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ABOUT AN INEQUALITY BY NGUYEN VAN CANH-XI

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1) In ΔABC the following relationship holds:

$$h_a^2+h_b^2+h_c^2+2r(R-2r)\le s^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution. ***Lemma. 2) In ΔABC the following relationship holds:***

$$\sum_{cyc} h_a^2=\frac{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}{4R^2}$$

Proof. *Using identity:* $h_a=\frac{2F}{a}$ *we get:*

$$\sum_{cyc} h_a=\frac{s^2+r^2+4Rr}{2R}\ \ and\ \sum_{cyc} h_bh_c=\frac{2s^2r}{R}, so\ we\ have:$$

$$\sum_{cyc} h_a^2=\left(\sum_{cyc} h_a\right)^2-2\sum_{cyc} h_bh_c=\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2s^2r}{R}=$$

$$=\frac{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}{4R^2}$$

Using Lemma inequality become as:

$$\frac{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}{4R^2}+2r(R-2r)\le s^2\Leftrightarrow$$

$s^4\le 2s^2(2R^2+8Rr-2r^2)-r(8R^3+8Rr^2+r^3)$, *which follows from*

$s^4\le 2s^2(2R^2+10Rr-r^2)-r(4R+r)^3$ *(See Lemma Blundon)*

$$s^4\overset{Blundon}{\le}2s^2(2R^2+10Rr-r^2)-r(4R+r)^3\overset{(1)}{\le}s^2(4R^2+8Rr-2r^2)-r(8R^3+8Rr^2+r^3),where\ (1)\Leftrightarrow$$

$$2s^2(2R^2+10Rr-r^2)-r(4R+r)^3\le s^2(4R^2+8Rr-2r^2)-r(8R^3+8Rr^2+r^3)$$

$\Leftrightarrow 12Rrs^2\le r(56R^3+48R^2r+4Rr^2)\Leftrightarrow 3s^2\le 14R^2+12Rr+r^2$, *which follows from:*

$$s^2\le 4R^2+4Rr+3r^2\ (Gerretsen)$$

Equality holds if and only if triangle is equilateral.

Lemma (Blundon) In ΔABC the following relationship holds:

$$s^4\le 2s^2(2R^2+10Rr-r^2)-r(4R+r)^3$$

Proof. *Using Blundon inequality:*

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\le s^2$$
$$\le 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr};(1)$$

Let $m=2R^2+10Rr-r^2$ *and* $n=2(R-2r)\sqrt{R^2-2Rr}$, *inequality* (1) *can be written as:* $m-n\le s^2\le m+n$, *then* $s^2-m+n\ge 0$ *and* $s^2-m-n\le 0$.

$$(\cdot)\to s^4-2ms^2+m^2-n^2\le 0\Leftrightarrow$$
$$s^4-2(2R^2+10Rr-r^2)s^2+(2R^2+10Rr-r^2)^2-4(R-2r)^2(R^2-2Rr)\ge 0$$
$$\Leftrightarrow s^4-2s^2(2R^2+10Rr-r^2)+r(4R+r)^3\le 0$$

Equality holds if and only if triangle is equilateral.

Remark: *Inequality can be developed.*

3) In ΔABC the following relationship holds:

$$\boldsymbol{h_a^2+h_b^2+h_c^2+\lambda r(R-2r)\le s^2,\qquad \lambda\le 4}$$

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Solution: *Using Lemma, inequality becomes as:*

$$\frac{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}{4R^2}+\lambda r(R-2r)\le s^2\Leftrightarrow$$

$s^4\le s^2(4R^2+8Rr-2r^2)-r[4\lambda R^3+(16-8\lambda)R^2r+8Rr^2+r^3]$, *which follows from*

$$s^4\le 2s^2(2R^2+10Rr-r^2)-r(4R+r)^2;(See\ Lemma\ Blundon)$$

$$s^4\overset{Blundon}{\leq}2s^2(2R^2+10Rr-r^2)-r(4R+r)^3\overset{(1)}{\leq}s^2(4R^2+8Rr-2r^2)-r(8R^3+8Rr^2+r^3),where\ (1)\Leftrightarrow$$

$$2s^2(2R^2+10Rr-r^2)-r(4R+r)^3$$
$$\le s^2(4R^2+8Rr-2r^2)-r[4\lambda R^3+(16-8\lambda)R^2r+8Rr^2+r^3]$$
$$12Rrs^2\le r[(64-4\lambda)R^3+(8\lambda+32)R^2r+4Rr^2]$$

$\Leftrightarrow 3s^2\le(16-\lambda)R^2+(2\lambda+8)Rr+r^2$, *which follows from*

$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$.*Remains to prove that:*

$$3(4R^2+4Rr+3r^2)\le(16-\lambda)R^2+(2\lambda+8)Rr+r^2$$

$\Leftrightarrow(4-\lambda)R^2+(2\lambda-4)Rr-8r^2\ge 0\Leftrightarrow(R-2r)[(4-\lambda)R+4r]\ge 0$, *which is true from* $R\ge 2r$ $(Euler)$ *and* $(4-\lambda)R+4r>$, *true from* $\lambda\le 4$.

Equality holds if and only if triangle is equilateral.

Remark:

1. For $\lambda=2$ we get Problem proposed by Nguyen Van Canh-Ben Tre-Vietnam from RMM 8/2020

2. The best inequality is obtained for $\lambda=4$.

REFERENCE:

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