$$We\ have:\ \mu(I_bAC)=\frac{\pi-A}{2}\to\mu(AI_bV)=\frac{\pi}{2}-\mu(I_bAC)=\frac{A}{2}\ and\ VI_b=2R$$

$$Now,\qquad \sin AI_bV=\frac{VK}{VI_b}\to VK=VI_b.\sin AI_bV=2R.\sin\frac{A}{2}\ (and\ analogs)$$

$$\to\sum\frac{a}{bc.VK^2}=\sum\frac{a}{bc.\left(2R.\sin\frac{A}{2}\right)^2}=4\sum\frac{a\cos^2\frac{A}{2}}{bc.\left(4R.\cos\frac{A}{2}\sin\frac{A}{2}\right)^2}=4\sum\frac{a\cos^2\frac{A}{2}}{bc.a^2}=$$

$$=\frac{4}{abc}\sum\cos^2\frac{A}{2}=\frac{2}{abc}\sum(1+\cos A)=\frac{1}{2RF}\left(4+\frac{r}{R}\right)=\frac{4R+r}{2R^2F}=\frac{r_a+r_b+r_c}{2R^2F}$$

Therefore,

$$\frac{a}{bc.VK^2}+\frac{b}{ca.VL^2}+\frac{c}{ab.VM^2}=\frac{r_a+r_b+r_c}{2R^2F}$$

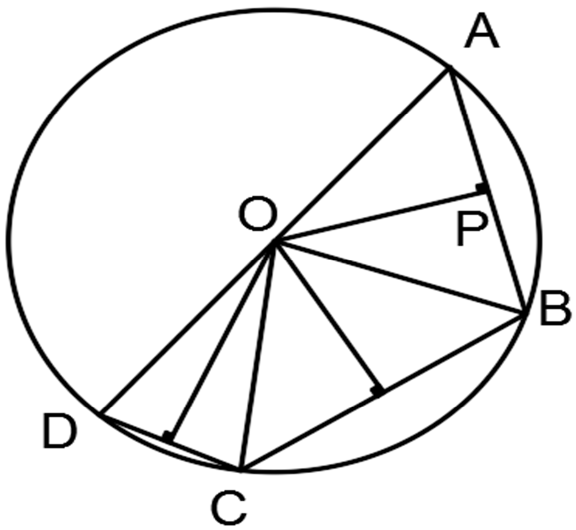
183. $ABCD$ **−cyclic quadrilateral,** R **−circumradii.**

If $AB=a,BC=b,CA=c,DA=2R,$

then : $R^3\geq abc$. **When equality holds ?**

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$Since\ AD=2R\ then\ AD\ is\ a\ diameter\ of\ the\ circumcircle\ of\ ABCD.$$

$$Let\ O\ be\ the\ crcumcenter\ of\ ABCD\ and\ P=pr_{AB}(O),Q=pr_{BC}(O),R=pr_{CD}(O)$$

$$Let\ \alpha=\mu(AOP)=\mu(POB),\beta=\mu(BOQ)=\mu(QOC),\gamma=\mu(COR)=\mu(ROD).$$

$$We\ have:\ \alpha+\beta+\gamma=\frac{\pi}{2}\ and\ a=2R\sin\alpha\ (and\ analogs)$$

$$\rightarrow abc=(2R)^3.\prod\sin\alpha\overset{AM-GM}{\leq}8R^3.\left(\frac{1}{3}\sum\sin\alpha\right)^3\overset{Jensen}{\leq}8R^3.\left(\sin\left(\frac{1}{3}\sum\alpha\right)\right)^3,(\because x\rightarrow\sin x-concave)$$

$$\rightarrow abc\leq 8R^3.\left(\sin\left(\frac{\pi}{6}\right)\right)^3=8R^3.\left(\frac{1}{2}\right)^3=R^3$$

$$Equality\ holds\ when:\ \sin\alpha=\sin\beta=\sin\gamma\leftrightarrow\alpha=\beta=\gamma=\frac{\pi}{6}\rightarrow a=2R\sin\frac{\pi}{6}=R\ (and\ analogs)$$

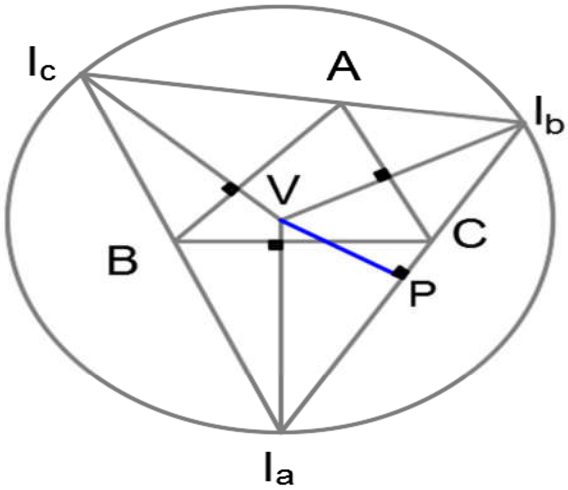
$$Therefore,\qquad R^3\geq abc,\qquad equality\ holds\ when\ a=b=b=R.$$

184. In $\Delta ABC, V$ −Bevan' spoint, I_a, I_b, I_c −excenters, R_a, R_b, R_c −circumradii f $\Delta VI_bI_c, \Delta VI_cI_a, \Delta VI_aI_b$ .Prove that :

$$\sum\frac{1}{{R_a}^2}=\frac{2R-r}{2R^3}$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$Let\ P\ be\ the\ feet\ of\ the\ perpendicular\ from\ V\ to\ I_aI_b.$$

$$We\ have:\ \mu(I_aCB)=\frac{\pi-C}{2}\rightarrow\mu(CI_aV)=\frac{\pi}{2}-\mu(I_aCB)=\frac{C}{2}$$

$$\Delta VI_aI_b \text{ is isosceles} \to \mu(I_aVI_b) = \pi - 2\mu(CI_aV) = \pi - C \text{ and } I_aP = \frac{I_aI_b}{2}$$

$$\to \cos VI_aI_b = \frac{I_aP}{I_aV} = \frac{I_aI_b}{2.2R} \to I_aI_b = 4R\cos\frac{C}{2}, \quad (\because I_aV = 2R)$$

$$\text{In } \Delta VI_aI_b, \text{we have: } \sin I_aVI_b = \frac{I_aI_b}{2R_c} \to R_c = \frac{I_aI_b}{2\sin I_aVI_b} = \frac{4R\cos\frac{C}{2}}{2\sin(\pi - C)} = \frac{2R\cos\frac{C}{2}}{\sin C}$$

$$= R\csc\frac{C}{2} \to R_a = R\csc\frac{A}{2} \ (\text{and analogs})$$

$$\text{Now,} \quad \sum\frac{1}{{R_a}^2} = \sum\frac{1}{\left(R\csc\frac{A}{2}\right)^2} = \frac{1}{R^2}\sum\sin^2\frac{A}{2} = \frac{1}{2R^2}\sum(1-\cos A)$$

$$= \frac{1}{2R^2}\left(2 - \frac{r}{R}\right)$$

Therefore,

$$\sum\frac{1}{{R_a}^2} = \frac{2R - r}{2R^3}$$

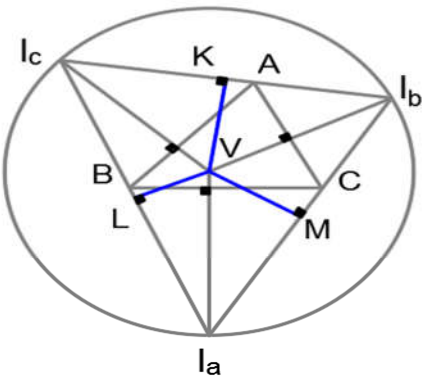
185. Prove that :

$$\frac{h_bh_c}{VK^2} + \frac{h_ch_a}{VL^2} + \frac{h_ah_b}{VM^2} = \frac{s^2}{R^2}$$

