RMM - Geometry Marathon 101 - 200

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

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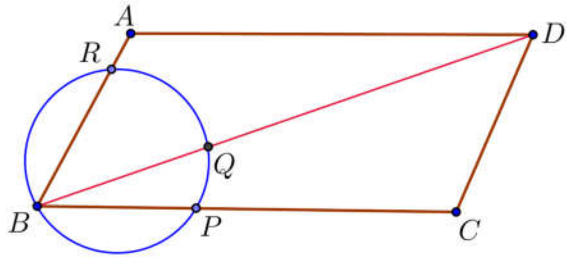
101. $ABCD$ −parallelogram.

$P \in BC, Q \in BD, R \in BA$

$BP \cdot BC + BR \cdot BA = BQ \cdot BD$

Prove that:

$BPQR$ −is cyclic.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil

Applying Ptolemy's theorem to the quadrilateral $BPQR$, we have:

$RP \cdot BQ = BR \cdot PQ + BP \cdot RQ \Rightarrow BPQR$ −is cyclic.

So, in ΔBRP, from law of sines: $\frac{RP}{sin(\alpha+\theta)} = \frac{BR}{sin\beta} = \frac{BP}{sin\varphi} = 2R; (1)$

In ΔQRP, from law of sines:

$$\begin{cases} \frac{RQ}{sin\alpha} = \frac{PQ}{sin\theta} = \frac{RP}{sin(\beta+\varphi)} = 2R \\ \frac{RQ}{sin\alpha} = \frac{PQ}{sin\theta} = \frac{RP}{sin(\alpha+\theta)} = 2R \end{cases} ; \beta + \varphi = 180^o - (\alpha + \theta)$$

$$\Rightarrow sin(\beta + \varphi) = sin(\alpha + \theta)$$

In ΔBDC, from law of sines:

$$\begin{cases} \frac{BC}{sin\alpha} = \frac{BA}{sin\theta} = \frac{BD}{sin[180 - (\alpha+\theta)]} = 2R_1 \\ \frac{BC}{sin\alpha} = \frac{BA}{sin\theta} = \frac{BD}{sin(\alpha+\theta)} = 2R_1 \end{cases}$$

Now, $BP \cdot BC + BR \cdot BA = BQ \cdot BD$

$$BP \cdot 2R_1 sin\alpha + BR \cdot 2R_1 sin\theta = BQ \cdot 2R_1 sin(\alpha + \theta)$$

$$BPsin\alpha + BRsin\theta = BQsin(\alpha + \theta)$$

From (1),(2): $sin\theta = \frac{RQ}{2R}, sin\theta = \frac{PQ}{2R}$ and $sin(\alpha + \theta) = \frac{RP}{2R}$

$$BP \cdot \frac{RQ}{2R} + BR \cdot \frac{PQ}{2R} = BQ \cdot \frac{RP}{2R}$$

Therefore, $BP \cdot BC + BR \cdot BA = BQ \cdot BD$ and $BPQR$ −is cyclic.

Solution 2 by proposer

Let $BP = p, BC = c, BR = r, BA = a, BC \equiv Bx, BA \equiv By$

$B(0,0), P(p,0), C(c,0), R(0,r), A(0,a), Q(q_1,q_2), D(c,a).$

Let circle $\omega \equiv (B,P,R), \omega \cap BD = Q'(q'_1,q'_2),$

$$q'_1 = \frac{c}{BD^2}(pc+ra), q'_2 = \frac{a}{BD^2}(pc+ra); (1)$$

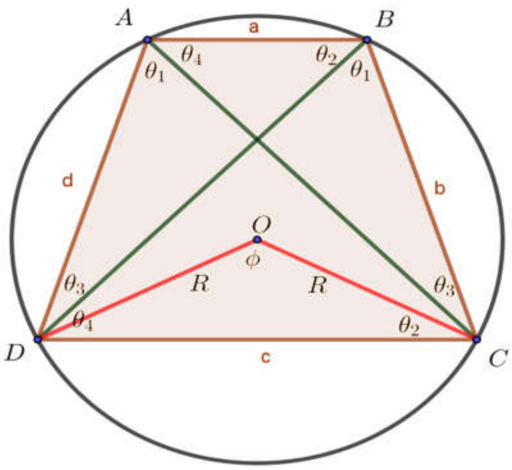
$$(\omega): x^2 + y^2 + 2xy\cos B - px - ry = 0, BD: \frac{x}{c} = \frac{y}{a}$$

$$BP \cdot BC + BR \cdot BA = BQ \cdot BD \Rightarrow pc + ra = \frac{BQ}{BD} \cdot BD^2 \Rightarrow$$

$$\begin{cases} pc + ra = \frac{q_1}{c} BD^2 \\ pc + ra = \frac{q_2}{d} BD^2 \end{cases} \Rightarrow \begin{cases} q_1 = \frac{c}{BD^2}(pc+ra) \\ q_2 = \frac{d}{BD^2}(pc+ra) \end{cases}; (2)$$

From (1),(2) it follows that $Q \equiv Q'$, therefore $BPQR$ −is cyclic.

102.



O −circumcenter, I −incenter, R −circumradii, r −radii, a, b, c, d −sides in a bicentric quadrilateral. Prove that:

$$2OI^2 + r\sum_{cyc}\sqrt{4R^2 - a^2} = 2(R^2 + 2r^2)$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } m = \text{product of the lengths of the diagonals, then}: \frac{m}{4r^2} - \frac{4R^2}{m} = 1$$

$$\Rightarrow m^2 - 4mr^2 - 16R^2r^2 = 0$$

$$\Rightarrow m = \frac{4r^2 \pm \sqrt{16r^4 + 64R^2r^2}}{2} = 2r^2 \pm 2r\sqrt{4R^2 + r^2}$$

$$= 2r\left(r + \sqrt{4R^2 + r^2}\right) \ (\because m > 0) \ \therefore m \overset{(a)}{=} 2r\left(r + \sqrt{4R^2 + r^2}\right)$$

$$\text{Sine rule on } \Delta\, ABC \Rightarrow \frac{a}{\sin\theta_3} = \frac{b}{\sin\theta_4}, \text{sine rule on } \Delta\, BCD \Rightarrow \frac{b}{\sin\theta_4}$$

$$= \frac{c}{\sin\theta_1} \text{ and sine rule on } \Delta\, CDA \Rightarrow \frac{c}{\sin\theta_1} = \frac{d}{\sin\theta_2} \text{ and } \therefore$$

$$\frac{a}{\sin\theta_3} = \frac{b}{\sin\theta_4} = \frac{c}{\sin\theta_1} = \frac{d}{\sin\theta_2} = 2R$$

$\because$ circumradius of each of triangles ABC, BCD and CDA is equal to $2R$

$$\text{Now, } 2R = \frac{b}{\sin\theta_4} = \frac{c}{\sin\theta_1} = \frac{b+c}{\sin\theta_4 + \sin\theta_1} = \frac{b+c}{2\sin\frac{\theta_4+\theta_1}{2}\cos\frac{\theta_4-\theta_1}{2}}$$

$$= \frac{b+c}{2\sin\frac{A}{2}\cos\frac{\theta_4-\theta_1}{2}} \Rightarrow \cos\frac{\theta_4-\theta_1}{2} \overset{(i)}{=} \frac{b+c}{4R\sin\frac{A}{2}}$$

$$\text{Again, } 2R = \frac{a}{\sin\theta_3} = \frac{d}{\sin\theta_2} = \frac{a+d}{\sin\theta_3 + \sin\theta_2} = \frac{a+d}{2\sin\frac{\theta_3+\theta_2}{2}\cos\frac{\theta_3-\theta_2}{2}}$$

$$= \frac{a+d}{2\sin\frac{C}{2}\cos\frac{\theta_3-\theta_2}{2}} \Rightarrow \cos\frac{\theta_3-\theta_2}{2} \overset{(ii)}{=} \frac{a+d}{4R\sin\frac{C}{2}}$$

$$\text{Cosine law on } \Delta\, OCD \Rightarrow R^2 + R^2 - 2R^2\cos\varphi = c^2 \Rightarrow 2R^2(1 - \cos\varphi) = c^2$$

$$\Rightarrow 4R^2\sin^2\frac{\varphi}{2} = c^2 \Rightarrow 4R^2\left(1 - \cos^2\frac{\varphi}{2}\right) = c^2$$

$$\Rightarrow 4R^2 - c^2 = 4R^2\cos^2\frac{\varphi}{2} \Rightarrow \sqrt{4R^2 - c^2} = 2R\cos\frac{\varphi}{2}$$

$= 2R\cos\theta_1$ ($\because$ angle at the center is twice angle at the circumference)

$$\Rightarrow \sqrt{4R^2-c^2} = 2R\cos\theta_1 \text{ and analogously}, \sqrt{4R^2-d^2} = 2R\cos\theta_2, \sqrt{4R^2-a^2} = 2R\cos\theta_3 \text{ and } \sqrt{4R^2-b^2} = 2R\cos\theta_4$$

$$\text{and summing up}, r\sum\sqrt{4R^2-a^2} = 2Rr\big((\cos\theta_1+\cos\theta_4)+(\cos\theta_2+\cos\theta_3)\big)$$

$$= 4Rr\left(\cos\frac{\theta_4+\theta_1}{2}\cos\frac{\theta_4-\theta_1}{2}+\cos\frac{\theta_3+\theta_2}{2}\cos\frac{\theta_3-\theta_2}{2}\right)$$

$$= 4Rr\left(\cos\frac{A}{2}\cos\frac{\theta_4-\theta_1}{2}+\cos\frac{C}{2}\cos\frac{\theta_3-\theta_2}{2}\right)$$

$$\overset{\text{via (i) and (ii)}}{\cong} 4Rr\left(\cos\frac{A}{2}\left(\frac{b+c}{4R\sin\frac{A}{2}}\right)+\cos\frac{C}{2}\left(\frac{a+d}{4R\sin\frac{C}{2}}\right)\right)$$

$$= r\left((b+c)\cot\frac{A}{2}+(a+d)\cot\frac{C}{2}\right) = r\left((b+c)\sqrt{\frac{ad}{bc}}+(a+d)\sqrt{\frac{bc}{ad}}\right)$$

$$= \frac{r\big(ad(b+c)+bc(a+d)\big)}{\sqrt{abcd}} =$$

$$= \frac{r\big(bd(a+c)+ac(b+d)\big)}{rs}\left(\because r = \frac{\sqrt{abcd}}{a+c} = \frac{\sqrt{abcd}}{b+d} \text{ and } a+c = b+d = s\right)$$

$$= \frac{s(ac+bd)}{s}$$

$$\overset{\text{via Ptolemy}}{\cong} m \overset{\text{via } (a)}{\cong} 2r\left(r+\sqrt{4R^2+r^2}\right) \Rightarrow r\sum\sqrt{4R^2-a^2} \overset{(l)}{\cong} 2r\left(r+\sqrt{4R^2+r^2}\right)$$

$$\text{Via Carlitz}, OI^2 = R^2 - \frac{2Rr}{s}\sqrt{\frac{(ab+cd)(ad+bc)}{ac+bd}} \text{ and consequently}, OI^2 \overset{\text{(iii)}}{\lessgtr} R^2$$

$$\text{Via Nicolaus Fuss (1792), if } t = OI, \text{ then}: \frac{1}{r^2} = \frac{1}{(R+t)^2}+\frac{1}{(R-t)^2} \Rightarrow \frac{1}{r^2}$$

$$= \frac{2R^2+2t^2}{(R^2-t^2)^2} \Rightarrow t^4 - 2t^2(R^2+r^2) + R^4 - 2R^2r^2 = 0$$

$$\Rightarrow t^2 = \frac{2(R^2+r^2) \pm \sqrt{4(R^2+r^2)^2 - 4(R^4-2R^2r^2)}}{2} = R^2+r^2 \pm r\sqrt{4R^2+r^2} \text{ and}$$

$$\because t^2 \overset{\text{via (iii)}}{\lessgtr} R^2 \therefore OI^2 = R^2+r^2-r\sqrt{4R^2+r^2}$$

$$\Rightarrow 2\mathrm{OI}^2 + \mathrm{r}\sum\sqrt{4\mathrm{R}^2 - a^2}$$

$$= 2\mathrm{R}^2 + 2\mathrm{r}^2 - 2\mathrm{r}\sqrt{4\mathrm{R}^2 + \mathrm{r}^2} + \mathrm{r}\sum\sqrt{4\mathrm{R}^2 - a^2} \overset{\text{via }(l)}{\triangleq} 2\mathrm{R}^2 + 2\mathrm{r}^2$$

$$- 2\mathrm{r}\sqrt{4\mathrm{R}^2 + \mathrm{r}^2} + 2\mathrm{r}\left(\mathrm{r} + \sqrt{4\mathrm{R}^2 + \mathrm{r}^2}\right)$$

$$= 2(\mathrm{R}^2 + 2\mathrm{r}^2)\ (\mathrm{QED})$$

103. In ΔABC the following relationship holds:

$$\frac{(a+b)(b+c)(c+a)}{8abc} \cdot \frac{w_a w_b w_c}{h_a h_b h_c} = \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

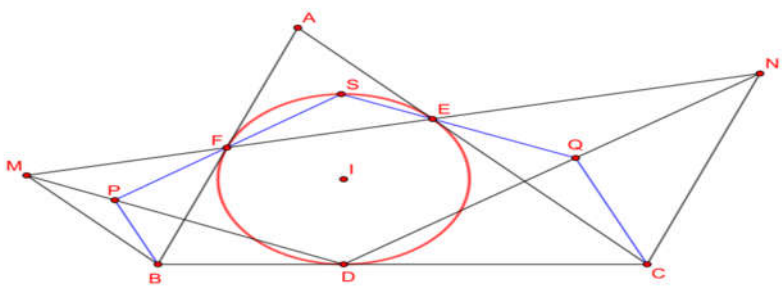
Solution by Daniel Sitaru-Romania

$$\frac{(a+b)(b+c)(c+a)}{8abc} \cdot \frac{w_a w_b w_c}{h_a h_b h_c} = \prod_{cyc}\left(\frac{a+b}{2c} \cdot \frac{w_c}{h_c}\right) =$$

$$= \prod_{cyc}\left(\frac{a+b}{2c} \cdot \frac{2}{a+b}\sqrt{abs(s-c)} \cdot \frac{c}{2F}\right) =$$

$$= \prod_{cyc}\frac{\sqrt{abs(s-c)}}{2F} = \frac{abcsF}{8F^3} = \frac{abcs}{8F^2} = \frac{4RFs}{8F^2} = \frac{Rs}{2F} = \frac{Rs}{2rs} = \frac{R}{2r}$$

104. Let ABC be a triangle with incentre I. The circle (I) is tangent to BC, CA and AB at points D, E and F, respectively. Let M, N be the points on the line EF such that BM is parallel to AC and CN is parallel to AB. Let P and Q be the points on DM and DN, respectively such that BP is parallel to CQ. Denote by S the intersection point of PF and QE. Prove that S lies on the circle (I).



Proposed by Luu Cong Dong-Vietnam

Solution by proposer:

We need a lemma.

Lemma. Let A, B, C, D, E and F be the points such that A, B, C, D, E lie on the circle (O). Denote by X, Y, Z the intersection points of the lines AE, CD, BF and BD, FA, EC, respectively. If X, Y, Z are collinear then F lies on the circle (O).

Proof of Lemma.

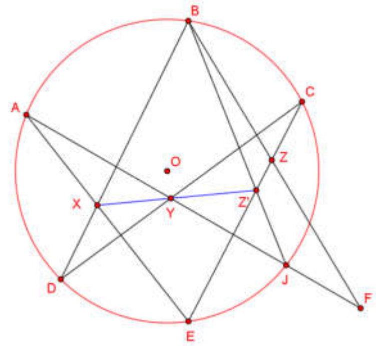


Figure 1

Let AF intersects the circle (O) again at J, AJ intersects CE at Z' (Figure 1).

Applying Pascal's Theorem to the array $\begin{pmatrix} ABC \\ DEJ \end{pmatrix}$, **we find that X, Y, Z' are collinear.**

Clearly, $Z = XY \cap CE$ **and** $Z' = XY \cap CE$**. We imply that** $Z \equiv Z'$.

It follows that $F \equiv J$. **Thus F lies on the circle (O).**

Come back to the solution of the problem

Let K be the intersection point of BP and CI; L be the intersection point of CQ and BI. Let DP intersects the circle (I) at U and DQ intersects the circle (I) at V (figure 2).

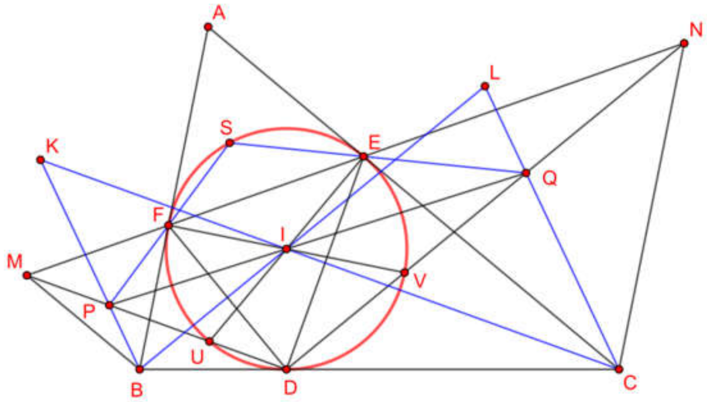


Figure 2.

It is easy to see that $BM = BF = BD$ and $\angle MBD = \angle A + \angle B$.

It follows that $\angle MDB = \dfrac{\angle C}{2} = \angle ICB$. We deduce that PD is parallel to IC.

Similarly, QD is parallel to IB.

We imply that $\dfrac{PB}{PK} = \dfrac{DB}{DC} = \dfrac{QL}{QC}$. Thus I, P, Q are collinear (because $BK // LC$).

Because $UD \parallel IC$ and $IC \perp DE$, $UD \perp DE$. It means that $I \in UE$.

Similarly, $I \in VF$.

Applying Lemma above to the array $\begin{pmatrix} EFD \\ VUS \end{pmatrix}$, and note that I, P, Q are collinear, we find that S belongs to the circle (I).

105. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos(B-C) - 2\cos A}{h_a} = 0$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Alex Szoros-Romania

Is well-known $b\cos C + c\cos B = a, \forall \Delta ABC$.

Lemma 1. In any ΔABC the following relationship holds:

$$\sum_{cyc} a\cos A = \frac{2sr}{R}$$

Proof. $\left(\sum_{cyc} a\right)\left(\sum_{cyc} \cos A\right) = \sum_{cyc} a\cos A + \sum_{cyc}(a\cos b + a\cos C) \Leftrightarrow$

$$2s\left(1 + \frac{r}{R}\right) = \sum_{cyc} a\cos A + \sum_{cyc}(a\cos B + b\cos A)$$

$$2s + \frac{2sr}{R} = \sum_{cyc} a\cos A + 2s \Rightarrow \sum_{cyc} a\cos A = \frac{2sr}{R}$$

Lemma 2. In any ΔABC the following relationship holds:

$$2\sum_{cyc} a\cos B\cos C = \sum_{cyc} a\cos A$$

Proof. $2\sum_{cyc} acosBcosC = acosBcosC + bcosAcosC + bcosAcosC + ccosAcosB + ccosAcosB + acosBcosC = \sum_{cyc} acosA$

$$\sum_{cyc}\frac{cos(B-C)}{h_a} = \sum_{cyc}\frac{a(cosBcosC + sinBsinC)}{2S} =$$

$$= \frac{1}{2S}\sum_{cyc} acosBcosC + \frac{1}{2S}\sum_{cyc} asinBsinC = \frac{1}{2S}\frac{\sum acosA}{2} + \frac{1}{2S}\frac{\sum abc}{4R^2} =$$

$$= \frac{1}{4S}\frac{2sr}{R} + \frac{1}{2S}\frac{3abc}{4R^2} = \frac{1}{2R} + \frac{12RS}{8R^2S} = \frac{2}{R}; (1)$$

$$\sum_{cyc}\frac{cosA}{h_a} = \sum_{cyc}\frac{acosA}{2S} = \frac{1}{2S}\sum_{cyc} acosA = \frac{1}{2S}\frac{2S}{R} \Rightarrow \sum_{cyc}\frac{cosA}{h_a} = \frac{1}{R}; (2)$$

From (1),(2) it follows that: $\sum_{cyc}\frac{cos(B-C)}{h_a} = 2\sum_{cyc}\frac{cosA}{h_a}$

Therefore, $\sum_{cyc}\frac{cos(B-C)-2cosA}{h_a} = 0$

Solution 2 by Izumi Ainsworth-Lima-Peru

$$\sum_{cyc} cotBcotC = 1; \sum_{cyc} sin2A = 4sinAsinBsinC$$

$$\sum_{cyc}\frac{cos(B-C)-2cosA}{h_a} = \sum_{cyc}\frac{cosBcosC + sinBsinC - 2cosA}{2RsinBsinC} =$$

$$= \frac{1}{2R}\sum_{cyc} cotBcotC + \frac{3}{2R} - \frac{1}{2RsinAsinBsinC}\sum_{cyc} sin2A = \frac{1}{2R} + \frac{3}{2R} - \frac{2}{R} = 0$$

106. In $\Delta ABC, \mu(\sphericalangle BAC) = 90^o, D \in (BC), AD \perp BC, r, r_1, r_2$ –inradii in $\Delta ABC, \Delta ACD, \Delta ABD$. Prove that:

$$r_a = \frac{r(r_1 + r_2 + r)}{r_1 + r_2 - r}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution 1 by Juan Jose Isach Macho-Spain

$$F_{ABC} = \frac{1}{2}bc = \frac{1}{2}ah_a, s = \frac{a+b+c}{2}, h_a = \frac{bc}{a}, BD = \frac{c^2}{a}, DC = \frac{b^2}{a}$$

$$F_{ACD}=\frac{1}{2}\frac{c^2}{a}\frac{bc}{a}=\frac{1}{2}\frac{bc^3}{a^2}, F_{ABD}=\frac{1}{2}\frac{b^3c}{a^2}$$

$$s_{ACD}=\frac{c}{2a}(a+b+c), s_{ABD}=\frac{b}{2a}(a+b+c)$$

$$r_1=\frac{F_{ACD}}{s_{ACD}}=\frac{bc^2}{a(a+b+c)}, r_1=\frac{F_{ABD}}{s_{ACD}}=\frac{b^2c}{a(a+b+c)}$$

$$(r_1+r_2+r)=\frac{bc}{a+b+c}\left(\frac{bc^2}{a(a+b+c)}+\frac{b^2c}{a(a+b+c)}+\frac{abc}{a(a+b+c)}\right)=$$

$$=\frac{b^2c^2(a+b+c)}{a(a+b+c)^2}=\frac{b^2c^2}{a(a+b+c)}$$

$$r_1+r_2-r=\frac{bc^2}{a(a+b+c)}+\frac{b^2c}{a(a+b+c)}-\frac{abc}{a(a+b+c)}=\frac{bc}{a(a+b+c)}(b+c-a)$$

$$r_a=\frac{bc(a+b+c)}{(-a+b+c)(a+b+c)}=\frac{bc}{-a+b+c}$$

$$\frac{r(r_1+r_2+r)}{r_1+r_2-r}=\frac{bc}{-a+b+c}$$

Therefore $r_a=\frac{r(r_1+r_2+r)}{r_1+r_2-r}$.

Solution 2 by Mansur Mansurov-Azerbaijan

$$\Delta ABC\sim\Delta DBA\sim\Delta DAC$$

$$\frac{r}{a}=\frac{r_2}{c}=\frac{r_1}{b}\Rightarrow\begin{cases}rb=r_1a\\rc=r_2a\end{cases}$$

$$r_a=\frac{2F_{ABC}}{b+c-a}=\frac{r(a+b+c)}{b+c-a}=\frac{ra+rb+rc}{\frac{ar_1}{r}+\frac{ar_2}{r}-a}=$$

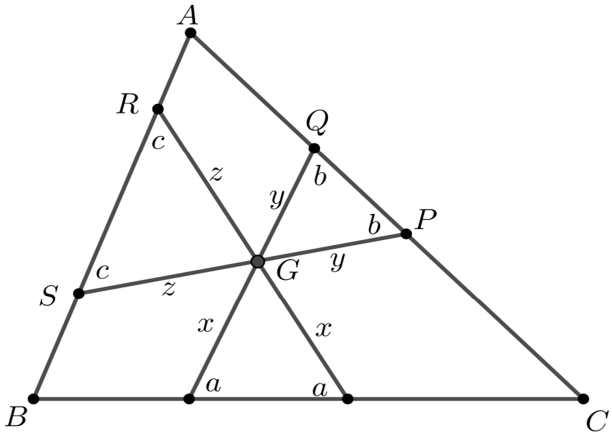
$$=\frac{ra+r_1a+r_2a}{\frac{a}{r}(r_1+r_2-r)}=\frac{a(r_1+r_2+r)}{\frac{a}{r}(r_1+r_2-r)}=\frac{r(r_1+r_2+r)}{r_1+r_2-r}$$

107. In $\Delta ABC, G$ –centroid, $M,N\in BC, P,Q\in CA, R,S\in AB$, $\overline{MGQ},\overline{NGR},\overline{SGP}$ –antiparallels, $\varphi_A,\varphi_B,\varphi_C$ –circumradii in $\Delta GMN,\Delta GPQ, GRS$. Prove that:

$$\varphi_A\varphi_B\varphi_C=\frac{R^3}{27}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Geanina Tudose-Romania



Well known $S-G-P\Rightarrow\frac{BS}{SA}+\frac{CP}{PA}=1$

$$\Delta ASP\sim\Delta ACB\Rightarrow\frac{SA}{b}=\frac{AP}{c}\Rightarrow SA=\frac{b\cdot AP}{c}\Rightarrow\frac{c-\frac{b\cdot AP}{c}}{b\frac{AP}{c}}+\frac{b-AP}{AP}=1\Rightarrow$$

$$\frac{c^2-bAP+b^2-bAP}{bAP}=1\Rightarrow 3bAP=c^2+b^2\Rightarrow AP=\frac{c^2+b^2}{3b}\Rightarrow SA=\frac{c^2+b^2}{3c}$$

$$\frac{SA}{b}=\frac{AP}{c}=\frac{z+y}{a}\Rightarrow z+y=\frac{a(b^2+c^2)}{3bc}$$

We have:

$$\begin{cases}z+y=\dfrac{a(b^2+c^2)}{3bc}\\ x+z=\dfrac{b(a^2+c^2)}{3ac}\\ x+y=\dfrac{c(a^2+b^2)}{3ab}\end{cases}\Rightarrow x+y+z=\frac{a^2b^2+b^2c^2+c^2a^2}{3abc}\Rightarrow x=\frac{b^2c^2}{3abc}=\frac{bc}{3a}$$

In ΔGMN, we have: $\frac{x}{sinA}=2\varphi_A\Rightarrow 2\varphi_A=\frac{x\cdot 2R}{a}\Rightarrow\varphi_A=\frac{\frac{bc}{3a}R}{a}=\frac{bc\cdot R}{3a^2}(and\ analogs)$

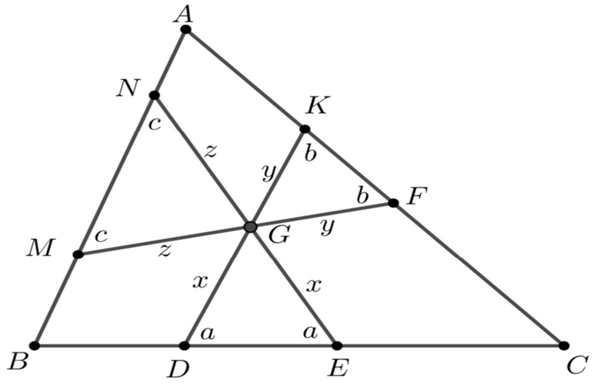
$$\varphi_B=\frac{ac\cdot R}{3b^2},\varphi_C=\frac{ab\cdot R}{3c^2}\Rightarrow\varphi_A\varphi_B\varphi_C=\frac{R^3}{27}$$

108. In $\Delta ABC, G$ –centroid, $D,E\in BC, F,K\in CA$, $M,N\in AB,\overline{DGK},\overline{EGN},\overline{MGF}$ –antiparallels, P_1,P_2,P_3 –perimeters of $\Delta GDE,\Delta GFK,\Delta GNM$. Prove that:

$$\frac{1}{P_1}+\frac{1}{P_2}+\frac{1}{P_3}=\frac{3(2R-r)}{2F}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Marian Ursărescu-Romania



Because $\overline{DGK}, \overline{EGN}, \overline{MGF}$ —antiparallels$\Rightarrow \begin{cases} \sphericalangle GDE = \sphericalangle GED = A \\ \sphericalangle GKF = \sphericalangle GFK = B \\ \sphericalangle GMN = \sphericalangle GNM = C \end{cases} \Rightarrow \begin{cases} GD = GE \\ GN = GM \\ GK = GF \end{cases}; (1)$

From Transversal theorem in ΔABC, means:

$\frac{BM}{MA} + \frac{CF}{FA} = 1$ **and because $\Delta AFM \sim \Delta EBD \sim \Delta DKC \sim \Delta ABC$, we have:**

$$GD = GE = \frac{bc}{3a}, GF = GK = \frac{ac}{3b}, GN = GM = \frac{ab}{3c}; (2)$$

Now, in ΔGDE we have: $DE^2 = GD^2 + GE^2 - 2GD \cdot GE\sin(DGE) =$

$$= 2\left(\frac{bc}{3a}\right)^2 \left(1 - \cos(DGE)\right) = 2\left(\frac{bc}{3a}\right)^2 \cdot 2\sin\left(\frac{DGE}{2}\right) = 4\left(\frac{bc}{3a}\right)^2 \sin^2\left(\frac{\pi - 2A}{2}\right) =$$

$$= 4\left(\frac{bc}{3a}\right)^2 \sin^2\left(\frac{\pi}{2} - A\right) = 4\left(\frac{bc}{3a}\right) \cos^2 A \Rightarrow DE = \frac{2bc}{3a}\cos A; (and\ analogs).$$

$$\Rightarrow P_1 = GD + GE + ED = \frac{2bc}{3a} + \frac{2bc}{3a}\cos A = \frac{2bc}{3a}(1 + \cos A)$$

$$= \frac{4bc}{3a}\cos^2\frac{A}{2} (and\ analogs)$$

$$\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{3}{4}\sum_{cyc} \frac{a}{bc\cos^2\frac{A}{2}} = \frac{3}{4}\sum_{cyc} \frac{a}{bc\frac{s(s-a)}{bc}} =$$

$$= \frac{3}{4s}\sum_{cyc} \frac{a}{s-a} \overset{\sum\frac{a}{s-a} = \frac{2(2R-r)}{r}}{=} \frac{3}{4s}\frac{2(2R-r)}{r} = \frac{3(2R-r)}{2F}$$

109. **Let ΔDEF be the orthic triangle of acute ΔABC. Prove that:**

$$\sum_{cyc} \frac{EF}{AH} \cdot \sum_{cyc} \frac{EF}{BE \cdot CF} = \frac{1}{r} + \frac{1}{R}$$

Proposed by Ertan Yldirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$EF = acosA, FD = bcosB, DE = ccosC$$

$$AH = 2RcosA, BH = 2RcosB, CH = 2RcosC$$

$$\sum_{cyc}\frac{EF}{AH}\cdot\sum_{cyc}\frac{EF}{BE\cdot CF} = \sum_{cyc}\frac{acosA}{2RcosA}\cdot\sum_{cyc}\frac{acosA}{h_b\cdot h_c} =$$

$$= \frac{1}{2R}\sum_{cyc}a\cdot\sum_{cyc}\frac{acosA}{\frac{2F}{b}\cdot\frac{2F}{c}} = \frac{2s}{2R}\cdot\frac{1}{4F^2}\cdot\sum_{cyc}abccosA = \frac{s}{4RF\cdot F}\cdot abc\sum_{cyc}cosA =$$

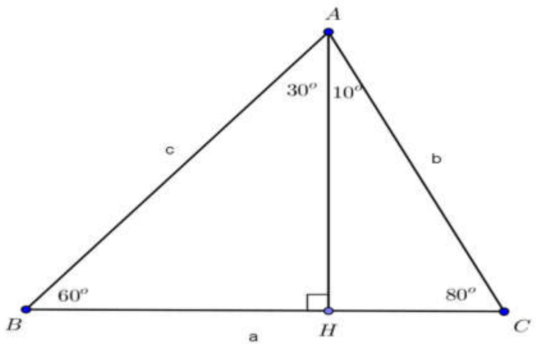
$$= \frac{s}{abc\cdot F}\cdot abc\left(1+\frac{r}{R}\right) = \frac{s}{rs}\left(1+\frac{r}{R}\right) = \frac{1}{r}+\frac{1}{R}$$

110. In $\Delta ABC, \mu(BAC) = 40^o, \mu(ABC) = 60^o$. Prove that:

$$\left(\frac{b}{c}\right)^3 + 3\left(\frac{b}{c}\right)^2 = 3$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$cos30^o = \frac{AH}{c}, cos10^o = \frac{AH}{b} \Rightarrow \frac{b}{c} = \frac{cos30^o}{cos10^o} = \frac{4cos^3 10^o - 3cos10^o}{cos10^o} \Rightarrow$$

$$\frac{b}{c} = 4cos^2 10^o - 3$$

$$\left(\frac{b}{c}\right)^3 + 3\left(\frac{b}{c}\right)^2 = (4cos^2 10^o - 3)^2(4cos^2 10^o - 3 + 3) =$$

$$= 4cos^2 10^o(4cos^2 10^o - 3)^2 = 4cos^2 30^o = 2(1 + cos60^o) = 3$$

Therefore,

$$\left(\frac{b}{c}\right)^3+3\left(\frac{b}{c}\right)^2=3$$

111. In ΔABC the following relationship holds:

$$ab\left(tan\frac{A}{2}+tan\frac{B}{2}\right)+bc\left(tan\frac{B}{2}+tan\frac{C}{2}\right)+ca\left(tan\frac{C}{2}+tan\frac{A}{2}\right)=\frac{abc}{r}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc} ab\left(tan\frac{A}{2}+tan\frac{B}{2}\right)=\sum_{cyc} ab\left(\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right)=$$

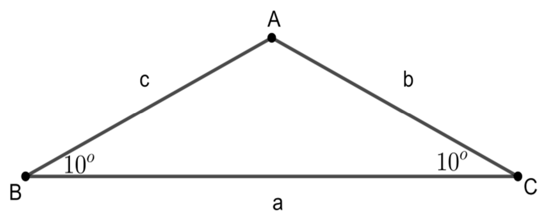
$$=\sum_{cyc}\frac{ab}{\sqrt{s}}\cdot\frac{(s-b)\sqrt{s-c}+(s-a)\sqrt{s-c}}{\sqrt{(s-a)(s-b)}}=\sum_{cyc} ab\cdot\frac{(s-b)(s-c)+(s-a)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\sum_{cyc}\frac{ab(s-c)(s-b+s-a)}{F}=\sum_{cyc}\frac{ab(s-c)(2s-b-a)}{rs}=$$

$$=\sum_{cyc}\frac{ab(s-c)(a+b+c-b-a)}{rs}=\frac{abc}{rs}\sum_{cyc}(s-c)=$$

$$=\frac{abc}{rs}(s-a+s-b+s-c)=\frac{abc}{rs}(3s-2s)=\frac{abc}{r}$$

112.



In ΔABC the following relationship holds:

$$\left(\frac{a}{b}\right)^6-6\left(\frac{a}{b}\right)^4+9\left(\frac{a}{b}\right)^2-3=0$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by proposer

$$\frac{a}{sin30^o}=\frac{b}{sin30^o}=k;(1)$$

$$sin30^o=sin20^o cos10^o+cos20^o sin10^o$$

$$a^2=2b^2-2b^2cos160^o\Rightarrow a^2=2b^2+2b^2cos20^o\Rightarrow$$

$$cos20^o=\frac{a^2-2b^2}{2b^2};(3)$$

$$\frac{1}{2}=\frac{a}{k}\cdot\frac{a}{2b}+\frac{b}{k}cos20^o\Rightarrow\frac{1}{2}=\frac{a^2}{2kb}+\frac{a^2-2b^2}{2b^2}\cdot\frac{b}{k}\Rightarrow k=\frac{2(a^2-b^2)}{b};(*)$$

$$sin30^o=sin(3\cdot 10^o)=3sin10^o-4sin^3 10^o\Rightarrow\frac{1}{2}=3\left(\frac{b}{k}\right)-4\left(\frac{b}{k}\right)^3\Rightarrow$$

$$k^3=6bk^2-8b^3;(**)$$

From $(*),(**)$ we get: $8b^3=k^2(6b-k)\Leftrightarrow$

$$8b^3=\left[\frac{2(a^2-b^2)}{b}\right]^2\left(6b-\frac{2(a^2-b^2)}{b}\right)\Leftrightarrow 8b^6=4(a^4+b^4-2a^2b^2)(8b^2-2a^2)\Leftrightarrow$$

$$a^6-3b^6-6a^4b^2+9a^2b^4=0\Leftrightarrow\left(\frac{a}{b}\right)^6-6\left(\frac{a}{b}\right)^4+9\left(\frac{a}{b}\right)^2-3=0$$

113. In ΔABC the following relationship holds:

$$\frac{a(b+c-a)}{r_b+r_c}+\frac{b(c+a-b)}{r_c+r_a}+\frac{c(a+b-c)}{r_a+r_b}=6r$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a(b+c-a)}{r_b+r_c}=\sum_{cyc}\frac{a(a+b+c-2a)}{\frac{F}{s-b}+\frac{F}{s-c}}=$$

$$=\frac{1}{F}\sum_{cyc}\frac{a(2s-2a)}{\frac{s-b+s-c}{(s-b)(s-c)}}=\frac{2}{F}\sum_{cyc}\frac{a(s-a)(s-b)(s-c)}{2s-b-c}=$$

$$=\frac{2}{F}\sum_{cyc}\frac{as(s-a)(s-b)(s-c)}{s(a+b+c-b-c)}=\frac{2}{F}\sum_{cyc}\frac{aF^2}{sa}=$$

$$=\frac{2}{F}\cdot\frac{1}{s}\cdot 3F^2=\frac{6F}{s}=\frac{6rs}{s}=6r$$

114. In any ΔABC the following relationship holds:

$$\sum \frac{m_a^2}{r_b r_c} + \frac{r_a^2 + r_b^2 + r_c^2}{r_a r_b + r_b r_c + r_c r_a} = \frac{ab}{w_c^2} + \frac{bc}{w_a^2} + \frac{ca}{w_b^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{bc}{w_a^2} = \sum \frac{bc(b+c)^2}{4bcs(s-a)} \overset{(i)}{=} \sum \frac{(b+c)^2}{4s(s-a)} \text{ and } \sum \frac{m_a^2}{r_b r_c} = \sum \frac{(b-c)^2 + 4s(s-a)}{4s(s-a)}$$

$$= 3 + \sum \frac{(b+c)^2 - 4bc}{4s(s-a)}$$

$$= 3 + \sum \frac{(b+c)^2}{4s(s-a)} - \sum \sec^2 \frac{A}{2} \overset{\text{via (i)}}{=} 3 + \sum \frac{bc}{w_a^2} - \frac{(4R+r)^2 + s^2}{s^2}$$

$$= \sum \frac{bc}{w_a^2} - \frac{(4R+r)^2 - 2s^2}{s^2} = \sum \frac{bc}{w_a^2} - \frac{r_a^2 + r_b^2 + r_c^2}{r_a r_b + r_b r_c + r_c r_a}$$

$$\Rightarrow \sum \frac{m_a^2}{r_b r_c} + \frac{r_a^2 + r_b^2 + r_c^2}{r_a r_b + r_b r_c + r_c r_a} = \frac{ab}{w_c^2} + \frac{bc}{w_a^2} + \frac{ca}{w_b^2} \ (Proved)$$

115. In any ΔABC the following relationship holds:

$$\frac{a(\cos B + \cos C)}{b+c} + \frac{b(\cos C + \cos A)}{c+a} + \frac{c(\cos A + \cos B)}{a+b} = \frac{2R - r}{R}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{a(\cos B + \cos C)}{b+c} + \frac{b(\cos C + \cos A)}{c+a} + \frac{c(\cos A + \cos B)}{a+b} =$$

$$= \sum_{cyc} \frac{a(\cos B + \cos C)}{b+c} = \sum_{cyc} \frac{a \cdot 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}{2R(\sin B + \sin C)} =$$

$$= \frac{1}{R}\sum_{cyc} \frac{a\cos\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} = \frac{1}{2R}\sum_{cyc} a\cot\frac{\pi - A}{2} = \frac{1}{2R}\sum_{cyc} a\tan\frac{A}{2} =$$

$$=\frac{1}{2R}\sum_{cyc} a\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{1}{2R}\sum_{cyc}\frac{a\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)}=$$

$$=\frac{1}{2R}\sum_{cyc}\frac{aF}{s(s-a)}=\frac{1}{2R}\sum_{cyc}\frac{ars}{s(s-a)}=\frac{r}{2R}\sum_{cyc}\frac{a}{s-a}=\frac{r}{2R}\cdot\frac{2(2R-r)}{r}=\frac{2R-r}{R}$$

116. In any ΔABC the following relationship holds:

$$\frac{a^2}{bc(r_a-r)}+\frac{b^2}{ca(r_b-r)}+\frac{c^2}{ab(r_c-r)}=\frac{1}{r}+\frac{1}{R}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a^2}{bc(r_a-r)}=\sum_{cyc}\frac{a^2}{bc\left(\frac{F}{s-a}-\frac{F}{s}\right)}=\frac{1}{F}\sum_{cyc}\frac{a^2}{bc\left(\frac{s-s+a}{s(s-a)}\right)}=$$

$$=\frac{1}{F}\sum_{cyc}\frac{a^2s(s-a)}{abc}=\frac{s}{abcF}\sum_{cyc}(sa^2-a^3)=\frac{s}{abcrs}\left(s\sum_{cyc}a^2-\sum_{cyc}a^3\right)=$$

$$=\frac{1}{abcr}\left(2s\left(s^2-r^2-4Rr\right)-2s(s^2-3r^2-6Rr)\right)=$$

$$=\frac{2s}{4rRF}(2r^2+2Rr)=\frac{s}{RF}(r+R)=\frac{s}{srR}(r+R)=\frac{1}{r}+\frac{1}{R}$$

117. In ΔABC the following relationship holds:

$$\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{\tan\frac{A}{2}}+\frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{C}{2}}=\frac{2}{r}(2R-r)$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{\tan\frac{A}{2}}=\sum_{cyc}\left(\tan\frac{B}{2}\cot\frac{A}{2}+\tan\frac{C}{2}\cot\frac{A}{2}\right)=$$

$$=\sum_{cyc}\left(\sqrt{\frac{(s-a)(s-c)}{s(s-b)}\cdot\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{(s-a)(s-b)}{s(s-c)}\cdot\frac{s(s-a)}{(s-b)(s-c)}}\right)=$$

$$=\sum_{cyc}\left(\frac{s-a}{s-b}+\frac{s-a}{s-c}\right)=\sum_{cyc}(s-a)\left(\frac{1}{s-b}+\frac{1}{s-c}\right)=$$

$$=\sum_{cyc}(s-a)\frac{s-c+s-b}{(s-b)(s-c)}=\sum_{cyc}(s-a)\frac{2s-c-b}{(s-b)(s-c)}=$$

$$=\sum_{cyc}(s-a)\frac{a+b+c-b-c}{(s-b)(s-c)}=\sum_{cyc}\frac{a(s-a)}{(s-b)(s-c)}=$$

$$=\frac{1}{(s-a)(s-b)(s-c)}\sum_{cyc}a(s-a)^2=$$

$$=\frac{1}{(s-a)(s-b)(s-c)}\left(s^2\sum_{cyc}a-2s\sum_{cyc}a^2+\sum_{cyc}a^3\right)=$$

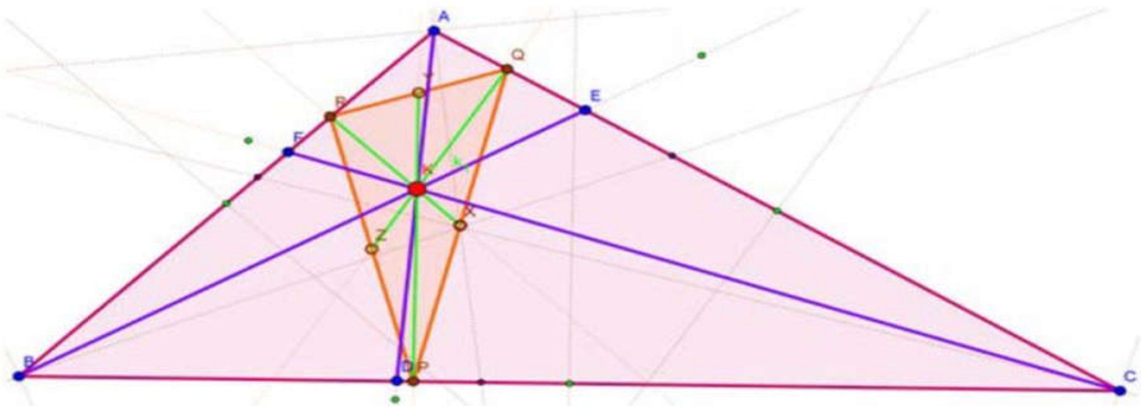
$$=\frac{s^2\cdot 2s-2s\cdot 2(s^2-r^2-4Rr)+2s(s^2-3r^2-6Rr)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{4Rrs-2sr^2}{(s-a)(s-b)(s-c)}=\frac{2s^2r(2R-r)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{2s^2r(2R-r)}{F^2}=\frac{2s^2r(2R-r)}{r^2s^2}=\frac{2}{r}(2R-r)$$

118. In any $\Delta ABC, m(\sphericalangle A)=90^\circ, \Delta PQR$ −pedal triangle of Lemoine's point the following relationship holds:

$$PQ^2+QR^2+RP^2=\frac{3}{2}h_a^2$$



Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$PQ = \frac{2F}{a^2+b^2+c^2}\sqrt{2a^2+2b^2-c^2}$$

$$QR = \frac{2F}{a^2+b^2+c^2}\sqrt{2b^2+2c^2-a^2}$$

$$RP = \frac{2F}{a^2+b^2+c^2}\sqrt{2c^2+2a^2-b^2}$$

Dasari Naga Vijay Krishna, Weitzenbock inequality – 2 proofs in a more geometrical way using the idea of "lemoine point" and "Fermat point", GeoGebra International Journal of Romania (GGIJRO), volume-4, No.1, 2015, page no.89, 90.

Dasari Naga Vijay Krishna, Jakub Kabat, "Several observations about Maneeals - a peculiar system of lines", Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica XV (2016), Ann. Univ. Paedagog. Crac. Stud. Math. 15 (2016), 51-68.

$$PQ^2+QR^2+RP^2 = \frac{4F^2}{(a^2+b^2+c^2)^2}\cdot 3(a^2+b^2+c^2) =$$

$$= \frac{12F^2}{a^2+b^2+c^2} = \frac{12\cdot\frac{b^2c^2}{4}}{a^2+a^2} = \frac{3}{2}\left(\frac{bc}{a}\right)^2 = \frac{3}{2}h_a^2$$

119. In ΔABC the following relationship holds:

$$\frac{c(cosA+cosB)}{h_a+h_b}+\frac{a(cosB+cosC)}{h_b+h_c}+\frac{b(cosC+cosA)}{h_c+h_a}=\frac{2(r+4R)}{a+b+c}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{c(cosA+cosB)}{h_a+h_b}+\frac{a(cosB+cosC)}{h_b+h_c}+\frac{b(cosC+cosA)}{h_c+h_a}=$$

$$=\sum_{cyc}\frac{c(cosA+cosB)}{h_a+h_b}=\sum_{cyc}\frac{c(cosA+cosB)}{\frac{2F}{a}+\frac{2F}{b}}=$$

$$=\frac{1}{2F}\sum_{cyc}\frac{abc(cosA+cosB)}{a+b}=\frac{abc}{4RF}\sum_{cyc}\frac{cosA+cosB}{sinA+sinB}=$$

$$=\frac{abc}{abc}\sum_{cyc}\frac{2cos\frac{A+B}{2}cos\frac{A-B}{2}}{2sin\frac{A+B}{2}cos\frac{A-B}{2}}=\sum_{cyc}cot\frac{A+B}{2}=\sum_{cyc}cot\frac{\pi-C}{2}=$$

$$=\sum_{cyc} tan\frac{C}{2}=\sum_{cyc}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=$$

$$=\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\sum_{cyc}(s-a)(s-b)=$$

$$=\frac{1}{F}\sum_{cyc}(s^2-sa-sb+ab)=$$

$$=\frac{1}{F}\left(3s^2-2s^2-2s^2+\sum_{cyc}ab\right)=\frac{1}{rs}(-s^2+s^2+r^2+4rR)=$$

$$=\frac{1}{rs}\cdot r(r+4R)=\frac{r+4R}{\frac{a+b+c}{2}}=\frac{2(r+4R)}{a+b+c}$$

120. In ΔABC the following relationship holds:

$$\frac{r_a+r_b}{ab}+\frac{r_b+r_c}{bc}+\frac{r_c+r_a}{ca}=\frac{1}{r}+\frac{1}{R}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{r_a+r_b}{ab}+\frac{r_b+r_c}{bc}+\frac{r_c+r_a}{ca}=$$

$$=\sum_{cyc}\frac{r_a+r_b}{ab}=\sum_{cyc}\frac{\frac{F}{s-a}+\frac{F}{s-b}}{ab}=F\sum_{cyc}\frac{\frac{1}{s-a}+\frac{1}{s-b}}{ab}=$$

$$=F\sum_{cyc}\frac{s-b+s-a}{ab(s-a)(s-b)}=F\sum_{cyc}\frac{c}{ab(s-a)(s-b)}=$$

$$=\frac{F}{abc(s-a)(s-b)(s-c)}\sum_{cyc}c^2(s-c)=$$

$$=\frac{Fs}{abcs(s-a)(s-b)(s-c)}\left(s\sum_{cyc}c^2-\sum_{cyc}c^3\right)=$$

$$=\frac{Fs}{abcF^2}\left(s\cdot 2(s^2-r^2-4rR)-2s(s^2-3r^2-6rR)\right)=$$

$$=\frac{s\cdot 2s}{abcF}(s^2-r^2-4rR-s^2+3r^2+6rR)=$$

$$=\frac{s\cdot 2s}{4Rrs\cdot rs}(2r^2+2rR)=\frac{1}{Rr}(R+r)=\frac{1}{r}+\frac{1}{R}$$

121. In any ΔABC the following relationship holds:

$$\frac{a\cos^2\frac{B}{2}+b\cos^2\frac{A}{2}}{\sin A\sin B}+\frac{b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}}{\sin B\sin C}+\frac{c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}}{\sin C\sin A}=\frac{2RF}{r^2}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{a\cos^2\frac{B}{2}+b\cos^2\frac{A}{2}}{\sin A\sin B}+\frac{b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}}{\sin B\sin C}+\frac{c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}}{\sin C\sin A}=$$

$$=\sum_{cyc}\frac{a\cos^2\frac{B}{2}+b\cos^2\frac{A}{2}}{\sin A\sin B}=\sum_{cyc}\frac{\frac{as(s-b)}{ac}+\frac{bs(s-a)}{bc}}{\frac{a}{2R}\cdot\frac{b}{2R}}=$$

$$=4R^2s\sum_{cyc}\frac{\frac{s-b+s-c}{c}}{ab}=\frac{4R^2s}{abc}\sum_{cyc}(2s-b-a)=$$

$$=\frac{4R^2s}{4RF}\sum_{cyc}(a+b+c-b-a)=\frac{Rs}{F}\sum_{cyc}c=\frac{Rs\cdot 2s}{rs}=\frac{2Rs}{r}=\frac{2Rrs}{r^2}=\frac{2RF}{r^2}$$

122. Let F^* –be area of pedal triangle of Bevan's point $(X(40))$ and F –area of ΔABC. Prove that:

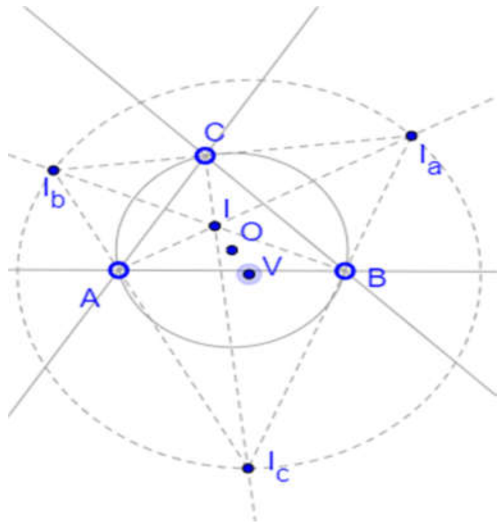
$$\frac{F^*}{F}=\frac{r}{2R}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$OX(40)=\sqrt{R^2-\frac{abc}{a+b+c}}-(J.Butterworth-1806)$$

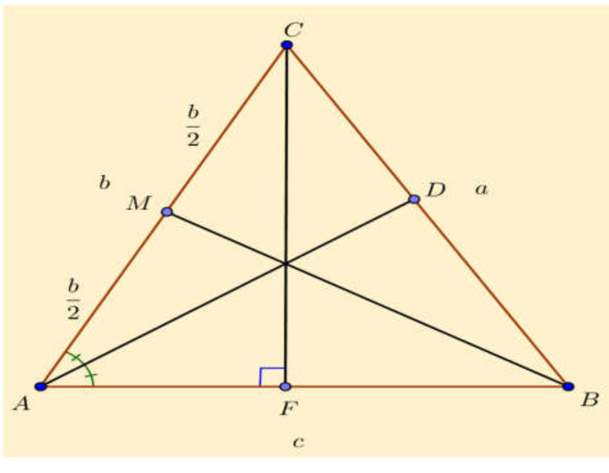
$$F^* = \frac{R^2 - OX(40)^2}{4R^2} \cdot F - (Coxeter, Greitzer - 1967)$$



$$\frac{F^*}{F} = \frac{R^2 - OX(40)^2}{4R^2} = \frac{1}{4} - \frac{1}{4R^2}\left(R^2 - \frac{abc}{a+b+c}\right) =$$

$$= \frac{1}{4R^2} \cdot \frac{abc}{a+b+c} = \frac{4RF}{4R^2 \cdot 2s} = \frac{F}{R \cdot 2s} = \frac{rs}{2sR} = \frac{r}{2R}$$

123.



ΔABC −random, AD −bisector, BM −median, CE −altitude,

AD, BM, CE −concurrent.

$BC = a, CA = b, AB = c$; $a, b, c \in N$. Prove that: $a = b = c$.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by proposer:

$$\frac{1}{cosA} = \frac{b}{c} + 1 \rightarrow \frac{1}{cosA} = \frac{b+c}{c} \rightarrow 2bc(b+c)cosA = 2bc \cdot c \rightarrow$$

$$(b+c)(b^2+c^2-a^2) = 2bc^2 \rightarrow b^3 + c^3 = a^2(b+c) + bc(c-b); (1)$$

$$a, b, c \in N \overset{(1)}{\rightarrow} (\text{wolframalpha}) a = b = c.$$

Solution 2 by proposer

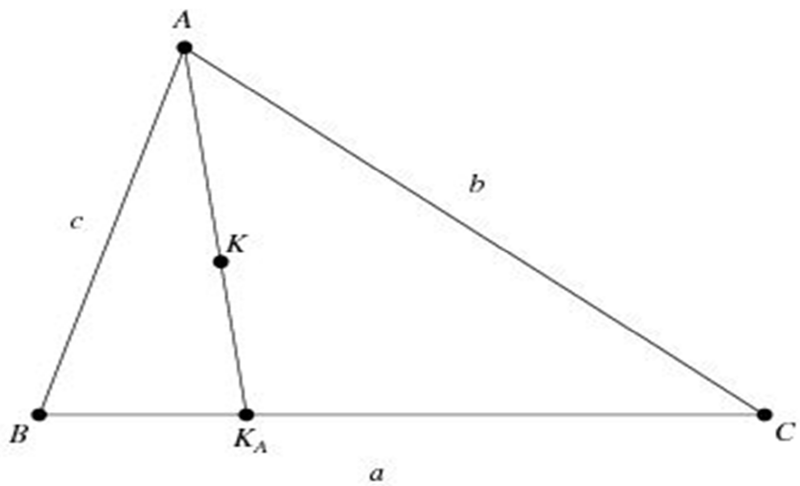
$$\text{PLAGIOGONAL SYSTEM: } AB \equiv Ax; AC \equiv Ay$$

$$AD: x - y = 0, BM: bx + 2cy = 0, CE: x + ycosA = bcosA$$

$$AD, BM, CE - \text{concurrent} \rightarrow \begin{vmatrix} 1 & -1 & 0 \\ b & 2c & bc \\ 1 & cosA & bcosA \end{vmatrix} = 0 \rightarrow cosA = \frac{c}{b+c}; (1)$$

$$\begin{cases} 0 < A < 180^o \\ a, b, c \in N \end{cases} \rightarrow cosA = \frac{1}{2} \overset{(1)}{\rightarrow} \begin{cases} b + c = 2c \\ \angle A = 60^o \end{cases} \rightarrow a = b = c.$$

124.



Let ΔDEF be the pedal triangle of symmedian point in ΔABC.
If $m(\sphericalangle A) = 90^\circ$ then:

$$[DEF] = \frac{3h_a^3}{8a}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2} = R^2 - \frac{3a^2b^2c^2}{(a^2+a^2)^2} = R^2 - \frac{3a^2b^2c^2}{4a^4} =$$

$$=\frac{a^2}{4}-\frac{3b^2c^2}{4a^2}$$

$$[DEF]=\frac{F}{4R^2}(R^2-OK^2)=\frac{F}{4\cdot\frac{a^2}{4}}\left(R^2-R^2+\frac{3b^2c^2}{4a^2}\right)=$$

$$=\frac{a\cdot h_a}{2a^2}\cdot\frac{3}{4}\cdot\left(\frac{bc}{a}\right)^2=\frac{3h_a}{8a}\cdot h_a^2=\frac{3h_a^3}{8a}$$

125. Let Δ_1,Δ_2 −be areas of the pedal triangles of Bevan's point and incenter in ΔABC. Prove that: $\Delta_1=\Delta_2$

Proposed by Mehmet Șahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\Delta_1=\frac{R^2-OV^2}{4R^2}\cdot F=\frac{R^2-R^2+\frac{abc}{a+b+c}}{4R^2}\cdot F=\frac{abc}{a+b+c}\cdot\frac{F}{4R^2}=\frac{4Rrs}{2s}\cdot\frac{F}{4R^2}=\frac{rF}{2R}$$

$$\Delta_2=\frac{R^2-OI^2}{4R^2}\cdot F=\frac{R^2-R^2+2Rr}{4R^2}\cdot F=\frac{rF}{2R}$$

$$V-Bevan's\ point, I-incenter$$

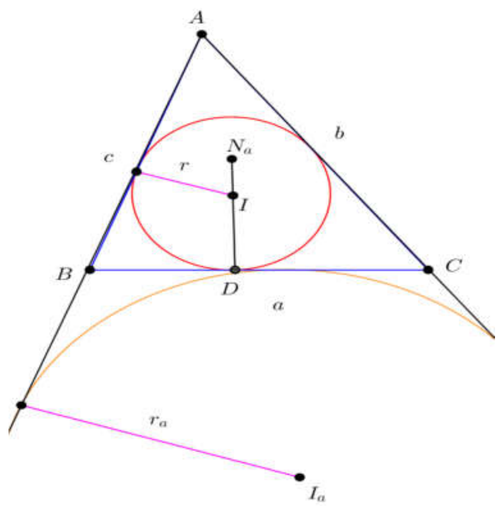
126. $\Delta ABC, F$ −area of $\Delta ABC, N_a$ −Nagel's point

$$DN_a\perp BC, DN_a=d$$

$$r, r_a\ \text{−inradius, exradius of } \Delta ABC$$

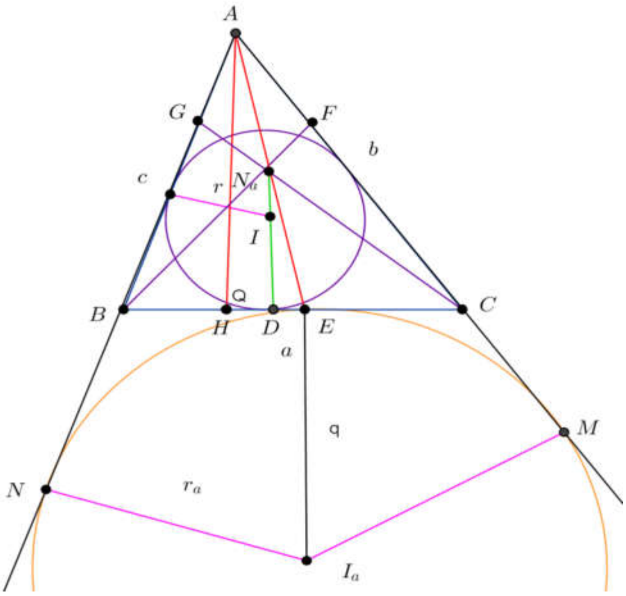
Prove that:

$$\frac{ad}{F}\cdot\frac{r_a}{r}=2$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil



$$2s = a + b + c; F = \frac{1}{2}BC \cdot AH = \frac{1}{2}ah_a; h_a = \frac{2F}{a}$$

$$AN = AM = s; BN = BE = AF = s - c$$

$$CM = CE = AG = s - b; CF = BG = s - a; AH = h_a$$

Applying Menelaus theorem in ΔABE, it follows that:

$$\frac{AG}{BG} \cdot \frac{BC}{CE} \cdot \frac{EN_a}{AN_a} = 1 \leftrightarrow \frac{s-b}{s-a} \cdot \frac{a}{s-b} \cdot \frac{EN_a}{AN_a} = 1 \to \frac{EN_a}{AN_a} = \frac{s-a}{a} \to \frac{AN_a}{EN_a} = \frac{a}{s-a}$$

$$AN_a + EN_a = AE; aEN_a = sAN_a - aAN_a \to \frac{AE}{AN_a} = \frac{s}{a} \to AN_a = \frac{a}{s-a} \cdot EN_a$$

$$\text{So,} \frac{AE}{\frac{a}{s-a} \cdot EN_a} = \frac{s}{a} \to \frac{AE}{EN_a} = \frac{s}{s-a}$$

Now, $\Delta AHE \sim \Delta N_aDE \to \frac{AH}{N_aD} = \frac{AE}{N_aE} \to \frac{h_a}{DN_a} = \frac{AE}{EN_a} \to \frac{h_a}{d} = \frac{s}{s-a} \to \frac{2F}{ad} = \frac{s}{s-a} \to \frac{ad}{F} = \frac{2(s-a)}{s}$

$$F + (s-a)r_a = sr \to \frac{r_a}{r} = \frac{s}{s-a}$$

Therefore, $\frac{ad}{F} \cdot \frac{r_a}{r} = \frac{2(s-a)}{s} \cdot \frac{s}{s-a} = 2 \to \frac{ad}{F} \cdot \frac{r_a}{r} = 2$

Solution 2 by Juan Jose Isach Mayo-Spain

Let D –be tha octogonal projection of A on BC.

$$BD = AB \cdot cosB = c \cdot \frac{a^2 + c^2 - b^2}{2ac} = \frac{a^2 + c^2 - b^2}{2a}; AD = \frac{2S_{\Delta ABC}}{a}$$

Let I − be the incenter of ΔABC, ω −the incircle of ABC, $G = \omega \cap BC$; $GH = 2r = \frac{2S_{\Delta ABC}}{s}$

$$DG = BG - BD = \frac{a - b + c}{2} - \frac{a^2 + c^2 - b^2}{2a} = \frac{(b - c)(b - a + c)}{2a} == \frac{b - c}{a} \cdot (s - a) = ST$$

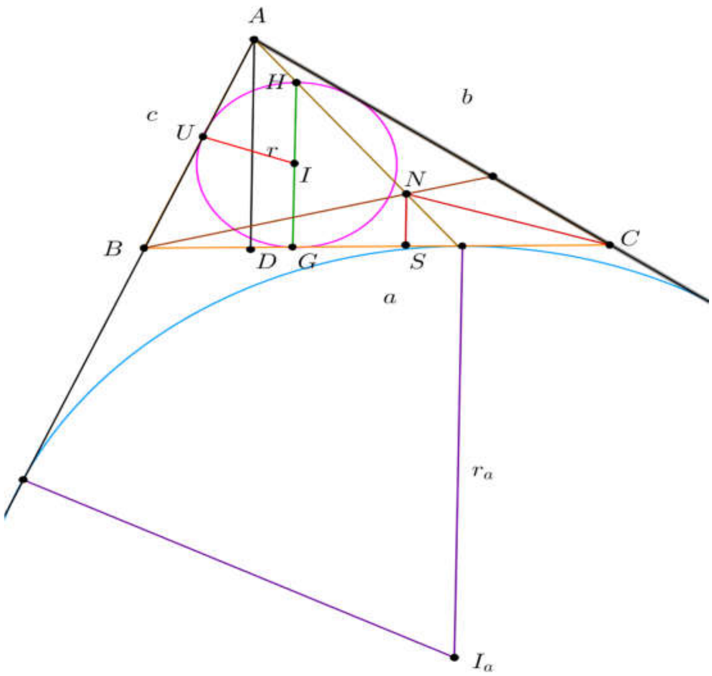
$$GT = b - c; DT = DG + GT = (b - c)\left(\frac{s - a}{a} + 1\right) = \frac{(b - c)s}{a}$$

$$\frac{NS}{AD} = \frac{ST}{DT} = \frac{\frac{b - c}{a}(s - a)}{\frac{(b - c)s}{a}} = \frac{s - a}{s}; NS = \frac{s - a}{s} \cdot \frac{2S_{\Delta ABC}}{a}$$

Let r − be the radius of the incircle of ΔABC; $\frac{r}{AU} = \frac{r_a}{AP} \leftrightarrow \frac{r}{r_a} = \frac{s-a}{s}$.

Let r_a −be the radius of the A −excircle of ABC.

$$\frac{a \cdot NS}{S_{ABC}} \cdot \frac{r_a}{r} = \frac{a(s - a)}{s \cdot S_{ABC}} \cdot \frac{2S_{ABC}}{a} \cdot \frac{s}{s - a} = 2$$



127. In scalene ΔABC the following relationship holds:

$$\frac{\sin^2 A-\sin^2 B}{r_a-r_b}+\frac{\sin^2 B-\sin^2 C}{r_b-r_c}+\frac{\sin^2 C-\sin^2 A}{r_c-r_a}=\frac{R+r}{R^2}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{\sin^2 A-\sin^2 B}{r_a-r_b}+\frac{\sin^2 B-\sin^2 C}{r_b-r_c}+\frac{\sin^2 C-\sin^2 A}{r_c-r_a}=$$

$$=\sum_{cyc}\frac{\sin^2 A-\sin^2 B}{r_a-r_b}=\sum_{cyc}\frac{\frac{a^2}{4R^2}-\frac{b^2}{4R^2}}{\frac{F}{s-a}-\frac{F}{s-b}}=\frac{1}{4R^2F}\sum_{cyc}\frac{a^2-b^2}{\frac{1}{s-a}-\frac{1}{s-b}}=$$

$$=\frac{1}{4R^2F}\sum_{cyc}\frac{(a-b)(a+b)(s-a)(s-b)}{s-b+s-a}=$$

$$=\frac{1}{4R^2F}\sum_{cyc}(a+b)(s-a)(s-b)=\frac{1}{4R^2F}\sum_{cyc}(a+b)(s^2-s(a+b)+ab)=$$

$$=\frac{1}{4R^2F}\sum_{cyc}(a+b)(s^2-s(2s-c)+ab)=\frac{1}{4R^2F}\sum_{cyc}(a+b)(s^2-2s^2+sc+ab)=$$

$$=\frac{1}{4R^2F}\left(s\sum_{cyc}c(a+b)-s^2\sum_{cyc}(a+b)+\sum_{cyc}ab(2s-c)\right)=$$

$$=\frac{1}{4R^2F}\left(2s\sum_{cyc}ab-4s^3+2s\sum_{cyc}ab-3abc\right)=$$

$$=\frac{1}{4R^2F}(4s(s^2+r^2+4Rr)-4s^3-12Rrs)=$$

$$=\frac{1}{4R^2F}(4s^3-4s^3+4sr^2+16Rrs-12Rrs)=$$

$$=\frac{1}{4R^2rs}(4sr^2+4Rrs)=\frac{1}{R^2}(r+R)=\frac{R+r}{R^2}$$

128. **In ΔABC the following relationship holds:**

$$\frac{h_a+h_b}{cosA+cosB}+\frac{h_b+h_c}{cosB+cosC}+\frac{h_c+h_a}{cosC+cosA}=2(4R+r)$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{h_a+h_b}{cosA+cosB}+\frac{h_b+h_c}{cosB+cosC}+\frac{h_c+h_a}{cosC+cosA}=$$

$$=\sum_{cyc}\frac{h_a+h_b}{cosA+cosB}=\sum_{cyc}\frac{\frac{2F}{a}+\frac{2F}{b}}{\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}}=$$

$$=2F\sum_{cyc}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)\cdot 2abc}{a(b^2+c^2-a^2)+b(c^2+a^2-b^2)}=$$

$$=4abcF\sum_{cyc}\frac{\frac{a+b}{ab}}{ab(a+b)+c^2(a+b)-(a^3+b^3)}=$$

$$=4abcF\sum_{cyc}\frac{\frac{1}{ab}}{ab+c^2-a^2-b^2+ab}=$$

$$=4abcF\sum_{cyc}\frac{1}{ab(c^2-(a-b)^2)}=4abcF\sum_{cyc}\frac{1}{ab(c-a+b)(c+a-b)}=$$

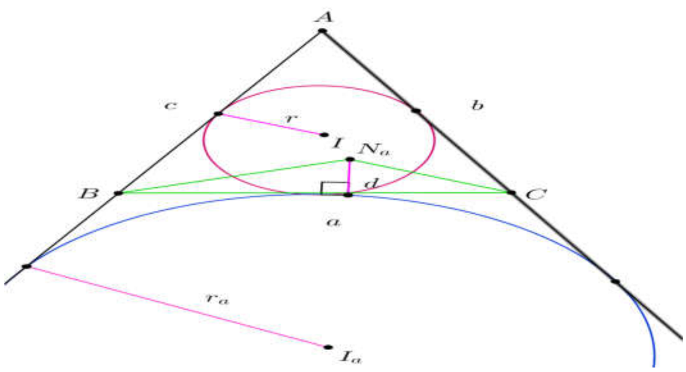
$$=4F\sum_{cyc}\frac{c}{4(s-a)(s-b)}=F\sum_{cyc}\frac{a}{(s-b)(s-c)}=F\cdot\frac{2(4R+r)}{F}=2(4R+r)$$

129.

N_a –Nagel's point.

Prove that:

$$\frac{[BCN_a]}{[BCA]}=\frac{r}{r_a}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil

$$BN = BE = AF = s - c$$

$$CM = CE = AG = s - b$$

$$GF = BG = s - a$$

Applying the Menelaus theorem to the ΔABE, we have:

$$\frac{AG}{BG}\cdot\frac{BC}{CE}\cdot\frac{EN_a}{AN_a} = 1 \Rightarrow \frac{s-b}{s-a}\cdot\frac{a}{s-b}\cdot\frac{EN_a}{AN_a} = 1 \Rightarrow \frac{EN_a}{AN_a} = \frac{s-a}{a} \Rightarrow \frac{AN_a}{EN_a} = \frac{a}{s-a}$$

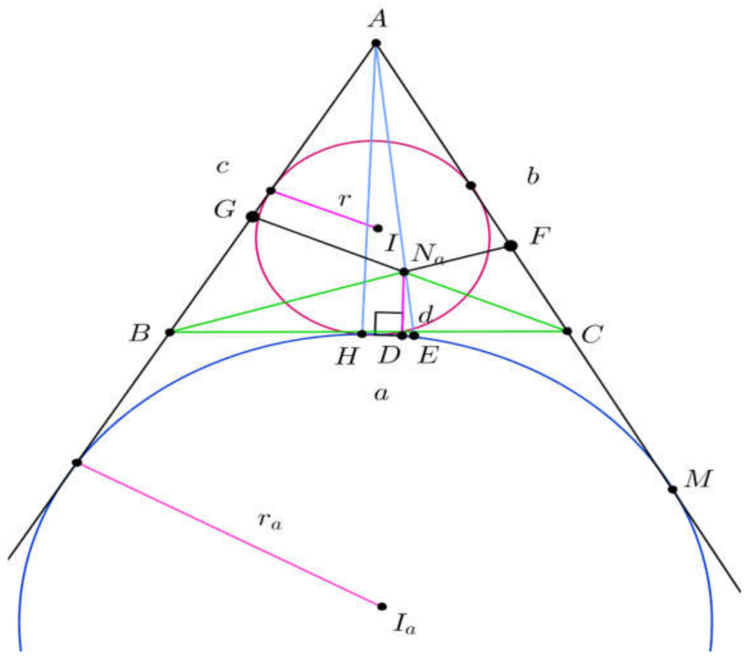
$$AN_a + EN_a = AE, aEN_a = sAN_a - aAN_a \Rightarrow \frac{AE}{AN_a} = \frac{s}{a} \Rightarrow AN_a = \frac{a}{s-a}EN_a$$

So, $\frac{AE}{\frac{a}{s-a}EN_a} = \frac{s}{a} \Rightarrow \frac{AE}{EN_a} = \frac{s}{s-a}$. **Now,**

$$\Delta AHE \sim \Delta N_aDE \Rightarrow \frac{N_aD}{AH} = \frac{N_aE}{AE} \Rightarrow \frac{N_aD}{AH} = \frac{s-a}{s}$$

$$\frac{[BCN_a]}{[BCA]} = \frac{\frac{1}{2}N_aD\cdot a}{\frac{1}{2}AH\cdot a} = \frac{N_aD}{AH} = \frac{s-a}{s}, [BCA] = sr = (s-a)r_a \Rightarrow \frac{s-a}{s} = \frac{r}{r_a}$$

Therefore: $\frac{[BCN_a]}{[BCA]} = \frac{r}{r_a}$



Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let x, y, z —be the complexe coordinates of points A, B and C respectively.

$$z_{N_a} = \frac{s-a}{s}x + \frac{s-b}{b}y + \frac{s-c}{c}z$$

$$[N_aBC] = \frac{1}{2}\left|Im\left(\overline{z_{N_a}}y + \overline{y}z + \overline{z}z_{N_a}\right)\right| =$$

$$= \frac{1}{2}\left|Im\left(\frac{s-a}{s}\overline{x}y + \left(1 - \frac{s-c}{s} - \frac{s-b}{s}\right)\overline{y}z + \frac{s-a}{s}\overline{z}x\right)\right| =$$

$$= \frac{s-a}{s} \cdot \frac{1}{2}\left|Im(\overline{x}y + \overline{y}z + \overline{z}x)\right| = \frac{r}{r_a}[ABC]$$

Therefore,

$$\frac{[BCN_a]}{[BCA]} = \frac{r}{r_a}$$

Solution 3 by proposer

Plagiogonal system: $BC = Bx, BA = By, B(0,0)$

$$N_a(n_1, n_2), n_2 = \frac{s-a}{s} \cdot c. d = n_2 sinB \Rightarrow d = \frac{s-a}{s} \cdot csinB; (1)$$

$$h_a = csinB \overset{(1)}{\Rightarrow} h_a = \frac{sd}{s-a} \Rightarrow \frac{d}{h_a} = \frac{s-a}{s} \Rightarrow \frac{[BCN_a]}{[BCA]} = \frac{r}{r_a}$$

Note: $\frac{[BCN_a]}{[ABC]} + \frac{[CAN_a]}{[ABC]} + \frac{[ABN_a]}{[ABC]} = \frac{r}{r_a} + \frac{r}{r_b} + \frac{r}{r_c} = 1 \rightarrow \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$

130. In any ΔABC the following relationship holds:

$$\sum_{cyc} tan\frac{A}{2} + 2020 \cdot \sum_{cyc} \frac{b-c}{a}cos^2\frac{A}{2} + 2021 \cdot \sum_{cyc} asin(B-C) = \frac{\sum cos^2\frac{A}{2}}{2\prod cos\frac{A}{2}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Samar Das-India

$$\sum_{cyc} tan\frac{A}{2} + 202 \cdot \sum_{cyc} \frac{b-c}{a}cos^2\frac{A}{2} + 2021 \cdot \sum_{cyc} asin(B-C) =$$

$$= \sum_{cyc} tan\frac{A}{2} + 2020 \cdot \sum_{cyc} \frac{b-c}{a} \cdot \frac{s(s-a)}{bc} + \sum_{cyc} asin(B-C) =$$

$$=\sum_{cyc}tan\frac{A}{2}+2020\frac{s}{abc}\sum_{cyc}(b-c)(s-a)+2020\cdot 2R\sum_{cyc}sinAsin(B-C)=$$

$$=\sum_{cyc}tan\frac{A}{2}+2020\frac{s}{abc}\sum_{cyc}(b-c)(s-a)+2020\cdot 2R\sum_{cyc}(sin^2B-sin^2C)=$$

$$=\sum_{cyc}tan\frac{A}{2}=\frac{\sum sin\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}}{\prod cos\frac{A}{2}}=\frac{2cos\frac{C}{2}sin\frac{A+B}{2}+2sin\frac{C}{2}cos\frac{A}{2}cos\frac{B}{2}}{\prod cos\frac{A}{2}}=$$

$$=\frac{2cos^2\frac{C}{2}+sin\frac{C}{2}\left(sin\frac{C}{2}+cos\frac{A-B}{2}\right)}{2\prod cos\frac{A}{2}}=\frac{2cos^2\frac{C}{2}+1-cos^2\frac{C}{2}+cos\frac{A+B}{2}cos\frac{A-B}{2}}{2\prod cos\frac{A}{2}}$$

$$=\frac{1+cos^2\frac{C}{2}+cos^2\frac{A}{2}-sin^2\frac{B}{2}}{2\prod cos\frac{A}{2}}=\frac{\sum cos^2\frac{A}{2}}{2\prod\frac{cosA}{2}}$$

Solution 2 by Daniel Sitaru-Romania

$$\sum_{cyc}asin(B-C)=2R\sum_{cyc}sinAsin(B-C)=$$

$$=R\sum_{cyc}(cos(A+B-C)-cos(A-B+C))=R\sum_{cyc}(cos(\pi-2C)-cos(\pi-2B))=$$

$$=-R\sum_{cyc}cos2C+R\sum_{cyc}cos2B=0$$

$$\sum_{cyc}\frac{b-c}{a}cos^2\frac{A}{2}=\sum_{cyc}\frac{b-c}{a}\cdot\frac{s(s-a)}{bc}=\frac{s}{abc}\sum_{cyc}(b-c)(s-a)=$$

$$=\frac{s}{abc}\left(s\sum_{cyc}(b-c)-\sum_{cyc}a(b-c)\right)=0$$

$$LHS=\sum_{cyc}tan\frac{A}{2}=\frac{4R+r}{s}$$

$$RHS=\sum_{cyc}cos^2\frac{A}{2}\cdot\left(2\prod_{cyc}cos\frac{A}{2}\right)^{-1}=\left(2+\frac{r}{2R}\right)\left(2\cdot\frac{s}{4R}\right)^{-1}=$$

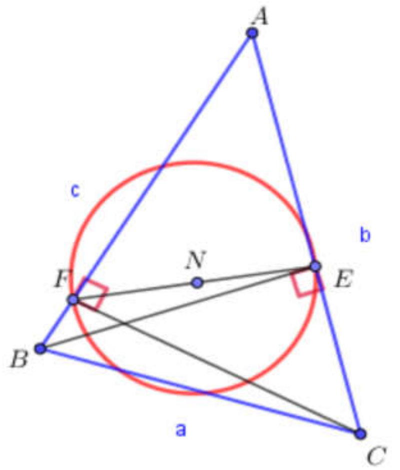
$$= \frac{4R+r}{2R} \cdot \frac{2R}{s} = \frac{4R+r}{s}$$

131. $\Delta ABC: BE \perp AC, CF \perp AB$

N −midpoint of FE

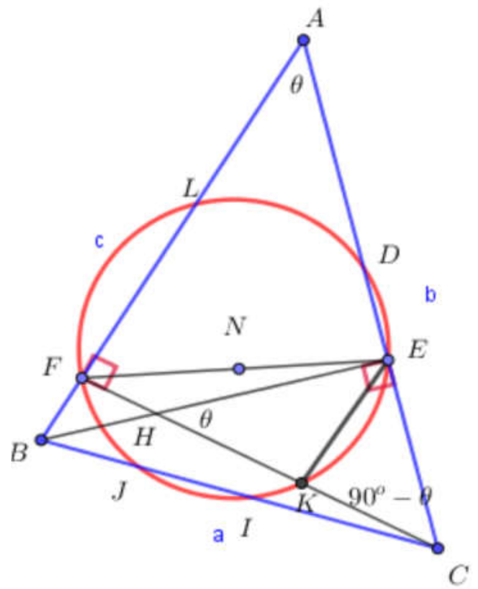
N −is center of nine points circle

Prove that: $\mu(\angle A) = 45^o$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreiera Queiroz-Olinda-Brazil



$$\angle A = \theta \to \angle FCE = 90^o - \theta, L, D, E, K, J, F - \text{cyclic}$$

FE −is the diameter of the nine point circle, $CK = KH$(property of the nine point circle)

In ΔEHC: $EK^2 = KH \cdot CK \to EK = KH = CK \to \theta = 45^o$

Solution 2 by proposer

$$AB = Ax, AC = Ay$$
$$AF = b \cdot cosA, AE = c \cdot cosA, A(0,0), F(b \cdot cosA, 0), E(0, c \cdot cosA),$$

$$M\left(\frac{b}{2}cosA, \frac{c}{2}cosA\right)$$

Let O_9 –the center of NPC, $O_9\left(\frac{bcosA-ccos2A}{4sin^2A}, \frac{ccosA-bcos2A}{4sin^2A}\right)$

Must: $\begin{cases}\frac{bcosA}{2} = \frac{bcosA-ccos2A}{4sin^2A}\\ \frac{ccosA}{2} = \frac{ccosA-bcos2A}{4sin^2A}\end{cases} \rightarrow \begin{cases}cos2A(c-bcosA) = 0\\ cos2A(b-cosA) = 0\end{cases}$

$$cos2A = 0; (1) or\ (c - bcosA = 0, b - ccosA = 0); (2)$$

$$(1) \rightarrow cos2a = 0 \rightarrow 2A = 90^o \rightarrow \mu(\angle A) = 45^o$$

$$(2) \rightarrow cosA = \frac{c}{b}, cosA = \frac{b}{c} \rightarrow cosA = 1; (0 < A < 180^o)$$

132.

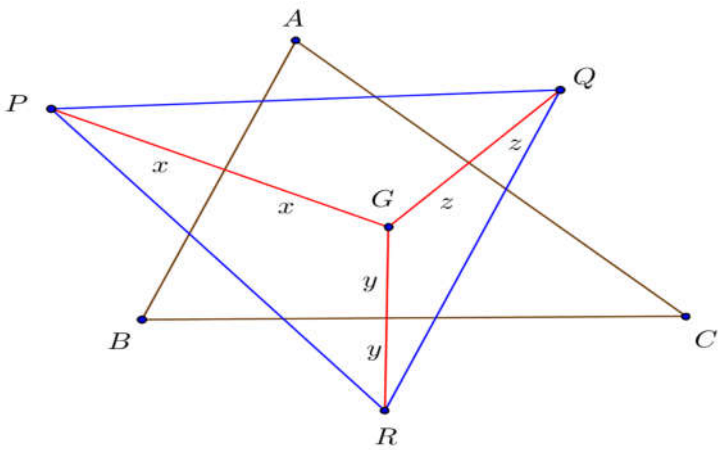
G –centroid of ΔABC

P, Q, R –symmetrical points of G

to BC, CA, AB respectively.

If $\frac{[PQR]}{[ABC]} = 1$, then

ΔABC –equilateral



If $\frac{[PQR]}{[ABC]} = \frac{8}{9}$, then ΔABC –right

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil