A SIMPLE PROOF FOR MAHLER'S INEQUALITY

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If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0; n \in \mathbb{N}; n \geq 2$ then:

$$\sqrt{(x_1+y_1)(x_2+y_2)} \geq \sqrt{x_1x_2} + \sqrt{y_1y_2}$$

$$\sqrt[3]{(x_1+y_1)(x_2+y_2)(x_3+y_3)} \geq \sqrt[3]{x_1x_2x_3} + \sqrt[3]{y_1y_2y_3}$$

$$\sqrt[4]{(x_1+y_1)(x_2+y_2)(x_3+y_3)(x_4+y_4)} \geq \sqrt[4]{x_1x_2x_3x_4} + \sqrt[4]{y_1y_2y_3y_4}$$

$$\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)} \geq \sqrt[n]{x_1x_2\cdot \ldots \cdot x_n} + \sqrt[n]{y_1y_2\cdot \ldots \cdot y_n}$$

Proof.

$$\frac{x_1}{x_1+y_1} + \frac{x_2}{x_2+y_2} + \ldots + \frac{x_n}{x_n+y_n} \overset{\text{AM-GM}}{\geq} n \cdot \sqrt[n]{\frac{x_1x_2\cdot \ldots \cdot x_n}{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)}}$$

$$(1) \quad \frac{x_1}{x_1+y_1} + \frac{x_2}{x_2+y_2} + \ldots + \frac{x_n}{x_n+y_n} \geq \frac{n\sqrt[n]{x_1x_2\cdot \ldots \cdot x_n}}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)}}$$

Analogous:

$$(2) \quad \frac{y_1}{x_1+y_1} + \frac{y_2}{x_2+y_2} + \ldots + \frac{y_n}{x_n+y_n} \geq \frac{n\sqrt[n]{y_1y_2\cdot \ldots \cdot y_n}}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)}}$$

By adding $(1);(2)$:

$$\frac{x_1+y_1}{x_1+y_1} + \frac{x_2+y_2}{x_2+y_2} + \ldots + \frac{x_n+y_n}{x_n+y_n} \geq \frac{n(\sqrt[n]{x_1x_2\cdot \ldots \cdot x_n} + \sqrt[n]{y_1y_2\cdot \ldots \cdot y_n})}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)}}$$

$$\underbrace{1+1+\ldots+1}_{\text{for "}n\text{" times}} \geq \frac{n(\sqrt[n]{x_1x_2\ldots x_n} + \sqrt[n]{y_1y_2\cdot \ldots \cdot y_n})}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)}}$$

$$n \geq \frac{n(\sqrt[n]{x_1x_2\cdot \ldots \cdot x_n} + \sqrt[n]{y_1y_2\cdot \ldots \cdot y_n})}{\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)}}$$

$$\sqrt[n]{(x_1+y_1)(x_2+y_2)\cdot \ldots \cdot(x_n+y_n)} \geq \sqrt[n]{x_1x_2\cdot \ldots \cdot x_n} + \sqrt[n]{y_1y_2\cdot \ldots \cdot y_n}$$

Observations:

1. If $y_1 = y_2 = \ldots = y_n = 1$ then:

$$\sqrt[n]{(1+x_1)(1+x_2)\cdot \ldots \cdot(1+x_n)} \geq 1 + \sqrt[n]{x_1x_2\cdot \ldots \cdot x_n}$$

$$(1+x_1)(1+x_2)\cdot \ldots \cdot(1+x_n) \geq (1 + \sqrt[n]{x_1x_2\cdot \ldots \cdot x_n})^n$$

2. If $y_1 = y_2 = \ldots = y_n = a$ then:

$$\sqrt[n]{(a+x_1)(a+x_2)\cdot \ldots \cdot(a+x_n)} \geq a + \sqrt[n]{x_1x_2\cdot \ldots \cdot x_n}$$

$$(a+x_1)(a+x_2)\cdot \ldots (a+x_n) \geq (a + \sqrt[n]{x_1x_2\cdot \ldots \cdot x_n})^n$$

3. If $y_1 = \frac{1}{x_1}; y_2 = \frac{1}{x_2}; \ldots; y_n = \frac{1}{x_n}$ then:

$$\sqrt[n]{\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)\cdot \ldots \cdot\left(x_n+\frac{1}{x_n}\right)} \geq \sqrt[n]{x_1x_2\cdot \ldots \cdot x_n} + \frac{1}{\sqrt[n]{x_1x_2\cdot \ldots \cdot x_n}}$$

$$\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)\cdot\ldots\cdot\left(x_n+\frac{1}{x_n}\right)\geq\left(\sqrt[n]{x_1x_2\cdot\ldots\cdot x_n}+\frac{1}{\sqrt[n]{x_1x_2\cdot\ldots\cdot x_n}}\right)^n$$

4. Equality holds for:

$$x_1=x_2=\ldots=x_n; y_1=y_2=\ldots=y_n$$

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References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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