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ABOUT AN INEQUALITY BY KOSTAS GERONIKOLAS-IV

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1) If $a,b,c>0$ such that $abc=1$ then:

$$\frac{a^4}{b^4+c^2}+\frac{b^4}{c^4+a^2}+\frac{c^4}{a^4+b^2}\ge\frac{3}{2}$$

Proposed by Kostas Geronikolas-Greece

Solution. *Using Bergstrom inequality, we get:*

$$LHS=\sum_{cyc}\frac{a^4}{b^4+c^2}\overset{AM-GM}{\ge}\sum_{cyc}\frac{a^4}{b^4+\frac{c^4+1}{2}}\ge 2\sum_{cyc}\frac{a^4}{2b^4+c^4+1}\overset{a^4=x}{=}$$

$$=2\sum_{cyc}\frac{x}{2y+z+1}\ge 2\frac{(\sum x)^2}{\sum(2xy+xz+x)}=\frac{2(\sum x^2+2\sum yz)}{3\sum yz+\sum x}\overset{(1)}{\ge}\frac{3}{2}=RHS,$$

$$(1)\Leftrightarrow\frac{2(\sum x^2+2\sum yz)}{3\sum yz+\sum x}\ge\frac{3}{2}\Leftrightarrow 4(\sum x^2+2\sum yz)\ge 9\sum yz+3\sum x\Leftrightarrow$$

$4\sum x^2\ge\sum yz+3\sum x$, *which follows from* $\sum x^2\ge\sum yz,\sum x^2\ge\sum x$ *and* $xyz=1$.

Equality holds if and only if $a=b=c=1$.

Remark: *Inequality can be developed.*

2) If $a,b,c>0$ such that $abc=2$ and $0\le\lambda\le 2$ then:

$$\frac{a^4}{b^4+\lambda c^2}+\frac{b^4}{c^4+\lambda a^2}+\frac{c^4}{a^4+\lambda b^2}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution. *Using Bergstrom inequality, we get:*

$$LHS=\sum_{cyc}\frac{a^4}{b^4+\lambda c^2}\overset{AM-GM}{\ge}\sum_{cyc}\frac{a^4}{b^4+\lambda\frac{c^2+1}{2}}\ge 2\sum_{cyc}\frac{a^4}{2b^4+\lambda c^4+\lambda}\overset{a^4=x}{=}$$

$$=2\sum_{cyc}\frac{x}{2y+\lambda z+\lambda}=2\sum_{cyc}\frac{x^2}{2xy+\lambda xz+\lambda x}\ge 2\frac{(\sum x)^2}{\sum(2xy+\lambda xz+\lambda x)}=$$

$$= \frac{2(\sum x^2 + 2\sum yz)}{(\lambda+2)\sum yz + \lambda x} \overset{(1)}{\geq} \frac{3}{\lambda+1} = RHS,$$

$$(1) \Leftrightarrow \frac{2(\sum x^2 + 2\sum yz)}{(\lambda+2)\sum yz + \lambda x} \geq \frac{3}{\lambda+1} \Leftrightarrow 2(\lambda+1)(\sum x^2 + 2\sum yz) \geq 3(\lambda+2)\sum yz + 3\lambda\sum x$$

$\Leftrightarrow (2\lambda+2)\sum x^2 \geq (2-\lambda)\sum yz + 3\lambda\sum x; (*)$, *which follows from*

$\sum x^2 \geq \sum yz, \sum x^2 \geq \sum x$ *and* $xyz = 1$.

$$\sum x^2 \geq \sum yz \to (2-\lambda)\sum x^2 \geq (2-\lambda)\sum yz; (2), 0 \leq \lambda \leq 2$$

$\sum x^2 \geq \sum yz, xyz = 1$ *which follows from:*

$$\sum x^2 \overset{Chebyshev's}{\geq} \frac{1}{3}\sum x\sum x \overset{AM-GM}{\geq} \sqrt[3]{xyz}(\sum x) = 1\cdot\sum x = \sum x \to 3\lambda\sum x^2 \geq 3\lambda\sum x; (3)$$

Summing relations $(2),(3)$ *it follows* $(*)$ *is true and the problem is proved.*

Equality holds if and only if $a = b = c = 1$.

Note:

For $\lambda = 1$ we get Problem 23/8/2020 proposed by Kostas Geronikolas.

REFERENCE:

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