R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT AN INEQUALITY BY ERTAN YILDIRIM-XIV

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1) In ΔABC the following relationship holds:

$$6r \le \sum_{cyc} \frac{a^2}{r_b + r_c} \le \frac{a^3 + b^3 + c^3}{4F}$$

Proposed by Ertan Yildirim-Izmir-Turkey

Solution. **Lemma. 2) In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{a^2}{r_b + r_c} = 2(2R - r)$$

Proof. *Using $r_a = \frac{F}{s-a}$, we get:*

$$\sum_{cyc} \frac{a^2}{r_b + r_c} = \sum_{cyc} \frac{a^2}{\frac{F}{s-b} + \frac{F}{s-c}} = \frac{1}{F} \sum_{cyc} \frac{a^2(s-b)(s-c)}{a} = \frac{1}{sr} \sum_{cyc} a(s-b)(s-c) =$$

$= \frac{1}{sr} \cdot 2sr(2R - r) = 2(2R - r)$, *which follows from* $\sum a(s-b)(s-c) = 2sr(2R - r)$

For RHS, using Lemma and $\sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$ *inequality becomes as:*

$$2(2R - r) \le \frac{2s(s^2 - 3r^2 - 6Rr)}{4sr} \Leftrightarrow 4r(2R - r) \le s^2 - 3r^2 - 6Rr \Leftrightarrow s^2 \ge 14Rr - r^2$$

Which follows from $s^2 \ge 16Rr - 5r^2\,(Gerretsen)$ *and* $R \ge 2r\ (Euler)$.

Equality holds if and only if triangle is equilateral. For LHS, using Lemma, we get:

$$2(2R - r) \ge 6r \Leftrightarrow R \ge 2r\ (Euler).$$

3) In ΔABC the following relationship holds:

$$6r \le \sum_{cyc} \frac{a^2}{h_b + h_c} \le 2(2R - r)$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 4) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{a^2}{h_b+h_c}=\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}$$

Proof. *Using* $h_a=\frac{2F}{a}$, *we get:*

$$\sum_{cyc}\frac{a^2}{h_b+h_c}=\sum_{cyc}\frac{a^2}{\frac{2F}{b}+\frac{2F}{c}}=\frac{1}{2F}\sum_{cyc}\frac{a^2bc}{b+c}=\frac{abc}{2F}\sum_{cyc}\frac{a}{b+c}=\frac{4RF}{2F}\cdot\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}=$$

$=\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}$, *which follows from* $\sum\frac{a}{b+c}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}$. *Let's get back to the main problem.*

For RHS, using Lemma and $\sum a^3=2s(s^2-3r^2-6Rr)$

$$\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}\le 2(2R-r)\Leftrightarrow 2R(s^2-r^2-Rr)\le(2R-r)(s^2+r^2+2Rr)$$

$\Leftrightarrow s^2\le 6R^2+2Rr-r^2$, *which follows from* $s^2\le 4R^2+4Rr+3r^2$ $(Gerretsen)$ *and* $R\ge 2r$ $(Euler)$. *Remains to prove that:* $4R^2+4Rr+3r^2\le 6R^2+2Rr-r^2\Leftrightarrow$ $R^2-Rr-2r^2\ge 0\Leftrightarrow(R-2r)(R+r)\ge 0$. *Equality holds if and only if triangle is equilateral. For LHS. Using Lemma, inequality becomes as:*

$$\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}\ge 6r\Leftrightarrow 2R(s^2-r^2-Rr)\ge 3r(s^2+r^2+2Rr)\Leftrightarrow$$

$s^2(22R-3r)\ge r(2R^2+8Rr+3r^2)$, *which follows from* $s^2\ge 16Rr-5r^2$ $(Gerretsen)$ *and* $R\ge 2r$ $(Euler)$. *Remains to prove that:*

$$(16Rr-5r^2)(2R-3r)\ge r(2R^2+8Rr+3r^2)\Leftrightarrow$$

$$(16R-5r)(2R-3r)\ge 2R^2+8Rr+3r^2\Leftrightarrow$$

$$32R^2-48Rr-10Rr+15r^2\ge 2R^2+8Rr+3r^2\Leftrightarrow$$

$$30R^2-6Rr+12r^2\ge 0\Leftrightarrow 5R^2-11Rr+2r^2\ge 0\Leftrightarrow(R-2r)(5R-r)\ge 0$$

5) In ΔABC the following relationship hods:

$$\sum_{cyc}\frac{a^2}{h_b+h_c}\le\sum_{cyc}\frac{a^2}{r_b+r_c}$$

Proposed by Marin Chirciu-Romania

Solution. *Using Lemma's we have:*

$$\sum_{cyc}\frac{a^2}{h_b+h_c}=\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr};\ \sum_{cyc}\frac{a^2}{r_b+r_c}=2(2R-r)$$

Inequality, becomes as: $\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}\le 2(2R-r)\Leftrightarrow$

$2R(s^2-r^2-Rr)\le(2R-r)(s^2+r^2+2Rr)\Leftrightarrow s^2\le 6R^2+2Rr-r^2$, *which follows from* $s^2\le 4R^2+4Rr+3r^2$ $(Gerretsen)$ *and* $R\ge 2r$ $(Euler)$. *Remains to prove that:*

$$4R^2+4Rr+3r^2\le 6R^2+2Rr-r^2\Leftrightarrow R^2-Rr-2r^2\ge 0\Leftrightarrow(R-2r)(R+r)\ge 0$$

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$6r\le\sum_{cyc}\frac{a^2}{h_b+h_c}\le\sum_{cyc}\frac{a^2}{r_b+r_c}\le\frac{a^3+b^3+c^3}{4F}$$

Solution. *See up these inequalities. Equality holds if and only if triangle is equilateral.*

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-INETRACTIVE JOURNAL-www.ssmrmh.ro