V –Bevan's point, $VK = d(V, BC), VL = d(V, CA), VM = d(V, AB)$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$\text{We have: } \mu(I_bAC) = \frac{\pi - A}{2} \to \mu(AI_bV) = \frac{\pi}{2} - \mu(I_bAC) = \frac{A}{2} \text{ and } VI_b = 2R.$$

$$Now,\quad \sin AI_bV=\frac{VK}{VI_b}\to VK=VI_b.\sin AI_bV=2R.\sin\frac{A}{2}\ (and\ analogs)$$

$$\to\sum\frac{h_bh_c}{VK^2}=(2sr)^2\sum\frac{1}{bc\left(2R.\sin\frac{A}{2}\right)^2}=\frac{(2sr)^2}{abc}\sum\frac{a}{\left(2R.\sin\frac{A}{2}\right)^2}$$

$$=\frac{(2sr)^2}{4sRr}\sum\frac{4R\cos\frac{A}{2}.\sin\frac{A}{2}}{4R^2\sin^2\frac{A}{2}}$$

$$=\frac{sr}{R^2}\sum\frac{1}{\tan\frac{A}{2}}\overset{r_a=s\tan\frac{A}{2}}{\cong}\frac{sr}{R^2}\sum\frac{s}{r_a}=\frac{s^2r}{R^2}.\frac{1}{r}=\frac{s^2}{R^2}.\ \left(\because\sum\frac{1}{r_a}=\frac{1}{r}\right)$$

$$Therefore,\quad \frac{h_bh_c}{VK^2}+\frac{h_ch_a}{VL^2}+\frac{h_ah_b}{VM^2}=\frac{s^2}{R^2}$$

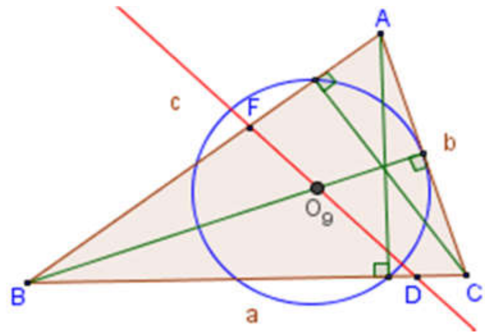
186. In ΔABC: $45^\circ<B<90^\circ$

O_9 –N.P.C. of $\Delta ABC, D\in BC, F\in BA$

$$BC\left(\frac{1}{BC}+\frac{1}{DF}\right)=\frac{4(1+cos\ B)}{1+2cos\ B}$$

Prove that:

ΔABC –isoscelles.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

Let $BD=d, BF=f$. *We know that* D, O_9, *and* F *are collinear, so*

$$\left(\frac{BC}{BF}+\frac{BA}{BD}\right)cosB-\left(\frac{BC}{BD}+\frac{BA}{BF}-2\right)cos2B=2$$

$$\left(\frac{a}{f}+\frac{c}{d}\right)cosB-\left(\frac{a}{d}+\frac{c}{f}-2\right)(2cos^2B-1)=1$$

$$\left(\frac{a}{f}+\frac{a}{d}+\frac{c}{d}-\frac{a}{d}\right)cosB-\left(\frac{a}{d}+\frac{a}{f}+\frac{c}{f}-\frac{a}{f}-2\right)(2cos^2B-1)=2$$

$$\left(\frac{a}{f}+\frac{a}{d}\right)cosB+\left(\frac{c}{d}-\frac{a}{d}\right)cosB-\left(\frac{a}{d}+\frac{a}{f}\right)(2cos^2B-1)-\left(\frac{c}{f}-\frac{a}{f}\right)(2cos^2B-1)$$
$$+2(2cos^2B-1)=2$$

$$cosB\cdot\frac{4(1+cosB)}{1+2cosB}+\left(\frac{c}{d}-\frac{a}{d}\right)cosB-\frac{4(1+cosB)}{1+2cosB}(2cos^2B-1)$$
$$+\left(\frac{c}{f}-\frac{a}{f}\right)(2cos^2B-1)+4(cos^2B-1)=0$$

$$\frac{c-a}{d}cosB-\frac{c-a}{f}(2cos^2B-1)+$$
$$+\frac{1+cosB}{1+2cosB}[4cosB-8cos^2B+4+4(cosB-1)(2cosB+1)]=0$$

$$\frac{c-a}{d}cosB-\frac{c-a}{f}(2cos^2B-1)+$$
$$+\frac{1+cosB}{1+2cosB}[4cosB-8cos^2B+4+8cos^2B-4cosB=4]=0$$

$$(c-a)\cdot\frac{cosB}{d}-2(c-a)\frac{cos^2B}{f}+\frac{c-a}{f}=0$$

$$(c-a)\left(\frac{2}{f}cos^2B-\frac{1}{d}cosB-\frac{1}{f}\right)=0$$

$$So,\frac{2}{f}cos^2B-\frac{1}{d}cosB-\frac{1}{f}=0\ or\ (c-a)=0\rightarrow a=c->$$

Therefore, ΔABC –isoscelles.

187. **V – Bevan's point, I_a, I_b, I_c – excenters in $\Delta ABC, R_a, R_b, R_c$ – circumradii of**

$\Delta VI_bI_c, \Delta VI_cI_a, \Delta VI_aI_b$. Prove that:

$$\frac{a}{R_a^2}+\frac{b}{R_b^2}+\frac{c}{R_c^2}=\frac{[I_aI_bI_c]-2F}{2R^3}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Izumi Ainsworth-Lima-Peru

We know that: $[I_aI_bI_c]=2pR\ ...(\psi)$

Now: $[VI_bI_c]=\frac{4R\cos\left(\frac{A}{2}\right)(2R)^2}{4R_a}=\frac{(2R)^2\sin(A)}{2}\Rightarrow R_a=\frac{2R\cos\left(\frac{A}{2}\right)}{\sin(A)}=\frac{R}{\sin\left(\frac{A}{2}\right)}$

In the problem:

$$\sum_{cyc}\frac{a}{R_a^2}=\frac{1}{R^2}\sum_{cyc}a\sin^2\left(\frac{A}{2}\right)=\frac{1}{R^2}\sum_{cyc}a\left(\frac{1-\cos(A)}{2}\right)$$

$$=\frac{1}{4R^3}\sum_{cyc}a(r_a-r)=\frac{1}{4R^3}\left(\sum_{cyc}ar_a-2pr\right)$$

$$=\frac{1}{4R^3}\left[pr\left(\frac{4pR}{F}-2\right)-2pr\right]=\frac{1}{2R^3}\left(2pR-2\underbrace{pr}_{F}\right)\dots(\lambda)$$

$$(\psi)\to(\lambda):$$

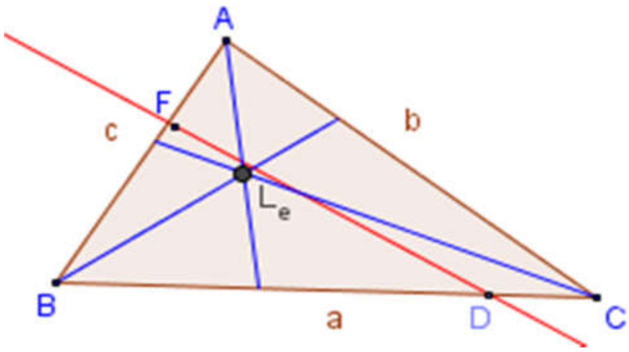
$$\therefore\sum_{cyc}\frac{a}{R_a^2}=\frac{[I_aI_bI_c]-2F}{2R^3}$$

188. L_e —Lemoine's point of ΔABC.

Prove that:

D,E,F —collinear⇔

$$\frac{BC}{BF}+\frac{BA}{BD}=\frac{AB^2+BC^2+CA^2}{BC\cdot BA}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Querioz-Olinda-Brazil

$$BC=a,AC=b,AB=c,BD=d,BF=y$$

$$AL_a \text{ is a symmedian, so: } \frac{BL_a}{L_aC}=\frac{c^2}{b^2} \text{ and } BL_a+L_aC=a, \text{ then}$$

$$BL_a=\frac{ac^2}{b^2+c^2},CL_a=\frac{ab^2}{b^2+c^2}$$

$$\text{In the same way: } AL_c=\frac{cb^2}{a^2+b^2} \text{ and } BL_c=\frac{ca^2}{a^2+b^2}$$

Applying the Menelaus Theorem, we have"

$$\frac{AL_c}{BL_c}\cdot\frac{BC}{CL_a}\cdot\frac{L_aL_e}{AL_e}=1\Leftrightarrow\frac{\frac{cb^2}{a^2+b^2}}{\frac{ca^2}{a^2+b^2}}\cdot\frac{a}{\frac{ab^2}{b^2+c^2}}\cdot\frac{L_aL_e}{AL_e}=1,so\ \frac{L_aL_e}{AL_e}=\frac{a^2}{b^2+c^2}$$

Now, consider D,L_e and F collinear, we will have:

$$\frac{AF}{FB}\cdot\frac{BD}{DL_a}\cdot\frac{L_aL_e}{AL_e}=1\ \ (Menelaus\ Theorem)$$

$$\frac{c-y}{y}\cdot\frac{d}{d-\frac{ac^2}{b^2+c^2}}\cdot\frac{a^2}{b^2+c^2}=1$$

$$a^2d(c-y)=y(db^2+dc^2-ac^2)$$

$$a^2dc-a^2dy=ydb^2+ydc^2-yac^2$$

$$a^2dc+ac^2y=yd(a^2+b^2+c^2)$$

$$\frac{a}{y}+\frac{c}{d}=\frac{a^2+b^2+c^2}{ac}.$$

Therefore,

$$\frac{BC}{BF}+\frac{BA}{BD}=\frac{AB^2+BC^2+CA^2}{BC\cdot BA}$$

Solution 2 by proposer

Plagiogonal system: $BC\equiv Bx,BA\equiv By$.

$$Let\ BD=D,BF=f,B(0,0),D(d,0),C(a,0),F(0,f),A(0,c),L_e(l_1,l_2)$$

$$l_1=\frac{ac^2}{a^2+b^2+c^2},l_2=\frac{a^2c}{a^2+b^2+c^2}$$

$$D,F,L_e-collinear\Leftrightarrow\begin{vmatrix}1&1&1\\d&0&l_1\\0&f&l_2\end{vmatrix}=0\Leftrightarrow\frac{a}{f}+\frac{c}{d}=\frac{a^2+b^2+c^2}{ac}$$

$$\Leftrightarrow ac\left(\frac{a}{f}+\frac{c}{d}\right)=a^2+b^2+c^2\Leftrightarrow a\left(\frac{1}{f}-\frac{1}{c}\right)+c\left(\frac{1}{d}-\frac{1}{a}\right)=\frac{b^2}{ac}$$

189. V **−Bevan's point,** I_a,I_b,I_c **−excenters in** ΔABC**. Prove that:**

$$aVA^2+bVB^2+cVC^2=4R([I_aI_bI_c]-3F)$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Izumi Ainsworth-Lima-Peru

By Stewart theorem:

$$(2R)^2 r_b \cdot csc\alpha + (2R)^2 r_c \cdot csc\alpha = (VA)^2 4R \cdot cos\frac{A}{2} + r_b r_c \cdot csc^2\alpha \cdot 4Rcos\frac{A}{2}$$

$$\rightarrow 4R^2 \underbrace{(r_b \cdot csc\alpha + r_c csc\,\alpha)}_{4Rcos\frac{A}{2}} = 4R \cdot cos\frac{A}{2}\left((VA)^2 + r_b r_c \cdot sec^2\frac{A}{2}\right)$$

$$\rightarrow 4R^2 = (VA)^2 + 4R^2 \cdot sinBsinC \rightarrow (VA)^2 = 4R^2(1 - sinBsinC)$$

$$\sum_{cyc} aVA^2 = \sum_{cyc} 4R^2(a - a \cdot sinBsinC) = 4R^2\left(2s - \sum_{cyc} a \cdot \frac{2F}{ac} \cdot \frac{2F}{ab}\right) =$$

$$= 4R^2\left(2s - \frac{3R}{R}\right) = 4R(2sR - 3F);\ \ (\mu)$$

$$F = \frac{abc}{4R} \rightarrow [I_a I_b I_c] = \frac{\left(4R \cdot cos\frac{A}{2}\right)\left(4R \cdot cos\frac{B}{2}\right)\left(4R \cdot cos\frac{C}{2}\right)}{4 \cdot 2R} =$$

$$\rightarrow [I_a I_b I_c] = 8R^2 \cdot cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2} = 8R^2 \cdot \frac{s}{4R} = 2sR;\ \ (\delta)$$

From $(\mu), (\delta)$ it follows that:

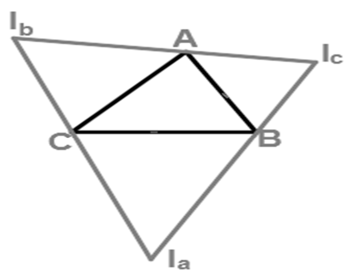
$$aVA^2 + bVB^2 + cVC^2 = 4R([I_a I_b I_c] - 3F)$$

190. I_a, I_b, I_c **−excenters in** $\Delta ABC, \varphi_a, \varphi_b, \varphi_c$ **−circumradii of** $\Delta BCI_a, \Delta CAI_a, \Delta ABI_c$**. Prove that :**

$$\sum \frac{1}{\varphi_a w_a} = \frac{1}{Rr}$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have:\ \mu(BI_aC) = \pi - \mu(BCI_a) - \mu(CBI_a) = \pi - \frac{\pi - C}{2} - \frac{\pi - B}{2} = \frac{\pi}{2} - \frac{A}{2}$$

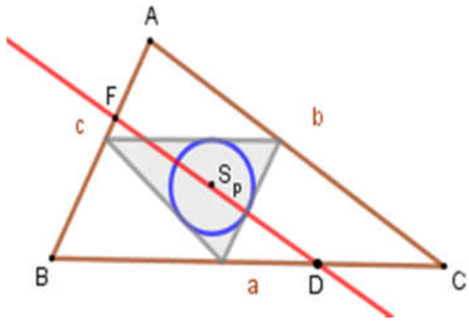
$$In\ \Delta BCI_a: 2\varphi_a = \frac{a}{\sin BI_aC} \to \varphi_a = \frac{a}{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}}$$

$$= 2R\sin\frac{A}{2}\ (and\ analogs)$$

$$\sum \frac{1}{\varphi_a w_a} = \sum \frac{b+c}{2bc.\cos\frac{A}{2}.2R\sin\frac{A}{2}} = \sum \frac{b+c}{abc} = \frac{4s}{4sRr} = \frac{1}{Rr}$$

$$Therefore,\ \sum \frac{1}{\varphi_a w_a} = \frac{1}{Rr}$$

191.

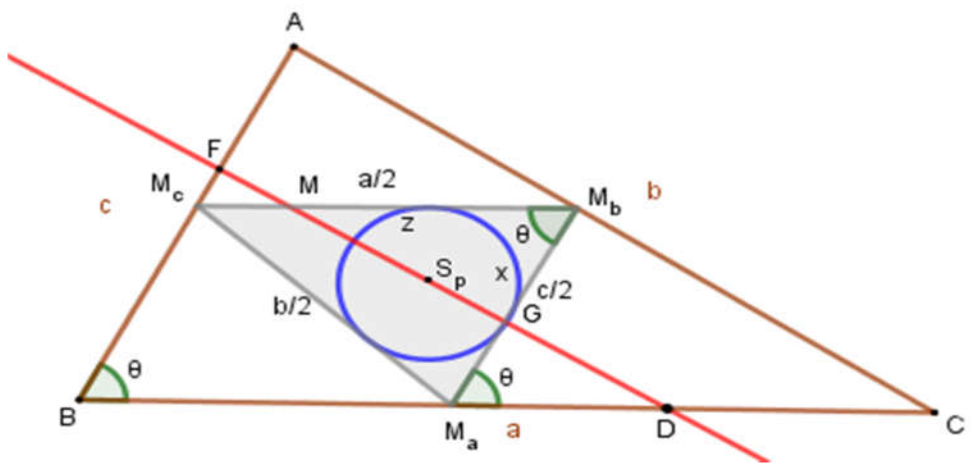


S_p *–Spieker's point of* ΔABC*. Prove that:*

$$D, F, S_p - collinear \Leftrightarrow \frac{BC}{BD}(AC + BC) + \frac{BA}{BF}(AB + AC) = 2(AB + BC + CA)$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$\Delta M_aM_bM_c$ −medial triangle.

$$BC=a,AB=c,AC=b,BF=y,BD=d,M_aM_b=\frac{c}{2},M_bM_c=\frac{a}{2},M_aM_c=\frac{b}{2}$$

$$GM_b=x,MM_b=z$$

ΔFBD is similar ΔGM_aD, so

$$\frac{FB}{GM_a}=\frac{BD}{M_aD}\to\frac{y}{\frac{c}{2}-x}=\frac{d}{d-\frac{a}{2}}\to\frac{2y}{c-2x}=\frac{2d}{2d-a}$$

$$2dy-ay=cd-2dx\to x=\frac{1}{2d}(cd+ay-2dy)$$

ΔGM_aD is similar ΔGM_bM

$$\frac{M_aD}{M_bM}=\frac{GM_a}{GM_b}\to\frac{d-\frac{a}{2}}{z}=\frac{\frac{c}{2}-x}{x}\to\frac{2d-a}{z}=\frac{c-2x}{x}$$

$$2dx-ax=cz-2xz\to z=\frac{x(2d-a)}{c-2x}=\frac{\frac{1}{2d}(cd+ay-2dy)(2d-a)}{c-\frac{1}{d}(cd+ay-2dy)}$$

$$z=\frac{(cd+ay-2dy)(2d-a)}{2(cd-cd-ay+2dy)}=\frac{(cd+ay-2dy)(2d-a)}{2y(2d-a)}$$

$$z=\frac{1}{2y}(cd+ay-2dy)$$

We know that if G,S_p and M are collinear, so

$$\left(\frac{1}{M_bM}-\frac{1}{M_bM_c}\right)+\left(\frac{1}{M_bG}-\frac{1}{M_aM_b}\right)=\frac{M_aM_b}{M_aM_b\cdot M_bM_c}$$

$$\left(\frac{1}{z}-\frac{1}{\frac{a}{2}}\right)+\left(\frac{1}{x}-\frac{1}{\frac{c}{2}}\right)=\frac{\frac{b}{2}}{\frac{a}{2}\cdot\frac{c}{2}}\Leftrightarrow$$

$$\frac{2y}{cd+ay-2dy}-\frac{2}{a}+\frac{2d}{cd+ay-2dy}=\frac{2b}{ac}\Leftrightarrow$$

$$yac-c(cd+ay-2dy)+dac-a(cd+ay-2dy)=b(cd+ay-2dy)\Leftrightarrow$$

$$yac-c^2d-yac+2cdy+dac-acd-a^2y+2ady=bcd+aby-2bdy\Leftrightarrow$$

$$aby+a^2y+c^2d+bcd=2dy(a+b+c)\Leftrightarrow$$

$$\frac{a}{d}(a+b)+\frac{c}{y}(b+c)=2(a+b+c)$$

Therefore,

$$\frac{BC}{BD}(AC+BC)+\frac{BA}{BF}(AB+AC)=2(AB+BC+CA)$$

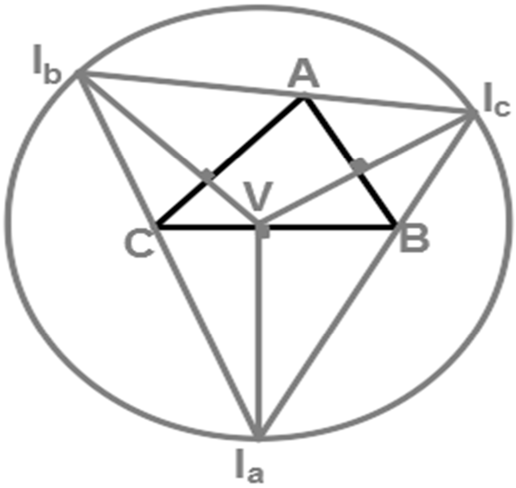
192. In $\Delta ABC, V$ –Bevan' s point, I_a, I_b, I_c – $excenters, R_a, R_b, R_c$ –circumradii of $\Delta VI_bI_c, \Delta VI_cI_a, \Delta VI_aI_b$,

Prove that :

$$\sum \frac{h_a}{{R_a}^2}=\frac{r(r_a+r_b+r_c)}{2R^3}$$

Proposed by Mehmet Șahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have:\ \mu(I_bAC)=\frac{\pi-A}{2}\to\mu(I_cI_bV)=\frac{\pi}{2}-\mu(I_bAC)=\frac{A}{2}\ and\ I_cV=2R.$$

$$In\ \Delta VI_bI_c, we\ have:\ \sin I_cI_bV=\frac{I_cV}{2R_a}\to R_a=R\csc\frac{A}{2}\ (and\ analogs)$$

$$\sum\frac{h_a}{{R_a}^2}=\frac{2sr}{R^2}\sum\frac{\sin^2\frac{A}{2}}{a}=\frac{2sr}{R^2}\sum\frac{\sin^2\frac{A}{2}}{4R\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{r}{2R^3}\sum s\tan\frac{A}{2}=\frac{r(r_a+r_b+r_c)}{2R^3}$$

$$Therefore,\quad \sum\frac{h_a}{{R_a}^2}=\frac{r(r_a+r_b+r_c)}{2R^3}$$

193. In $\Delta ABC, n_a$ −Nagel's cevian, the following relationship holds:

$$2\left(\frac{R}{r}-1\right)\sum_{cyc}\frac{h_a}{n_a}=\sum_{cyc}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's\ prove\ that: 2\left(\frac{R}{r}-1\right)\sum_{cyc}\frac{h_a}{n_a}=\sum_{cyc}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)$$

$$\frac{n_a}{r_a}+\frac{r_a}{n_a}=\frac{n_a^2+r_a^2}{n_ar_a}=\frac{s^2-2r_ah_a+r_a^2}{n_ar_a}$$

$$\frac{s^2-2r_ah_a+r_a^2}{r_a}=\frac{s^2}{r_a}-\frac{4sr}{a}+\frac{sr}{s-a}=\frac{as^2(s-a)-4s^2r^2+\frac{as^2r^2}{s-a}}{a(s-a)r_a}=$$

$$=\frac{as^2(s-a)^2-4s^2r^2(s-a)+s^2r^2a}{asr(s-a)}=$$

$$=s\cdot\frac{a(s-a)^2\cos\frac{A}{2}-4r^2(s-a)\cos\frac{A}{2}+ar^2\cos\frac{A}{2}}{ar(s-a)\cos\frac{A}{2}}=$$

$$=s\cdot\frac{a(s-a)^2\cos\frac{A}{2}-4r^2(s-a)\cos\frac{A}{2}+ar(s-a)\sin\frac{A}{2}}{ar(s-a)\cos\frac{A}{2}}=$$

$$=s\cdot\frac{a(s-a)\cos\frac{A}{2}-4r^2\cos\frac{A}{2}+ar\cdot\sin\frac{A}{2}}{ar\cdot\cos\frac{A}{2}}=$$

$$=s\cdot\frac{16R^2\cdot\cos^3\frac{A}{2}\prod\left(\sin\frac{A}{2}\right)-4r^2\cdot\cos\frac{A}{2}+ar\cdot\sin\frac{A}{2}}{ar\cdot\cos\frac{A}{2}}$$

$$\to\frac{n_a^2+r_a^2}{r_a}=s\cdot\frac{4Rr\cdot\cos^3\frac{A}{2}-4r^2\cdot\cos\frac{A}{2}+4Rr\cdot\cos\frac{A}{2}\sin^2\frac{A}{2}}{ar\cdot\cos\frac{A}{2}}=$$

$$=s\cdot\frac{4R\left(1-\sin^2\frac{A}{2}\right)-4r+4R\cdot\sin^2\frac{A}{2}}{a}=\frac{4s(R-r)}{a}=2\left(\frac{R}{r}-1\right)h_a$$

$$\rightarrow \sum_{cyc}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)=\sum_{cyc}\frac{n_a^2+r_a^2}{r_a n_a}=2\left(\frac{R}{r}-1\right)\sum_{cyc}\frac{h_a}{n_a}$$

194. V – Bevan's point, I_a, I_b, I_c – excenters in $\Delta ABC, R_a, R_b, R_c$ – circumradii of $\Delta VI_bI_c, \Delta VI_cI_a, \Delta VI_aI_b, \varphi_a, \varphi_b, \varphi_c$ – circumradii of $\Delta BCI_a, \Delta CAI_b, \Delta ABI_c$.

Prove that:

$$\varphi_a R_a = \varphi_b R_b = \varphi_c R_c = 2R^2$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Izumi Ainsworth-Lima-Peru

$$[VI_bI_c]=\frac{4R\cos\left(\frac{A}{2}\right)(2R)^2}{4R_a}=\frac{(2R)^2\sin(A)}{2}\Rightarrow R_a=\frac{R}{\sin\left(\frac{A}{2}\right)} \quad (\alpha)$$

$$[BI_aC]=\frac{ar_a}{2}=\frac{ar_a\csc\left(\frac{180-C}{2}\right)r_a\csc\left(\frac{180-B}{2}\right)}{4\varphi_a}\Rightarrow$$

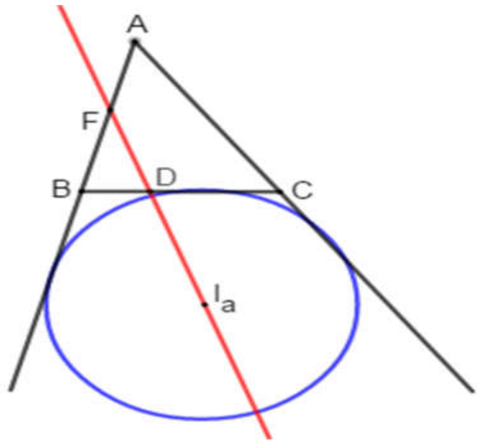
$$\varphi_a=\frac{r_a}{2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)} \quad (\beta)$$

$(\alpha)\times(\beta)$:

$$\Rightarrow \varphi_a R_a=\frac{Rr_a}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}=\frac{Rr_a}{2\frac{r_a}{4R}}=2R^2$$

Analogously: $\varphi_a R_b=\varphi_c R_c=2R^2$

195.

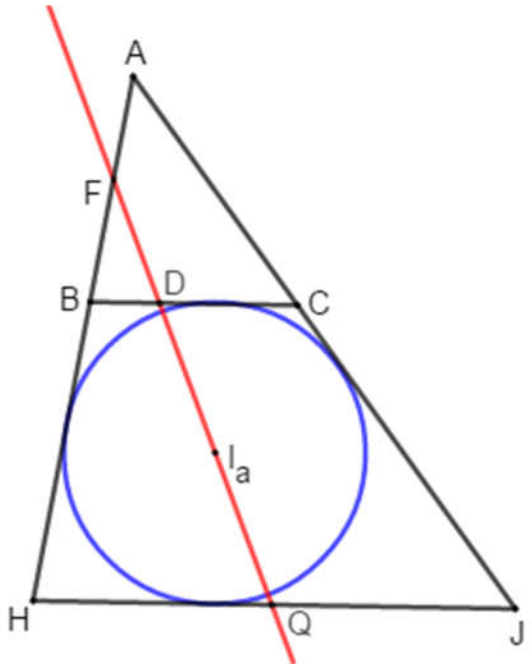


D, F, I_a –collinears if

$$\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(-\frac{1}{BF}+\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



We extend FI_a to the HJ side of the ΔAHJ at point Q.

In the triangle AHJ, we have: I_a –incenter of a triangle.

We know that if Q, I_a and F are collinear, so

$$\frac{1}{HQ}-\frac{1}{HJ}+\frac{1}{HF}-\frac{1}{AH}=\frac{AJ}{AH\cdot HJ};(I)$$

ΔABC is similar to ΔAHJ, with similarity ratio $k=\frac{s}{s-a}$, then

$$AH=\frac{cs}{s-a}, AJ=\frac{bs}{s-a}, HJ=\frac{as}{s-a}$$

Now, ΔFBD is similar toΔFHQ, so $\frac{BD}{HQ}=\frac{FB}{FH}$; $AB+BH=AH$ and $c+BH=\frac{ac}{s-a}\rightarrow$

$$BH=\frac{ac}{s-a}, FH=FB+BH=f+\frac{ac}{s-a}=\frac{f(s-a)+ac}{s-a}$$

$$\frac{d}{x}=\frac{f}{\frac{f(s-a)+ac}{s-a}}\rightarrow\frac{1}{x}=\frac{f(s-a)}{d[f(s-a)+ac]}$$

Replacing in (I) it follows that:

$$\frac{1}{x}-\frac{1}{\frac{as}{s-a}}+\frac{1}{\frac{f(s-a)+ac}{s-a}}-\frac{1}{\frac{cs}{s-a}}=\frac{\frac{bs}{s-a}}{\frac{as}{s-a}\cdot\frac{cs}{s-a}}\leftrightarrow$$

$$\frac{f}{d[f(s-a)+ac]}-\frac{1}{as}+\frac{1}{f(s-a)+ac}-\frac{1}{cs}=\frac{b}{acs}\leftrightarrow$$

$$\frac{1}{f(s-a)+ac}\cdot\left(\frac{f}{d}+1\right)=\frac{2}{ac}\leftrightarrow\frac{f}{d}+1=\frac{2}{ac}[f(s-a)+ac)]\leftrightarrow$$

$$\frac{f}{d}+1=2+\frac{f}{ac}(b+c-a)\leftrightarrow\frac{1}{d}-\frac{1}{f}=\frac{b}{ac}+\frac{1}{a}-\frac{1}{c}$$

$$\left(\frac{1}{d}-\frac{1}{a}\right)+\left(-\frac{1}{f}+\frac{1}{c}\right)=\frac{b}{ac}$$

Therefore,

$$\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(-\frac{1}{BF}+\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}$$

196. S_p **–Spieker point in** $\Delta ABC, d_a=d(S_p,BC), d_b=d(S_p,CA)$, $d_c=d(S_p,AB)$. **Prove that:**

$$d_a+d_b+d_c=\frac{s^2+r^2-2Rr}{4R}$$

Proposed by Mehmet Șahin-Ankara-Turkyie

Solution 1 by Izumi Ainsworth-Lima-Peru

If M,N,P –midpoints of AB,BC,CA respectively.$D\in MP$ such that N,S_p and D be collinear.By property: $\frac{DS_p}{S_pN}=\frac{\frac{a}{2}}{\frac{b}{2}+\frac{c}{2}}=\frac{a}{b+c}\to$

$$\frac{DN}{S_pN}=\frac{2s}{b+c}=\frac{\frac{h_a}{2}}{d_a}\to d_a=\frac{h_a(b+c)}{4s}=\frac{bc(b+c)}{8Rs}$$

In the problem:

$$\sum_{cyc}d_a=\sum_{cyc}\frac{bc(b+c)}{8Rs}=\frac{F}{2s}\sum_{cyc}\frac{2s-a}{a}=F\left(\sum_{cyc}\frac{1}{a}-\frac{3}{2s}\right)=$$

$$=F\left(\frac{s^2+r^2+4Rr}{4FR}-\frac{3r}{2F}\cdot\frac{2R}{2R}\right)=\frac{s^2+r^2-2Rr}{4R}$$

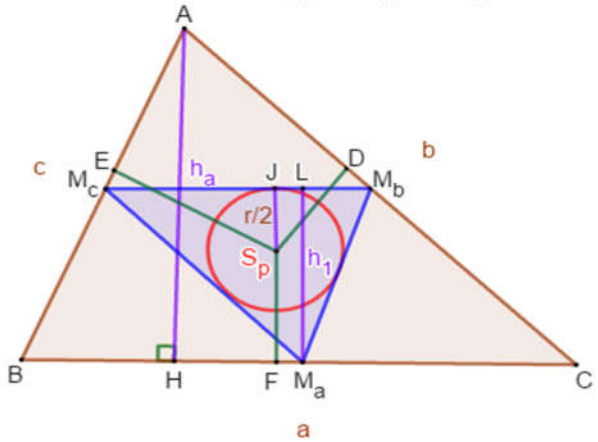
Solution 2 by Jose Ferreira Queiroz-Olinda-Brazil

$$BC = a, CA = b, AB = c, AH = h_a, M_aL = h_1 = \frac{h_a}{2}$$

$$d_a = h_1 - \frac{r}{2} = \frac{h_a}{2} - \frac{r}{2}; d_b = \frac{h_b}{2} - \frac{r}{2}; d_c = \frac{h_c}{2} - \frac{r}{2}; F = \frac{1}{2}a \cdot h_a = sr = \frac{abc}{4R}$$

$$d_a + d_b + d_c = \frac{h_a}{2} - \frac{r}{2} + \frac{h_b}{2} - \frac{r}{2} + \frac{h_c}{2} - \frac{r}{2}$$

$$2(d_a + d_b + d_c) = \frac{2sr}{a} + \frac{2sr}{b} + \frac{2sr}{c} - 3r$$



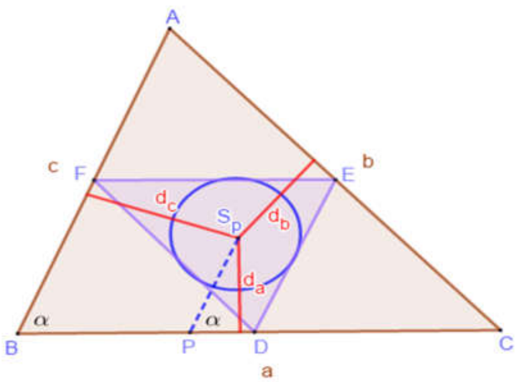
$$2(d_a + d_b + d_c) = 2sr\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3r$$

$$2(d_a + d_b + d_c) = 2sr \cdot \frac{ab + bc + cd}{abc} - 3r$$

$$2(d_a + d_b + d_c) = 2sr \cdot \frac{ab + bc + cd}{4Rrs} - 3r$$

$$2(d_a + d_b + d_c) = \frac{(s^2 + r^2 + 4Rr) - 6Rr}{2R}$$

Solution 3 by Thanasis Gakopoulos-Farsala-Greece



$PS_p = \frac{c(b+c)}{2(a+b+c)}$; (1) from theory of plagiogonal system.

$$\sum_{cyc} ab(a+b) = 2s(s^2 - 2Rr + r^2); (2)$$

$$\Delta PDS_p:\ d_a = Pcdot\ sin\ B = PS_p \cdot \frac{b}{2R} \overset{(1)}{\cong} \frac{bc(b+c)}{4R(a+b+c)} = \frac{bc(b+c)}{4R \cdot 2s}$$

$$d_a + d_b + d_c = \frac{1}{4R \cdot 2s}\sum_{cyc} ab(a+b) \overset{(2)}{\cong} \frac{2s(s^2 - 2Rr + r^2)}{4R \cdot 2s} = \frac{s^2 + r^2 - 2Rr}{4R}$$

Therefore,

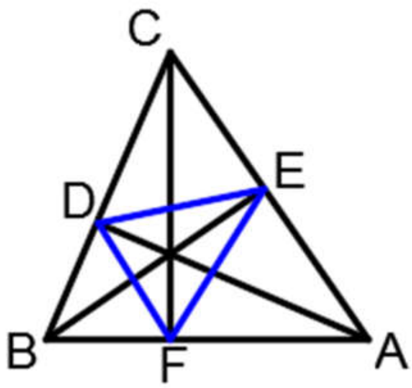
$$d_a + d_b + d_c = \frac{s^2 + r^2 - 2Rr}{4R}$$

197. O **−circumcenter,** Ψ **−area of orthic triangle of acute** $\Delta ABC, O_a, O_b, O_c$ **−circumcenters of** $\Delta BOC, \Delta COA, \Delta AOB$**. Prove that:**

$$F = 2\sqrt{\Psi.[O_aO_bO_c]}$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco



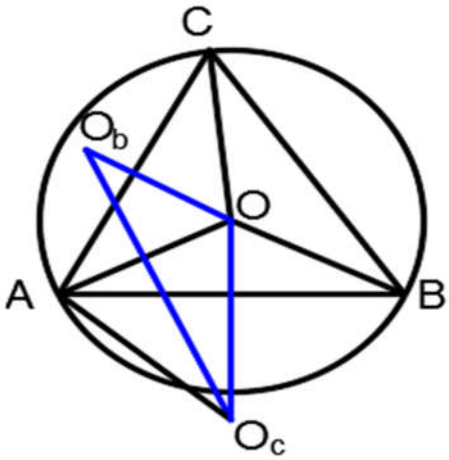
$$We\ have : AE = c.\cos A\ \ and\ AF = b.\cos A \to [AEF] = \frac{1}{2}AE.AF.\sin A$$

$$= \frac{1}{2}.c.\cos A.b.\cos A.\frac{a}{2R}$$

$$\to [AEF] = \frac{abc}{4R}.\cos^2 A = F\cos^2 A\ \ (and\ analogs).$$

$$\to \Psi = F - \sum [AEF] = F - \sum F\cos^2 A = F\left(1 - \sum \cos^2 A\right) = 2F\prod \cos A$$

$$\rightarrow \Psi = 2F\prod \cos A\,(1)$$



$$We\ have:\ \mu(AOB) = 2C \rightarrow \mu(AOO_c) \overset{(a)}{=} C\ and\ \mu(OBA) = \frac{\pi}{2} - C$$

$$Now, in\ \Delta AOB, \sin OBA = \frac{R}{2OO_c} \rightarrow OO_c = \frac{R}{2\sin\left(\frac{\pi}{2} - C\right)} \overset{(b)}{=} \frac{R}{2\cos C}$$

$$Similar\ to\ (a)\ and\ (b), we\ have:\ \mu(O_bOA) = B\ and\ OO_b = \frac{R}{2\cos B} \rightarrow \mu(O_bOO_c)$$

$$= B + C = \pi - A$$

$$\rightarrow [O_bOO_c] = \frac{1}{2}.OO_b.OO_c.\sin O_bOO_c = \frac{1}{2}\cdot\frac{R}{2\cos B}\cdot\frac{R}{2\cos C}.\sin A$$

$$= \frac{R^2\sin 2A}{16\prod\cos A}\ (and\ analogs)$$

$$\rightarrow [O_aO_bO_c] = \sum[O_bOO_c] = \sum\frac{R^2\sin 2A}{16\prod\cos A} = \frac{R^2}{16\prod\cos A}\sum\sin 2A$$

$$= \frac{R^2}{16\prod\cos A}.4\prod\sin A =$$

$$= \frac{R^2}{4\prod\cos A}\cdot\frac{F}{2R^2} \rightarrow [O_aO_bO_c] = \frac{F}{8\prod\cos A}\quad(2)$$

$$From\ (1),(2):\ 2\sqrt{\Psi.[O_aO_bO_c]} = 2\sqrt{2F\prod\cos A.\frac{F}{8\prod\cos A}} = F$$

$$Therefore,\qquad F = 2\sqrt{\Psi.[O_aO_bO_c]}$$

Solution 2 by Izumi Ainsworth-Lima-Peru

We know that:

$$\Psi=\frac{abc\cdot \cos A\cos B\cos C}{2R};(\alpha)$$

If M,N,P –midpoints of $\overline{OA},\overline{OB},\overline{OC}$ respectively, and

$$\begin{cases}\angle BOO_a=\angle O_aOC=A\\ \angle COO_b=\angle O_bOA=B\\ \angle AOO_c=\angle O_cOB=C\end{cases}\rightarrow [O_aO_bO_c]=[O_aNOP]+[O_bPOM]+[O_cMON]=$$

$$=\frac{R}{2}\cdot\frac{R}{2}\cdot\tan A+\frac{R}{2}\cdot\frac{R}{2}\cdot\tan B+\frac{R}{2}\cdot\frac{R}{2}\cdot\tan C=$$

$$=\frac{R^2\cdot\tan A\tan B\tan C}{4};(\beta)$$

In the problem: $2\sqrt{\Psi.[O_aO_bO_c]}\overset{(\alpha,\beta)}{\triangleq}$

$$=2\sqrt{\frac{abc\cdot\cos A\cos B\cos C}{2R}\cdot\frac{R^2\cdot\tan A\tan B\tan C}{4}}=$$

$$=\sqrt{\frac{R}{2}\cdot abc\cdot\sin A\sin B\sin C}=\sqrt{\frac{Rabc}{2}\cdot\frac{2F}{bc}\cdot\frac{2F}{ca}\cdot\frac{2F}{ab}}=\sqrt{F^2}=F$$

198. Find:

$$\Omega=\frac{\tan\frac{\pi}{9}\cdot\cot\frac{\pi}{18}-\tan\frac{2\pi}{9}\cdot\cot\frac{\pi}{18}-\tan\frac{\pi}{9}\cdot\tan\frac{2\pi}{9}}{\cot^2\frac{7\pi}{18}+\cot^2\frac{5\pi}{18}+\cot^2\frac{\pi}{18}}$$

Proposed by Samir Cabiyev-Azerbaijan

Solution 1 by Asmat Qatea-Afghanistan

$$\frac{\tan\frac{\pi}{9}\cdot\cot\frac{\pi}{18}-\tan\frac{2\pi}{9}\cdot\cot\frac{\pi}{18}-\tan\frac{\pi}{9}\cdot\tan\frac{2\pi}{9}}{\cot^2\frac{7\pi}{18}+\cot^2\frac{5\pi}{18}+\cot^2\frac{\pi}{18}}=$$

$$=\frac{\tan\frac{\pi}{9}\tan\frac{4\pi}{9}-\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}-\tan\frac{\pi}{9}\tan\frac{2\pi}{9}}{\tan^2\frac{\pi}{9}+\tan^2\frac{2\pi}{9}+\tan^2\frac{4\pi}{9}}$$

$$\sqrt{3}=tan(3x)=\frac{3tanx-tan^3x}{1-3tan^2x}\to\sqrt{3}=tan(3x)\to x_1=\frac{\pi}{9},x_2=\frac{4\pi}{9},x_3=-\frac{2\pi}{9}$$

$$\sqrt{3}=\frac{3t-t^3}{1-3t^2}\to t^3-3\sqrt{3}t^2-3t+\sqrt{3}=0\to$$

$$t_1\cdot t_2+t_1\cdot t_3+t_2\cdot t_3=-3;t_1=tan\frac{\pi}{9},t_2=tan\frac{4\pi}{9},t_3=tan\frac{2\pi}{9}$$

$$3=tan^2(3x)=\left(\frac{3tanx-tan^3x}{1-3tan^2x}\right)^2\to 3=tan^2(3x)\to x_1=\frac{\pi}{9},x_2=\frac{2\pi}{9},x_3=\frac{4\pi}{9}$$

$$3=\left(\frac{3t-t^3}{1-3t^2}\right)^2\to t^6-33t^4+27t^2-3=0\to t_1^2+t_2^2+t_3^2=33$$

Therefore,

$$\Omega=\frac{tan\frac{\pi}{9}\cdot cot\frac{\pi}{18}-tan\frac{2\pi}{9}\cdot cot\frac{\pi}{18}-tan\frac{\pi}{9}\cdot tan\frac{2\pi}{9}}{cot^2\frac{7\pi}{18}+cot^2\frac{5\pi}{18}+cot^2\frac{\pi}{18}}=-\frac{1}{11}$$

Solution 2 by proposer

$$\Omega=\frac{tan\frac{\pi}{9}\cdot cot\frac{\pi}{18}-tan\frac{2\pi}{9}\cdot cot\frac{\pi}{18}-tan\frac{\pi}{9}\cdot tan\frac{2\pi}{9}}{cot^2\frac{7\pi}{18}+cot^2\frac{5\pi}{18}+cot^2\frac{\pi}{18}}$$

$$tan\frac{\pi}{9}\cdot cot\frac{\pi}{18}-tan\frac{2\pi}{9}\cdot cot\frac{\pi}{18}-tan\frac{\pi}{9}\cdot tan\frac{2\pi}{9}:(I)$$

$$cot^2\frac{7\pi}{18}+cot^2\frac{5\pi}{18}+cot^2\frac{\pi}{18}:(II)$$

$$20^\circ=\alpha\to 3\alpha=60^\circ\to tan3\alpha=\sqrt{3}\to\sqrt{3}=\frac{3tan\alpha-tan^3\alpha}{1-3tan^2\alpha}$$

$$tan^3\alpha-3\sqrt{3}tan^2\alpha-3tan\alpha+\sqrt{3}=0;(1),\alpha=20^\circ,\beta=40^\circ\to 3\beta=120^\circ$$

$$\to tan3\beta=-\sqrt{3}\to -tan^3\alpha-3\sqrt{3}tan^2\alpha+3tan\alpha+\sqrt{3}=0$$

$\to\beta=-40^\circ$ (1) true, $\gamma=80^\circ\to 3\gamma=240^\circ\to$

$tan^3\gamma-3\sqrt{3}tan^2\gamma-3tan\gamma+\sqrt{3}=0\to$ (1) is true.

$$tan^3\alpha-3\sqrt{3}tan^2\alpha-3tan\alpha+\sqrt{3}=0\to\alpha_1=20^\circ,\alpha_2=-40^\circ,\alpha_3=80^\circ$$

$$ax^3+bx^2+cx+d=0$$

$$x_1^2+x_2^2+x_3^2=\frac{b^2-2ac}{a^2}\to tan^2 20^\circ+tan^2(-40^\circ)+tan^2 80^\circ=$$

$$=tan^2 20^\circ+tan^2 40^\circ+tan^2 80^\circ=\frac{\left(-3\sqrt{3}\right)^2-2\cdot 1\cdot(-3)}{1^2}=33$$

$$tan\,20^\circ tan(-40^\circ)+tan(-40^\circ)tan80^\circ+tan20^\circ tan80^\circ=$$

$$= x_1x_2 + x_2x_3 + x_1x_3 = \frac{(x_1+x_2+x_3) - (x_1^2+x_2^2+x_3^2)}{2} = -3$$

Therefore,

$$\Omega = \frac{tan\frac{\pi}{9}\cdot cot\frac{\pi}{18} - tan\frac{2\pi}{9}\cdot cot\frac{\pi}{18} - tan\frac{\pi}{9}\cdot tan\frac{2\pi}{9}}{cot^2\frac{7\pi}{18} + cot^2\frac{5\pi}{18} + cot^2\frac{\pi}{18}} = -\frac{1}{11}$$

199. $BD_1 = d_1, BD_2 = d_2$

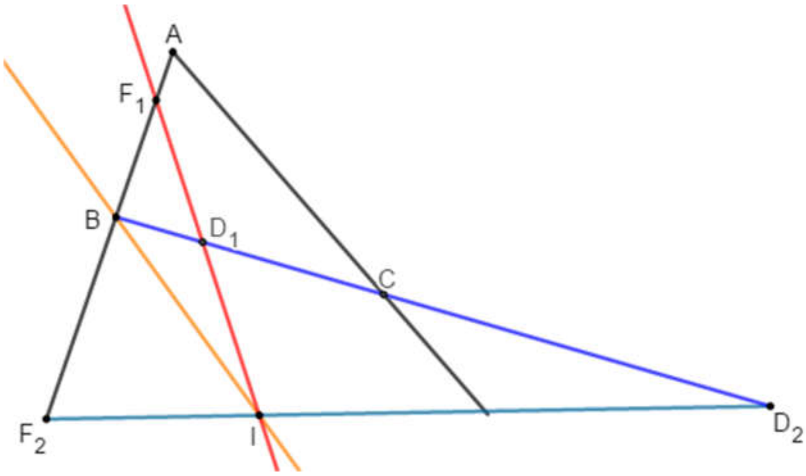
$BF_1 = f_1, BF_2 = f_2$

$$\frac{1}{d_1} - \frac{1}{d_2} = \frac{1}{f_1} + \frac{1}{f_2}$$

$D_1F_1 \cap D_2F_2 = I$

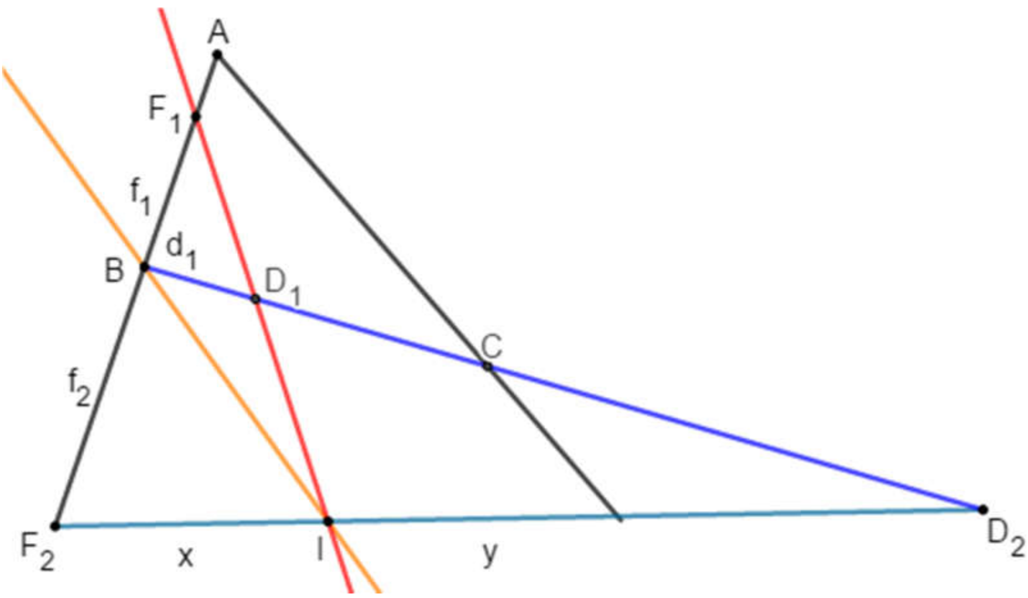
Prove:

BI is bisector of $\angle D_1BF_2$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira-Olinda-Brazil



$$\frac{1}{d_1}-\frac{1}{d_2}=\frac{1}{f_1}+\frac{1}{f_2}\rightarrow\frac{d_2-d_1}{d_1d_2}=\frac{f_1+f_2}{f_1f_2};(1)$$

$$D_1D_2=d_2-d_1$$

$$F_2I=x,D_2I=y$$

Applying Menelaus Theorem (F_1,D_1,I) −collinear

$$\frac{D_2I}{F_2I}\cdot\frac{F_1F_2}{BF_1}\cdot\frac{BD_1}{D_1D_2}=1\leftrightarrow\frac{y}{x}\cdot\frac{f_1+f_2}{f_1}\cdot\frac{d_1}{d_2-d_1}=1$$

$(I)\leftrightarrow\frac{y}{x}\cdot\frac{f_2(d_2-d_1)}{d_1d_2}=\frac{d_2-d_1}{d_1}\leftrightarrow\frac{y}{x}=\frac{d_2}{f_2}\rightarrow BI$ is bisector of $\angle D_1BF_2$

200. S_p −Spieker point, O −circumcenter in ΔABC. If $d(O,BC)=d_a$, $d(O,CA)=d_b$, $d(O,AB)=d_c$, $d(S_p,BC)=d_1$, $d(S_p,CA)=d_2$, $d(S_p,AB)=d_3$ then:

$$d_1d_a+d_2d_b+d_3d_c=\frac{s^2-6Rr-3r^2}{4}$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Izumi Ainsworth-Lima-Peru

We know that: $d_1=\frac{bc(b+c)}{8Rs};(\alpha)$

Now, ΔBOC is isosceles, $BO=OC=R\rightarrow\widehat{BC}=2A\rightarrow\angle BOC=2A$

$d_a=RcosA;(\beta)$. From $(\alpha),(\beta)$ it follows that:

$$\sum_{cyc}d_1d_a=\sum_{cyc}\frac{Rbc(b+c)cosA}{8Rs}=\frac{FR}{2s}\sum_{cyc}\frac{(2s-a)cosA}{a}=$$

$$=\frac{FR}{2s}\left(2s\sum_{cyc}\frac{cosA}{a}-\sum_{cyc}cosA\right)=\frac{FR}{2s}\left[2s\left(\frac{s^2-r^2-4Rr}{4RF}\right)-\left(1+\frac{r}{R}\right)\right]=$$

$$=\frac{s^2-4Rr-r^2}{4}-\frac{FR}{2s}\left(\frac{R+r}{R}\cdot\frac{2sr}{2F}\right)=\frac{s^2-6Rr-3r^2}{4}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru