16}{3}$$

2195. In ΔABC the following relationship holds:

$$\frac{9r}{4R^2} \leq \sum_{cyc} \frac{r_a}{bc}\cdot cos^2\frac{A}{2} \leq \frac{9}{16r}$$

Proposed by Ertan Yildirim-Turkey

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc} \frac{r_a}{bc}\cdot cos^2\frac{A}{2} = \sum_{cyc} \frac{sr}{bc(s-a)}\cdot\frac{s(s-a)}{bc} = \frac{s^2 r}{16R^2 r^2 s^2}\sum_{cyc} a^2 \rightarrow$$

$$\sum_{cyc} \frac{r_a}{bc}\cdot cos^2\frac{A}{2} = \frac{\sum a^2}{16R^2 r} = \frac{s^2 - r^2 - 4Rr}{8R^2 r}$$

By Leibniz: $\sum_{cyc} \frac{r_a}{bc} \cdot \cos^2 \frac{A}{2} \le \frac{9R^2}{16R^2 r} = \frac{9}{16r}$

By Gerretsen: $\sum_{cyc} \frac{r_a}{bc} \cdot \cos^2 \frac{A}{2} \ge \frac{(16Rr - 5r^2) - r^2 - 4Rr}{8R^2 r}$

$$\sum_{cyc} \frac{r_a}{bc} \cdot \cos^2 \frac{A}{2} \ge \frac{6R - 3r}{4R^2} \overset{Euler}{\ge} \frac{12r - 3r}{4R^2} = \frac{9r}{4R^2}$$

Solution 2 by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{r_a}{bc} \cdot \cos^2 \frac{A}{2} = \sum_{cyc} \frac{F}{(s-a)bc} \cdot \frac{s(s-a)}{bc} = s^2 r \sum_{cyc} \frac{1}{b^2 c^2} = \frac{s^2 r}{a^2 b^2 c^2} \sum_{cyc} a^2 =$$

$$= \frac{1}{16R^2 r} \sum_{cyc} a^2 \overset{Leibniz}{\le} \frac{1}{16R^2 r} \cdot 9R^2 = \frac{9}{16r} = RHS$$

$$\sum_{cyc} \frac{r_a}{bc} \cdot \cos^2 \frac{A}{2} \ge 3 \sqrt[3]{\frac{r_a r_b r_c}{a^2 b^2 c^2} \cdot \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}; (1)$$

$$r_a r_b r_c = s^2 r, abc = 4Rrs, \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} = \frac{s^2}{16R^2}; (2)$$

From (1),(2) we get:

$$\sum_{cyc} \frac{r_a}{bc} \cdot \cos^2 \frac{A}{2} \ge 3 \sqrt[3]{\frac{s^2 r \cdot s^2}{16R^2 r^2 s^2 \cdot 16R^2}}$$

We must show that:

$$\sqrt[3]{\frac{s^2}{4^4 R^4 r}} \ge \frac{9r}{4R^2} \Leftrightarrow \frac{s^2}{4^4 R^4 r} \ge \frac{3^3 r^3}{4^3 R^6} \Leftrightarrow s^2 R^2 \ge 4 \cdot 2 + r^4 \text{ true because}$$

$s^2 \ge 27r^2$ **and** $R^2 \ge 4r^2$ **.**

2196. In ΔABC the following relationship holds:

$$\frac{16r}{R} \le \frac{(h_a + r_a)(h_b + r_b)(h_c + r_c)}{s_a s_b s_c} \le \frac{R^3}{r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$(h_a+r_a)(h_b+r_b)(h_c+r_c)=\frac{F^3(a+b)(b+c)(c+a)}{abc(s-a)(s-b)(s-c)}=\frac{s^3r^3\cdot 2s(s^2+r^2+2Rr)}{4Rrs\cdot sr^2}=$$

$=\frac{s^2(s^2+r^2+2Rr)}{2R}$, we must show that:

$$\frac{16r}{R}\le\frac{s^2(s^2+r^2+2Rr)}{2Rs_as_bs_c}\le\frac{R^3}{r^3}$$

For LHS: $s_as_bs_c\le\le m_am_bm_c\le\frac{Rs^2}{2}$, we must show that:

$$\frac{16r}{R}\le\frac{s^2(s^2+r^2+2Rr)}{2R\cdot\frac{Rs^2}{2}}\Leftrightarrow s^2+r^2+2Rr\ge 16Rr\Leftrightarrow s^2\ge 14Rr-r^2;(1)$$

From Gerretsen: $s^2\ge 16Rr-5r^2;(2)$

From (1),(2) we must show that: $16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow$

$$2Rr\ge 4r^2\Leftrightarrow R\ge 2r(Euler).$$

For RHS: $s_as_bs_c\ge h_ah_bh_c=\frac{8F^3}{abc}=\frac{8s^3r^3}{4Rrs}=\frac{2s^2r^2}{R}$. We must show that:

$$\frac{s^2+r^2+2Rr}{4r^2}\le\frac{R^3}{r^3}\Leftrightarrow r(s^2+r^2+2Rr)\le 4R^3;(R\ge 2r)\Leftrightarrow$$

$$s^2+r^2+2Rr\le 8R^2\Leftrightarrow s^2\le 8R^2-2Rr-r^2;(3)$$

From Gerretsen: $s^2\le 4R^2+4Rr+3r^3;(4)$

From (3),(4) we must to show: $4R^2+4Rr+3r^2\le 8R^2-2Rr-r^2\Leftrightarrow$

$6Rr+4r^2\le 4R^2$ true.

2197. In ΔABC the following relationship holds:

$$8\le\frac{(h_a+r_a)(h_b+r_b)(h_c+r_c)}{w_aw_bw_c}\le\frac{2R^2}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$h_a+\frac{r_a2F}{a}+\frac{F}{s-a}=F\left(\frac{2s-2a+a}{a(s-a)}\right)=\frac{F(b+c)}{a(s-a)}$$

$$(h_a+r_a)(h_b+r_b)(h_c+r_c)=\frac{F^3(a+b)(b+c)(c+a)}{abc(s-a)(s-b)(s-c)}=\frac{s^3r^3\cdot 2s(s^2+r^2+2Rr)}{4Rrs\cdot sr^2}=$$

$=\frac{s^2(s^2+r^2+2Rr)}{2R}$, we must show that:

$$8\le\frac{s^2(s^2+r^2+2Rr)}{2Rw_aw_bw_c}\le\frac{2R^2}{r^2}$$

For LHS: $w_a\le\sqrt{s(s-a)}\to w_aw_bw_c\le sF=s^2r$, we must show that:

$$8\le\frac{s^2(s^2+r^2+2Rr)}{2R\cdot s^2r}\Leftrightarrow s^2+r^2+2Rr\ge 16Rr\Leftrightarrow s^2\ge 14Rr-r^2;(1)$$

From Gerretsen: $s^2\ge 16Rr-5r^2;(2)$

From (1),(2) we must show that: $16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow$

$$2Rr\ge 4r^2\Leftrightarrow R\ge 2r(Euler).$$

For RHS: $w_aw_bw_c\ge h_ah_bh_c=\frac{8F^3}{abc}=\frac{8s^3r^3}{4Rrs}=\frac{2s^2r^2}{R}$. We must show that:

$$\frac{s^2+r^2+2Rr}{4r^2}\le\frac{2R^2}{r^2}\Leftrightarrow s^2+r^2+2Rr\le 8R^2\Leftrightarrow s^2\le 8R^2-2Rr-r^2;(3)$$

From Gerretsen: $s^2\le 4R^2+4Rr+3r^3;(4)$

From (3),(4) we must to show: $4R^2+4Rr+3r^2\le 8R^2-2Rr-r^2\Leftrightarrow$

$$6Rr+4r^2\le 4R^2\Leftrightarrow R\ge 2r(Euler).$$

2198. In ΔABC the following relationship holds:

$$\sqrt{\frac{a^4+b^4}{2}}+\sqrt{\frac{b^4+c^4}{2}}+\sqrt{\frac{c^4+a^4}{3}}\le 4(3R^2-2Rr+r^2)$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma: $\sqrt{\frac{x^4+y^4}{2}}\le x^2-xy+y^2,\forall x,y>0;(1)$

Proof. $(1)\Leftrightarrow 2(x^2-xy+y^2)^2\ge x^4+y^4\Leftrightarrow(x-y)^4\ge 0$ is true.

$$\sum_{cyc}\sqrt{\frac{a^4+b^4}{2}}\overset{(1)}{\le}\sum_{cyc}(a^2-ab+b^2)=2\sum_{cyc}a^2-\sum_{cyc}ab=$$

$$=4s^2-4r^2-16Rr-s^2-r^2-4Rr=3s^2-20Rr-5r^2\overset{Gerresen}{\le}$$

$$\le 3(4R^2+4Rr+3r^2)-20Rr-5r^2=4(3R^2-2Rr+r^2)$$

Therefore, $\sqrt{\frac{a^4+b^4}{2}}+\sqrt{\frac{b^4+c^4}{2}}+\sqrt{\frac{c^4+a^4}{3}}\le 4(3R^2-2Rr+r^2)$

2199. In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a^2+r_b^2}{2}}+\sqrt{\frac{r_b^2+r_c^2}{2}}+\sqrt{\frac{r_c^2+r_a^2}{2}}\le 8R-7r$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\sqrt{\frac{r_a^2+r_b^2}{2}}\overset{BCS}{\le}\sqrt{3\sum_{cyc}\frac{r_a^2+r_b^2}{2}}=\sqrt{3\sum_{cyc}r_a^2}=\sqrt{3}\cdot\sqrt{(4R+r)^2-2s^2}\overset{Gerretsen}{\le}$$

$$\le\sqrt{3}\cdot\sqrt{(4R+r)^2-2(16Rr-5r^2)}=\sqrt{3}\cdot\sqrt{16R^2-24Rr+11r^2}\overset{(1)}{\le}8R-7r\Leftrightarrow$$

$$3(16R^2-24Rr+11r^2)\le 64R^2-112Rr+49r^2\Leftrightarrow$$

$16R^2-40Rr+16r^2\ge 0\Leftrightarrow 8(R-2r)(2R-r)\ge 0$, which is true from

$R\ge 2r(Euler)$. Therefore,

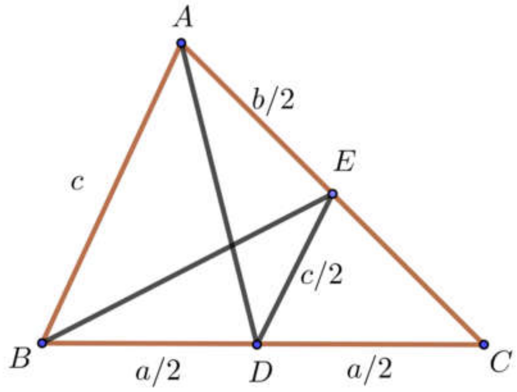
$$\sqrt{\frac{r_a^2+r_b^2}{2}}+\sqrt{\frac{r_b^2+r_c^2}{2}}+\sqrt{\frac{r_c^2+r_a^2}{2}}\le 8R-7r$$

2200. In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\ge\frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}}>1+\frac{2}{s}$$

Proposed by Florică Anastase-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco



Ptolemy's inequality in $ABDE$: $c\cdot\frac{c}{2}+\frac{a}{2}\cdot\frac{b}{2}\ge m_am_b\Rightarrow$

$$m_am_b\overset{(1)}{\le}\frac{1}{4}(2c^2+ab)(and\ analogs)$$

$$\frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}}=1+\frac{2\sum sin\frac{A}{2}}{\sum asin\frac{A}{2}}$$

$a\ge b\ge c\Rightarrow sin\frac{A}{2}\ge sin\frac{B}{2}\ge sin\frac{C}{2}$. **By Chebyshev's inequality:**

$$\sum_{cyc}a\cdot sin\frac{A}{2}\ge\frac{1}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}sin\frac{A}{2}\right)\Rightarrow\frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}}\le 1+\frac{3}{s}$$

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)}\overset{AM-GM}{\ge}\sqrt[3]{\prod_{cyc}\left(1+\frac{2}{\sqrt{m_am_b}}\right)}\overset{(1)}{\ge}\sqrt[3]{\prod_{cyc}\left(1+\frac{4}{\sqrt{2c^2+ab}}\right)}\overset{Holder}{\ge}$$

$$\ge 1+\sqrt[3]{\prod_{cyc}\frac{4}{\sqrt{2c^2+ab}}}=1+\frac{4}{\sqrt[6]{\prod_{cyc}(2c^2+ab)}}\overset{(*)}{\ge}1+\frac{3}{s}$$

$$(*)\Leftrightarrow 16s^2\ge 9\sqrt[3]{\prod_{cyc}(2c^2+ab)}$$

$$3\sqrt[3]{\prod_{cyc}(2c^2+ab)}\overset{AM-GM}{\le}\sum_{cyc}(2c^2+ab)=2\sum_{cyc}a^2+\sum_{cyc}ab$$

$$\Rightarrow 9\sqrt[3]{\prod_{cyc}(2c^2+ab)}\le 6\sum_{cyc}a^2+3\sum_{cyc}ab=15s^2-9r^2-36Rr\Rightarrow$$

$$16s^2 \geq 9\sqrt[3]{\prod_{cyc}(2c^2+ab)}$$

$$\Rightarrow \sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)} \geq \frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}}$$

$$\frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}} = 1+\frac{2\sum sin\frac{A}{2}}{\sum a\cdot sin\frac{A}{2}} \overset{a<s}{>} 1+\frac{2\sum sin\frac{A}{2}}{s\cdot\sum sin\frac{A}{2}} \Rightarrow$$

$$\frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}} > 1+\frac{2}{s}$$

Therefore,

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)} \geq \frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}} > 1+\frac{2}{s}$$

Solution 2 by proposer

$$\because \sum_{i=1}^{2}(a_i+b_i)\sum_{i=1}^{2}\frac{a_ib_i}{a_i+b_i} \leq \left(\sum_{i=1}^{2}a_i\right)\left(\sum_{i=1}^{2}b_i\right) \ (Milne's\ ineq.n=2)$$

$$\sum_{cyc}(m_a+m_b)\sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right) \leq \left(\sum_{cyc}m_a\right)\left(\sum_{cyc}m_b\right) \Rightarrow$$

$$\sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right) \geq \frac{(\sum m_a)^2}{2\sum m_a} \Rightarrow \sum_{cyc}\left(\frac{m_am_b}{m_a+m_b}\right) \leq \frac{1}{2}\sum_{cyc}m_a \leq \frac{1}{2}\sum_{cyc}\frac{b+c}{2} = \frac{2s}{2} = s$$

$$\because m_a \leq \frac{b+c}{2}, m_b \leq \frac{c+a}{2}, m_c \leq \frac{a+b}{2}$$

$$\sum_{cyc}\frac{1}{x+\frac{m_am_b}{m_a+m_b}} \overset{BCS}{\geq} \frac{9}{3x+\sum\frac{m_am_b}{m_a+m_b}} \geq \frac{9}{3x+s} = \frac{3}{x+\frac{s}{3}} \Rightarrow$$

$$\int_0^1\sum_{cyc}\frac{1}{x+\frac{m_am_b}{m_a+m_b}}dx \geq \int_0^1\frac{3}{x+\frac{s}{3}}dx \Rightarrow$$

$$\sum_{cyc} log\left(1+\frac{m_a+m_b}{m_am_b}\right) \ge 3log\left(1+\frac{3}{s}\right) \Rightarrow \sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)} \ge 1+\frac{3}{s}; (1)$$

Now,

$$(a+b+c)\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right) \overset{Chebyshev's}{\le} 3asin\frac{A}{2}+3bsin\frac{B}{2}+3csin\frac{C}{2}$$

$$\Leftrightarrow \sum_{cyc}(a-b)\left(sin\frac{A}{2}-sin\frac{B}{2}\right) \ge 0, (2)$$

On the other hand,

$$\sum_{cyc}(a+b-c)sin\frac{C}{2} > 0, (3)$$

From (2),(3) it follows that:

$$\frac{1}{2}\cdot\frac{3}{s} \ge \frac{sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}}{asin\frac{A}{2}+bsin\frac{B}{2}+csin\frac{C}{2}} > \frac{1}{s}; (4)$$

From (1),(4) it follows that:

$$\sqrt[3]{\prod_{cyc}\left(1+\frac{1}{m_a}+\frac{1}{m_b}\right)} \ge 1+\frac{3}{s} \ge 1+\frac{2\sum sin\frac{A}{2}}{\sum asin\frac{A}{2}} = \frac{\sum(a+2)sin\frac{A}{2}}{\sum asin\frac{A}{2}} > 1+\frac{2}{s}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru