$$AH = h_a, [ABC] = F, h_a = \frac{2F}{a}, h_b = \frac{2F}{b}, h_c = \frac{2F}{c}, sinA = \frac{2F}{bc}, sinB = \frac{2F}{ac},$$

$$sinC = \frac{2F}{ab}, F = \frac{abc}{4R}$$

$\Delta AHT \sim \Delta GTM_a \Rightarrow GT = \frac{h_a}{3}$. In the same way: $cosQ = \frac{h_b}{3}$ and $GZ = \frac{h_c}{3}$.

$$[PQR] = [PGR] + [PGQ] + [RGQ]$$

$$[PQR] = \frac{1}{2} \cdot \frac{2}{3} h_a \cdot \frac{2}{3} h_c sinB + \frac{1}{2} \cdot \frac{2}{3} h_a \cdot \frac{2}{3} h_b sinC + \frac{1}{2} \cdot \frac{2}{3} h_b \cdot \frac{2}{3} h_c sinA$$

$$[PQR] = \frac{2}{9}(h_b h_c sinA + h_a h_c sinB + h_a h_b sinC) =$$

$$= \frac{2}{9}\left(\frac{2F}{b} \cdot \frac{2F}{c} sinA + \frac{2F}{c} \cdot \frac{2F}{a} sinB + \frac{2F}{a} \cdot \frac{2F}{b} sinC\right) =$$

$$= \frac{8F^2}{9}\left(\frac{1}{bc} sinA + \frac{1}{ca} sinB + \frac{1}{ab} sinC\right) = \frac{8F^2}{9}\left(\frac{1}{bc} \cdot \frac{2F}{bc} + \frac{1}{ca} \cdot \frac{2F}{ca} + \frac{1}{ab} \cdot \frac{2F}{ab}\right) =$$

$$\frac{16F^3}{9} \cdot \frac{a^2 + b^2 + c^2}{a^2 b^2 c^2} = \frac{16F^3}{9} \cdot \frac{a^2 + b^2 + c^2}{16R^2 F^2} = \frac{F}{9R^2} \cdot (a^2 + b^2 + c^2)$$

$$i) \frac{[PQR]}{[ABC]} = 1 \Rightarrow \frac{F}{9R^2} \cdot (a^2 + b^2 + c^2) = F \Rightarrow a^2 + b^2 + c^2 = 9F$$

Euler line: Distance between the circumcenter and the orthocentre triangle:

$$OH^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

$$\begin{cases} OH^2 = 9R^2 - 9R^2 = 0 \\ GO^2 = R^2 - \frac{1}{9} \cdot 9R^2 = 0 \end{cases} \Rightarrow G = O = H \Rightarrow \Delta ABC \text{ −equilateral.}$$

$$ii) \frac{[PQR]}{[ABC]} = \frac{8}{9} \Rightarrow \frac{F}{9R^2} \cdot (a^2 + b^2 + c^2) = \frac{8}{9} F \Rightarrow a^2 + b^2 + c^2 = 8R^2$$

$$\text{So, } \begin{cases} OH^2 = 9R^2 - 8R^2 = R^2 \\ GO^2 = R^2 - \frac{1}{9} \cdot 8R^2 = \frac{R^2}{9} \end{cases} \Rightarrow \Delta ABC \text{ −right.}$$

Solution 2 by proposer

Plagiogonal system: $BC \equiv Bx, BA \equiv By, B(0,0), G\left(\frac{a}{3}, \frac{a}{3}\right), P(p_1, p_2),$

$$p_1 = \frac{a + 2c\cos B}{3}, p_2 = -\frac{c}{3}, R(r_1, r_2), r_1 = -\frac{a}{3}, r_2 = \frac{c + 2a\cos B}{3}$$

$$[GPR]=\frac{sinB}{2}\left\|\begin{vmatrix}1 & 1 & 1\\ \frac{a}{3} & r_1 & p_1\\ \frac{c}{3} & r_2 & p_2\end{vmatrix}\right\|=\frac{sinB}{2}\cdot ac\cdot\frac{4}{9}sin^2B$$

$\frac{[GPR]}{[ABC]}=\frac{4}{9}sin^2B;(1)$, similarly $\frac{[GPQ]}{[ABC]}=\frac{4}{9}sin^2C;(2),\frac{[GQR]}{[ABC]}=\frac{4}{9}sin^2A;(3)$

From (1),(2),(3), it follows that: $\frac{[PQR]}{[ABC]}=\frac{4}{9}(sin^2A+sin^2B+sin^2C);(4)$

$$i)\ \frac{[PQR]}{[ABC]}=1\Rightarrow sin^2A+sin^2B+sin^2C=\frac{9}{4}\Rightarrow 2+2cosAcosBcosC=\frac{9}{4}\Rightarrow$$

$$cosAcosBcosC=\frac{1}{8}\Rightarrow \Delta ABC-\text{equilateral}$$

$$ii)\ \frac{[PQR]}{[ABC]}=\frac{8}{9}\Rightarrow sin^2A+sin^2B+sin^2C=\frac{8}{9}\cdot\frac{9}{4}\Rightarrow 2+2cosAcosBcosC=2\Rightarrow$$

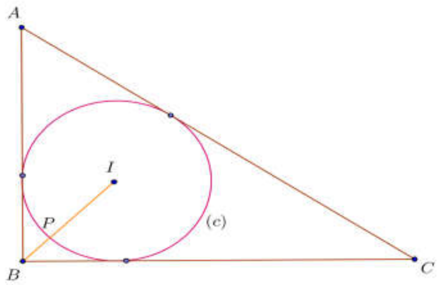
$$cosAcosBcosC=0\Rightarrow \Delta ABC-\text{right.}$$

133.

In ΔABC:

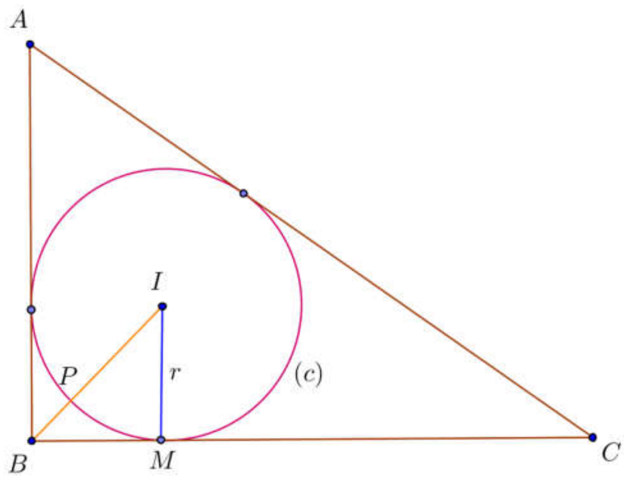
$\frac{BP}{PI}=\sqrt{2}-1.$

Prove: $\mu(B)=90^\circ$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Adrian Popa-Romania



$$\frac{BP}{PI}=\frac{BI-PI}{PI}=\frac{BI}{PI}-1$$

$$\Delta BIM: \sin\frac{B}{2}=\frac{r}{BI}\Rightarrow r=BI\sin\frac{B}{2}\Rightarrow BI=\frac{r}{\sin\frac{B}{2}}, PI=r\Rightarrow$$

$$\frac{BP}{PI}=\frac{r}{r\sin\frac{B}{2}}-1=\frac{1}{\sin\frac{B}{2}}-1=\sqrt{2}-1\Rightarrow \sin\frac{B}{2}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\Rightarrow\frac{B}{2}=45^\circ\Rightarrow\mu(B)=90^\circ$$

Solution 2 by Juan Jose Isach Mayo-Spain

$$r(I)=IP=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}=\frac{1}{2}\sqrt{[(b+c)^2-a^2][a^2-(b^2+c^2)]}$$

$$AI=\frac{\sqrt{bcs(s-a)}}{s}=\frac{\sqrt{bc[(b+c)^2-a^2]}}{a+b+c}$$

$$AP=AI-IP=\frac{\sqrt{(b+c)^2-a^2}}{a+b+c}\left(\sqrt{bc}-\frac{1}{2}\sqrt{a^2-(b-c)^2}\right)$$

$$\frac{AP}{IP}=\frac{2(\sqrt{bc}-\frac{1}{2}\sqrt{a^2-(b-c)^2}}{\sqrt{a^2-(b-c)^2}}$$

If ΔABC is rectangle in $A\Leftrightarrow a^2=b^2+c^2\Leftrightarrow\frac{AP}{PI}=\frac{2}{\sqrt{2}}-1=\sqrt{2}-1.$

Solution 3 by proposer

Plagiogonal system: $BC\equiv Bx; BA\equiv By$

$$(c): x^2+y^2+\frac{a^2-b^2+c^2}{ac}xy-(a-b+c)(x+y)+\frac{(a-b+c)^2}{4}=0$$

$$BI: x=y; I(i,i), i=\frac{ac}{a+b+c}$$

$$\frac{BP}{BI}=\frac{p-0}{i-p}=2\sqrt{\frac{ac}{b^2-(a-c)^2}}-1=\sqrt{2}-1\Rightarrow$$

$$2ac=b^2-(a-c)^2\Rightarrow b^2=a^2+c^2\Rightarrow\mu(B)=90^\circ$$

134.

$\widehat{xOy}=45^o$

$A\in Ox, B\in Oy, OA=a, OB=b$

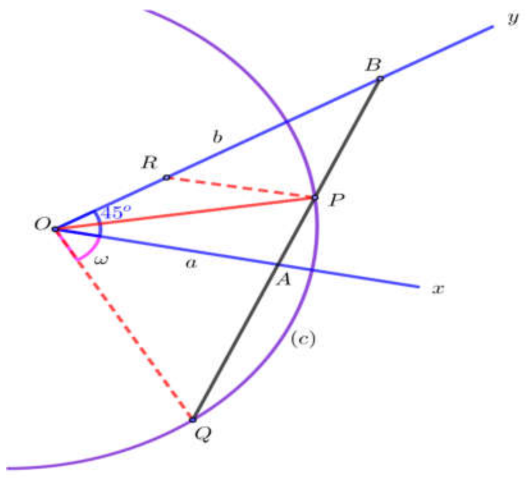
$P\in AB, OP$ –bisector of $\widehat{xOy}$

$O(c)=(O,OP)$

$(c)\cap AB=Q$

$\widehat{\omega}=\widehat{AOQ}; PR\parallel Ox, OR=p$

Find: $\widehat{\omega}=f(a,b)$

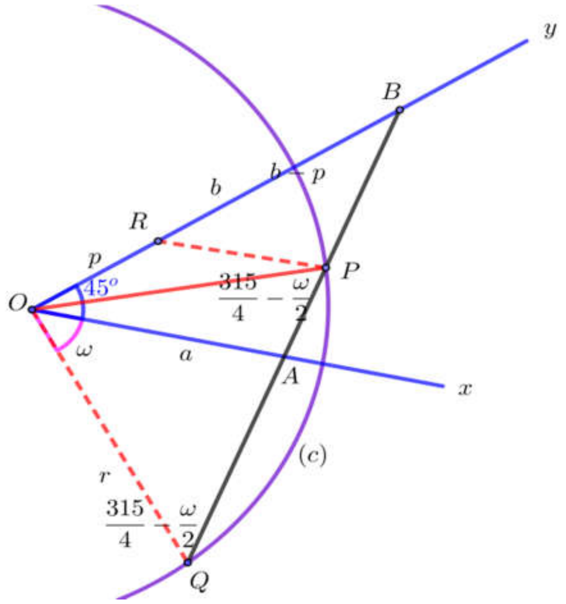


$$\frac{b}{a}=\varphi=\frac{\sqrt{5}+1}{2}, \widehat{\omega}=?$$

Prove that: $\frac{1}{a}+\frac{1}{b}=\frac{1}{p}$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Ogwuche Moses-Nigeria



Let $AP=x, PB=y$. Considering ΔAOB. Since OP –is the bisector of $\angle xOy$, then $\frac{x}{y}=\frac{a}{b}\Rightarrow y=\frac{bx}{a}$. By law of cosines: $(x+y)^2=a^2+b^2-2ab\cos 45^o=a^2+b^2-ab\sqrt{2}$.

Put $y=\frac{bx}{a}\Rightarrow\left(x+\frac{bx}{a}\right)^2=a^2+b^2-ab\sqrt{2}\Rightarrow x=\frac{a\sqrt{a^2+b^2-ab\sqrt{2}}}{a+b}; y=\frac{b\sqrt{a^2+b^2-ab\sqrt{2}}}{a+b}$.

Considering ΔOPA: $\angle OPA=\frac{315^o}{4}-\frac{\omega}{2}$, by law of sines: $\frac{\sin\left(\frac{315^o}{4}-\frac{\omega}{2}\right)}{a}=\frac{\sin\left(\frac{45^o}{2}\right)}{x}\Rightarrow$

$$\omega=\frac{315^o}{2}-2\sin^{-1}\left(\frac{(a+b)\sin\left(\frac{45^o}{2}\right)}{\sqrt{a^2+b^2-ab\sqrt{2}}}\right)$$

If $\frac{b}{a}=\frac{\sqrt{5}+1}{2}$, then $\omega=\frac{315^o}{2}-2\sin^{-1}\left(\frac{\left(1+\frac{b}{a}\right)\sin\left(\frac{45^o}{2}\right)}{\sqrt{1+\left(\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)\sqrt{2}}}\right)$

$$\omega = \frac{315^o}{2} - 2sin^{-1}\left(\frac{\frac{3+\sqrt{5}}{2}\cdot\frac{\sqrt{2-\sqrt{2}}}{2}}{\frac{5+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\cdot\sqrt{2}}\right) = 96.85923223487^o$$

Since, $PR \parallel OA$, then ΔBRP is similar to ΔBOA. Hence,

$$\frac{b-p}{b} = \frac{y}{x+y} \Rightarrow p = \frac{bx}{x+y} = \frac{b\left(\frac{a\sqrt{a^2+b^2-ab\sqrt{2}}}{a+b}\right)}{\sqrt{a^2+b^2-ab\sqrt{2}}}$$

$$p = \frac{ab}{a+b} \Rightarrow \frac{1}{p} = \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b}$$

Solution 2 by Juan Jose Isach Mayo-Spain

In ΔOAB we have that $OA = a, OB = b$ and $\angle AOB = 45^o$. Then

$$AB = \sqrt{OB^2 + OA^2 - 2OA\cdot OB\cdot cos45^o} = \sqrt{a^2+b^2-ab\sqrt{2}}$$

Let P be a point of AB such that OP is angle bisector of $\angle AOB$. Bisector theorem aplicated in vertex B of ΔOAB: $\frac{PB}{OB} = \frac{AB-PB}{OA} \Leftrightarrow PB = \frac{b\cdot AB}{a+b}$. How $AP = AB - PB \Rightarrow AP = \frac{a\cdot AB}{a+b}$.

Applying Stewart theorem in ΔAOB for the cevian OP:

$$OP^2\cdot AB = OA^2\cdot PB + OB^2\cdot AP - PB\cdot PB\cdot AB$$

$$OP^2\cdot AB = OA^2\cdot\frac{b\cdot AB}{a+b} + OB^2\cdot\frac{a\cdot AB}{a+b} - \frac{b\cdot AB}{a+b}\cdot\frac{a\cdot AB}{a+b}\cdot AB$$

$$OP^2 = \frac{a^2b}{a+b} + \frac{b^2a}{a+b} - \frac{ab\left(a^2+b^2-ab\sqrt{2}\right)}{(a+b)^2} = \left(2+\sqrt{2}\right)\frac{a^2b^2}{(a+b)^2} = \frac{ab\sqrt{\sqrt{2}+2}}{a+b}$$

How the triangles ΔOAB and ΔRPB are similarity and $\frac{PB}{AB} = \frac{b}{a+b}$ then

$$RP = OA\cdot\frac{PB}{AB} = \frac{ab}{a+b}$$

If denominate $p = RP = \frac{ab}{a+b}$. It's verify: $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{1}{p}$.

If now we consider the ΔOAP and $\alpha = \angle OPA$ then:

$$\cos\alpha=\frac{OP^2+AP^2-OA^2}{2\cdot OP\cdot AP}=\frac{\frac{b^2(\sqrt{2}+2)+(a^2+b^2-ab\sqrt{2})-(a+b)^2}{(a+b)^2}}{2\frac{\sqrt{(\sqrt{2}+2)b}}{a+b}\cdot\frac{\sqrt{a^2+b^2-ab\sqrt{2}}}{a+b}}$$

$$\cos\alpha=\frac{(b-a)(\sqrt{2}+2)}{2\sqrt{(\sqrt{2}+2)}\sqrt{a^2+b^2-ab\sqrt{2}}}=\frac{(b-a)\sqrt{\sqrt{2}+2}}{2\sqrt{a^2+b^2-ab\sqrt{2}}}$$

Let ω be the angle $\angle QOA$. The triangle ΔQOA is isosceles, then $\omega=157,5^o-2\alpha$.

In particular if $\frac{b}{a}=\varphi=\frac{1+\sqrt{5}}{2}$, then:

$$\cos\alpha=\frac{\left(\frac{(1+\sqrt{5})a}{2}-a\right)(2+\sqrt{2})}{2\sqrt{(\sqrt{2}+2)}\sqrt{a^2+\frac{(1+\sqrt{5})^2a^2}{4}-\sqrt{2}a\frac{(1+\sqrt{5})a}{2}}}=$$

$$=\frac{\frac{1}{4}\sqrt{2}\sqrt{\sqrt{2}+2}(\sqrt{5}-1)}{\sqrt{\sqrt{5}-\sqrt{2}-\sqrt{10}+5}}$$

$$\alpha=\frac{180^o}{\pi}\cos^{-1}\left(\frac{\frac{1}{4}\sqrt{2}\sqrt{\sqrt{2}+2}(\sqrt{5}-1)}{\sqrt{\sqrt{5}-\sqrt{2}-\sqrt{10}+5}}\right)\cong 60,32038383882\ldots^o$$

$$\omega=157,5^o-2\alpha=36,8592322348\ldots^o$$

Solution 3 by proposer

Plagiogonal system: $(Ox,Oy,\vartheta),\vartheta=\widehat{xOy}$

$$\begin{cases}OP^2=\frac{2a^2b^2}{(a+b)^2}(1+\cos\vartheta)\\(c):x^2+y^2+2xy\cos\vartheta=OP^2\Rightarrow Q(q_1,q_2),\\AB:\frac{x}{a}+\frac{y}{b}=1\end{cases}$$

$$q_1=\frac{ab(-2a^2\cos\vartheta-a^2+2ab+b^2}{(a+b)(a^2-2ab\cos\vartheta+b^2)};\ q_2=\frac{ab(-2b^2\cos\vartheta+a^2+2ab-b^2)}{(a+b)(a^2-2ab\cos\vartheta+b^2)}$$

$$\lambda_{OQ}=\lambda=\frac{q_2}{q_1}=\frac{a^2-b^2+2ab-2b^2\cos\vartheta}{-a^2+b^2+2ab-2a^2\cos\vartheta},\qquad \lambda_{OA}=0$$

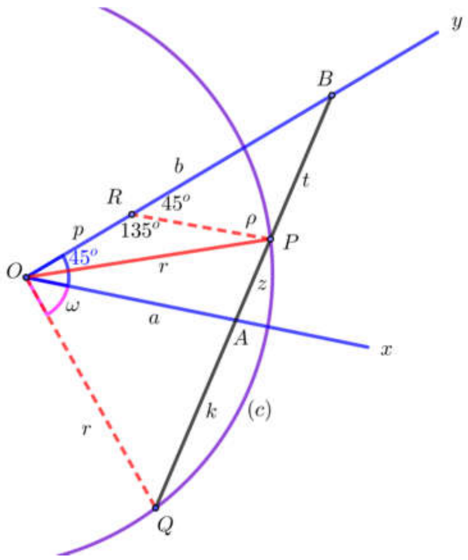
$$tan\omega=\frac{(0-\lambda)sin\vartheta}{(0+\lambda)cos\vartheta+1}\overset{\vartheta=45^o}{\Longrightarrow}$$

$$\omega=tan^{-1}\left\{\frac{2ab+a^2-(1+\sqrt{2})b^2}{-2(1+\sqrt{2})ab+(1+\sqrt{2})a^2+b^2}\right\}$$

$$\frac{b}{a}=\varphi=\frac{1+\sqrt{5}}{2}\Rightarrow\omega=tan^{-1}\left(\frac{-1+3\sqrt{2}-\sqrt{5}+\sqrt{10}}{-3+\sqrt{5}+2\sqrt{10}}\right)\Rightarrow\omega=36,86^o$$

$$\{OP: y=x, AB:\frac{x}{a}+\frac{y}{b}=1\Rightarrow P(p,p), p=\frac{ab}{a+b}\Rightarrow\frac{1}{a}+\frac{1}{b}=\frac{1}{p}$$

Solution 4 by Jose Ferreira Queiroz-Olinda-Brazil



$$AP=z, PB=t, cos\frac{\pi}{8}=\frac{\sqrt{2+\sqrt{2}}}{2}, \varphi^2=1+\varphi, AQ=k.$$

$\angle ROP=\angle RPO=22,5^o=\frac{\pi}{8}\Rightarrow\Delta ROP$ –isosceles, then $OR=RP, OQ=OP=r$.

$$OP^2=OR^2+RP^2-2OR\cdot RP\cdot cos135^o$$

$$r^2=p^2+p^2-2p^2\left(-\frac{\sqrt{2}}{2}\right)=2r^2+r^2\sqrt{2}\Rightarrow r=p\sqrt{2+\sqrt{2}}$$

ΔBOA and ΔBRP are similarity, so $\frac{b-p}{b}=\frac{p}{a}=\frac{t}{t+z}$, consider $\frac{b}{a}=\varphi$, then:

$$a=\varphi p, b=\varphi^2 p, t=\varphi z\Rightarrow\frac{b-p}{b}=\frac{p}{a}\Rightarrow ab-pa=pb\Rightarrow ab=p(a+b)$$

$$\Rightarrow\frac{1}{p}=\frac{1}{a}+\frac{1}{b}$$

In ΔPOA: $z^2 = a^2 + r^2 - 2ar \cdot \cos\frac{\pi}{8} = \varphi^2 p^2 + 2p^2 + p^2\sqrt{2} - 2\varphi p \cdot p\sqrt{2+\sqrt{2}} \cdot \frac{\sqrt{2+\sqrt{2}}}{2}$

$$z^2 = p^2\left(3 + \sqrt{2} - (1+\sqrt{2})\varphi\right)$$

Applying Stewart's theorem in ΔOPQ: $r^2 z + r^2 k - a^2(z+k) = (z+k)zk$

$$r(z+k) - a^2(z+k) = zk(z+k) \Leftrightarrow r^2 - a^2 = zk.$$

$$k = \frac{r^2 - a^2}{z} = \frac{2p^2 + p^2\sqrt{2} - p^2\varphi^2}{2} = \frac{p^2\left(1 + \sqrt{2} - \varphi\right)}{z}$$

$$k^2 = p^2 \cdot \frac{4 + 2\sqrt{2} - \left(1 + 2\sqrt{2}\right)\varphi}{3 + \sqrt{2} - \left(1 + \sqrt{2}\right)\varphi}; (I)$$

In ΔAOQ: $k^2 = r^2 + a^2 - 2ar \cdot \cos w$

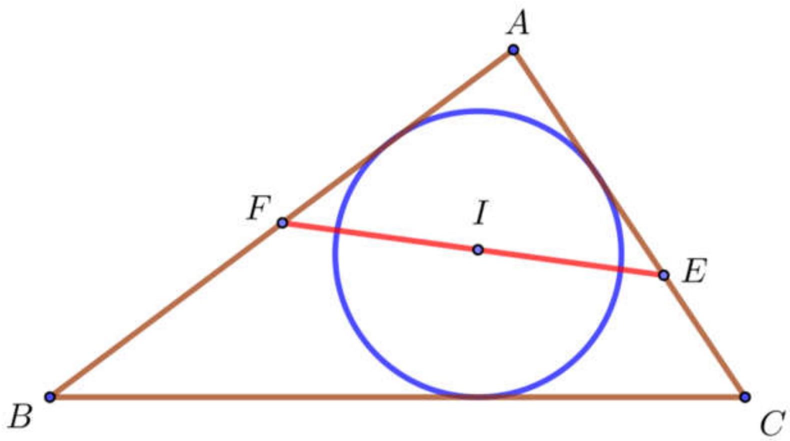
$$k^2 = 2p^2 + p^2\sqrt{2} + \varphi^2 p^2 - 2\varphi p \cdot \sqrt{2 + \sqrt{2}}; (II)$$

From $(I), (II)$ it follows that:

$$\cos\omega = \frac{6 + 3\sqrt{2} - \varphi(2 + 2\sqrt{2})}{3 + \sqrt{2} - \varphi\left(1 + \sqrt{2}\right)2\varphi\sqrt{2 + \sqrt{2}}}, \qquad \varphi = \frac{b}{a}; \ \varphi = \frac{\sqrt{5}+1}{2}$$

$$\cos\omega = \frac{\left(\sqrt{10} + 3\sqrt{5} - 1\right)\sqrt{4 - 2\sqrt{2}}}{12} \Rightarrow \omega \cong 36,86^o$$

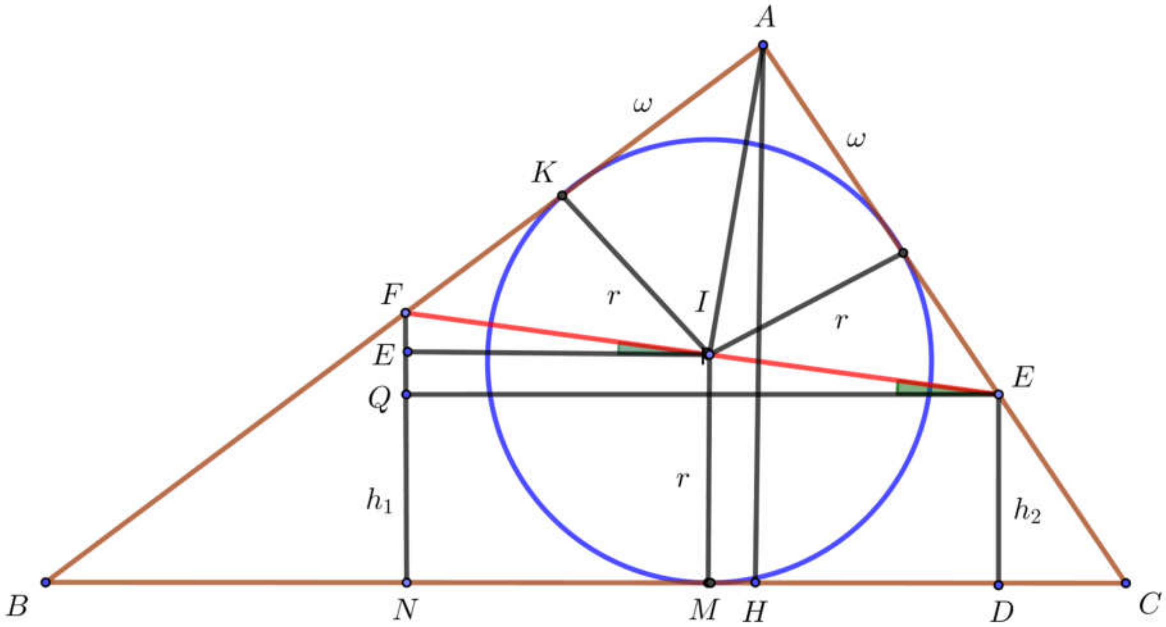
135.



$CA = b; AB = c; AE = e; AF = f$.Find: EF^2

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Brazil



$$AH = h_a; IM = r; FN = h_1; ED = h_2; EF = l; IF = z; IE = l - z$$

$$2s = a + b + c; F = \frac{1}{2}ah_a = sr; r = \frac{ah_a}{2s}$$

Applying angle bisector theorem, we have:

$$\frac{IF}{AF} = \frac{IE}{AE} \Rightarrow \frac{z}{t} = \frac{l - z}{e} \Rightarrow z = \frac{lt}{e + t} \Rightarrow \frac{l}{z} = \frac{e + t}{t}; (1)$$

ΔAHB **is similar** ΔFNB**, so** $\frac{AH}{FN} = \frac{AB}{FB} \Rightarrow \frac{h_a}{h_1} = \frac{c}{c-t} \Rightarrow h_1 = \frac{h_a}{c}(c - t)$

ΔAHC **is similar** ΔEDC**, so** $\frac{AH}{ED} = \frac{AC}{EC} \Rightarrow \frac{h_a}{h_2} = \frac{b}{b-e} \Rightarrow h_2 = \frac{h_a}{b}(b - e)$

$$FQ = h_1 - h_2 = \frac{h_a}{bc}(ec - bt)$$

$$FP = h_1 - r = \frac{h_a}{c}(c - t) - \frac{ah_a}{2s} = \frac{h_a}{2sc}(2s(c - t) - ac)$$

ΔFQE **is similar** ΔFPI: $\frac{FQ}{FP} = \frac{EF}{IF} \Rightarrow \frac{\frac{h_a}{bc}(ec-bt)}{\frac{h_a}{2sc}(2s(c-t)-ac)} = \frac{l}{z} \overset{(1)}{\Rightarrow}$

$$\frac{2s(ec - bt)}{2sb(2s(c - t) - abc)} = \frac{e + t}{t} \Rightarrow a = \frac{bc}{et}(e + t) - (b + c)$$

Now, in ΔIKF and ΔILE: $\begin{cases} IF^2 = IK^2 + KF^2 \\ IE^2 = IL^2 + LE^2 \end{cases} \Rightarrow \begin{cases} z^2 = r^2 = (t-\omega)^2 \\ (l-z)^2 = r^2 + (e-\omega)^2 \end{cases}$

$$z^2 - (t-\omega)^2 = (l-z)^2 - (e-\omega)^2 \Leftrightarrow e^2 - t^2 + 2\omega(t-e) = l^2 - 2lz$$

$$e^2 - t^2 + 2\omega(t-e) = l^2 - 2l\cdot\frac{lt}{e+t} \Leftrightarrow e^2 - t^2 + 2\omega(t-e) = \frac{l^2(e-t)}{e+t}$$

$$e + t - 2\omega = \frac{l^2}{e+t} \Rightarrow l^2 = (e+t)^2 - 2\omega(e+t)$$

$$l^2 = (e+t)^2 - (b+c-a)(e+t)$$

$$l^2 = (e+t)\left[(e+t) - (b+c) + \frac{bc}{et}(e+t) - (b+c)\right]$$

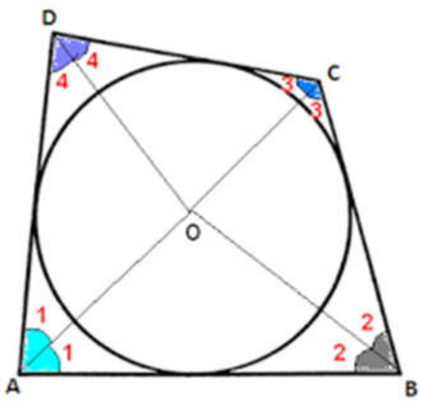
$$l^2 = (e+t)\left[\frac{(e+t)(bc+et)}{et} - 2(b+c)\right]$$

Therefore, $EF^2 = \frac{1}{et}(e+t)^2(bc+et) - 2(e+t)(b+c)$

136. $ABCD$ **–tangential quadrilateral,** O **–incenter,**

A_1, B_1, C_1, D_1 **–intersection points of incircle with** AO, BO, CO, DO**. Prove that:**

$$AA_1 + BB_1 + CC_1 + DD_1 \geq 4(\sqrt{2}-1)R$$



Proposed by Ionuţ Florin Voinea-Romania

Solution by George Florin Şerban-Romania

$$C(O,R) \cap AB = \{M\} \Rightarrow OM \perp AB$$

In $\Delta AOM, \mu(\sphericalangle AMO) = 90^o, sin\,(MAO) = sin\frac{A}{2} = \frac{OM}{AO} = \frac{R}{AO} \Rightarrow AO = \frac{R}{sin\frac{A}{2}}$

$$\sum_{cyc} AA_1 = \sum_{cyc}(AO - OA_1) = \sum_{cyc}\left(\frac{R}{sin\frac{A}{2}} - R\right) = \sum_{cyc}\frac{R}{sin\frac{A}{2}} - \sum_{cyc} R =$$

$$= R\sum_{cyc}\frac{1}{sin\frac{A}{2}} = 4R \overset{(1)}{\geq} 4(\sqrt{2}-1)R$$

$$(1) \Leftrightarrow R\sum_{cyc}\frac{1}{sin\frac{A}{2}} \geq 4\sqrt{2}R \Rightarrow \sum_{cyc}\frac{1}{sin\frac{A}{2}} \geq 4\sqrt{2}; (2)$$

$$\sum_{cyc}\mu(A) = 2\pi \Rightarrow \sum_{cyc}\frac{\mu(A)}{2} = \pi \Rightarrow \frac{\mu(A)}{2}, \frac{\mu(B)}{2}, \frac{\mu(C)}{2}, \frac{\mu(D)}{2} \in (0,\pi)$$

$$sin\frac{x}{2} > 0, \forall x \in (0,\pi) \Rightarrow \sum_{cyc}\frac{1}{sin\frac{A}{2}} \overset{Bergstrom}{\geq} \frac{16}{\sum_{cyc} sin\frac{A}{2}}; (3)$$

Let $f: (0,2\pi) \to \mathbb{R}, f(x) = sin\frac{x}{2}, f'(x) = \frac{1}{2}cos\frac{x}{2}, f''(x) = -\frac{1}{4}sin\frac{x}{2} < 0, \forall x \in (0,2\pi)$

f –concave, applying Jensen inequality, it follows that:

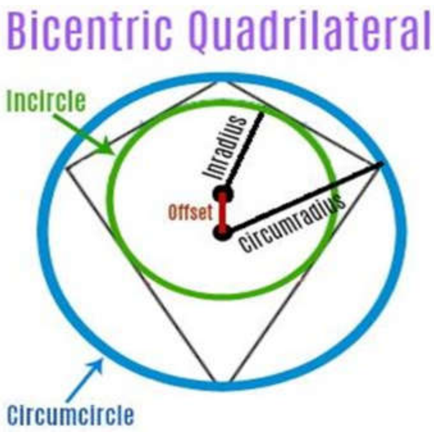
$$\sum_{cyc} sin\frac{A}{2} \leq 4sin\frac{\pi}{4} = 2\sqrt{2}; (4)$$

From (3),(4) it follows that: $\sum_{cyc}\frac{1}{sin\frac{A}{2}} \geq 4\sqrt{2} \Rightarrow (1) true.$

Therefore,

$$AA_1 + BB_1 + CC_1 + DD_1 \geq 4(\sqrt{2}-1)R$$

137.



a, b, c, d –sides, –inradii in a bicentric quadrilateral. Prove that:

$$\frac{a^4}{b} + \frac{b^4}{c} + \frac{c^4}{d} + \frac{d^4}{a} \geq 32r^3$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{d}+\frac{d^4}{a}\overset{BCS}{\geq}\frac{(\sum a)^4}{4^2\sum a}=\frac{(\sum a)^3}{16}\overset{(1)}{\geq}32r^3$$

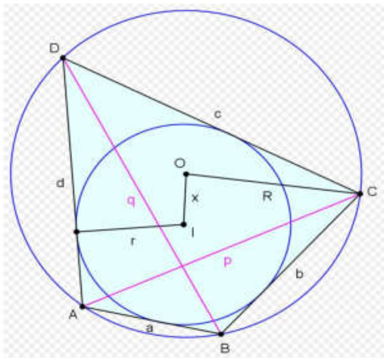
$$(1)\Leftrightarrow \sum a\geq 8r$$

We know that in any bicentric quadrilateral $a+c=b+d$ and $r=\frac{\sqrt{abcd}}{a+c}=\frac{\sqrt{abcd}}{b+d}$

$$(1)\Leftrightarrow 2(a+c)\geq\frac{8\sqrt{abcd}}{b+d}\Leftrightarrow(a+c)(b+d)\geq 4\sqrt{abcd}$$

Which is clearly true from AM-GM: $a+c\geq 2\sqrt{ac}$ and $b+d\geq 2\sqrt{bd}$

138.



R –circumradii, r –inradii, s –semiperimeter in a bicentric quadrilateral.

Prove that: $r+\sqrt{r+4R^2}\geq\frac{32R^2r}{s^2}$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Blundon and Eddy, we have:

$$s^2\geq 8r\left(\sqrt{4R^2+r^2}-r\right)$$

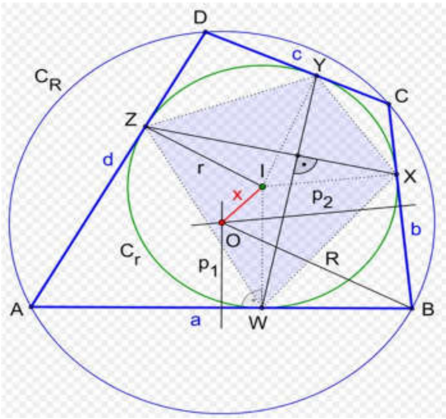
$$\Rightarrow\frac{32R^2r}{s^2}\leq\frac{4R^2}{\sqrt{4R^2+r^2}-r}=\frac{4R^2\left(\sqrt{r^2+4R^2}+r\right)}{(4R^2+r^2)-r^2}$$

Therefore,

$$r+\sqrt{r+4R^2}\geq\frac{32R^2r}{s^2}$$

139. a, b, c, d **–sides,** r **–inradii in a bicentric quadrilateral. Prove that:**

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\geq 8r$$



Proposed by Daniel Sitaru-Romania

Solution 1 by Alex Szoros-Romania

If we denote F –area of this quadrilateral, then: $4r^2\leq F=\sqrt{abcd}$

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\overset{Bergstrom}{\geq}\frac{(a+b+c+d)^2}{a+b+c+d}=a+b+c+d\geq$$

$$4\sqrt[4]{abcd}=4\sqrt{F}\geq 4\sqrt{4r^2}=8r$$

Observation:

$ABCD$ –inscriptible quadrilateral, then $F=\sqrt{(s-a)(s-b)(s-c)(s-d)}$

$$2s=a+b+c+d=2(a+c)=2(b+d)\Rightarrow$$

$$F=\sqrt{(a+c-a)(b+d-b)(a+c-c)(b+d-d)}=\sqrt{abcd}$$

$$\because r=\frac{F}{s}\Leftrightarrow r=\frac{2\sqrt{abcd}}{a+b+c+d}$$

But: $(a+b+c+d)^2=[(a+b)+(c+d)]^2\geq 4(a+b)(c+d)\geq$

$$\geq 4\cdot 2\sqrt{ab}\cdot 2\sqrt{cd}=16\sqrt{abcd}$$

$$(2s)^2\geq 16F\Rightarrow s^2\geq 4F\Rightarrow\left(\frac{F}{r}\right)^2\geq 4F\Rightarrow F\geq 4r^2$$

Therefore, $F\geq\sqrt{abcd}\geq 4r^2$

Solution 2 by Adrian Popa-Romania

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\overset{Bergstrom}{\geq}\frac{(a+b+c+d)^2}{a+b+c+d}=a+b+c+d\geq$$

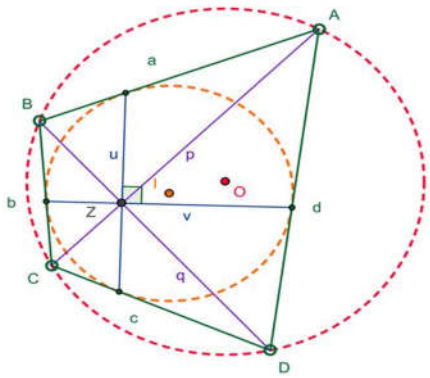
$$r=\frac{\sqrt{abcd}}{a+c}=\frac{\sqrt{abcd}}{\frac{a+b+c+d}{2}}=\frac{2\sqrt{abcd}}{a+b+c+d}$$

We must to prove that:

$$a+b+c+d\geq\frac{16\sqrt{abcd}}{a+b+c+d}\Leftrightarrow(a+b+c+d)^2\geq 16\sqrt{abcd}$$

which is true from AM-GM: $\frac{a+b+c+d}{4}\overset{AM-GM}{\geq}\sqrt[4]{abcd}$

140.



a,b,c,d −sides, r −inradii, R −circumradii, F −area in a bicentric quadrilateral. Prove that:

$$a^4+b^4+c^4+d^4\geq 8F^2\left(1-\sqrt{\frac{r}{R}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that: $a+c=b+d; r=\frac{\sqrt{abcd}}{a+c}; F=\sqrt{abcd}=sr$

$s=\frac{\sum a}{2}=b+d\overset{(1)}{\geq}4r=4\frac{\sqrt{abcd}}{a+c}; (1)\Leftrightarrow(a+c)(b+d)\geq 4\sqrt{abcd}$, which is true from

AM-GM: $a+c\geq 2\sqrt{ac}, b+d\geq 2\sqrt{bd}\Rightarrow s\geq 4r; (I)$

$$\sum_{cyc} a^4 \overset{Holder}{\geq} \frac{(\sum a)^4}{4^3} = \frac{s^4}{4} \overset{(2)}{\geq} 8F^2\left(1-\sqrt{\frac{r}{R}}\right);\ (2) \Leftrightarrow s^2 \geq 32r^2\left(1-\sqrt{\frac{r}{s}}\right) \Leftrightarrow$$

$$s^2 + 32r^2\sqrt{\frac{r}{s}} \geq 32r^2$$

By AM-GM: $s^2 + 32r^2\sqrt{\frac{r}{s}} \geq 2\sqrt{s^2 \cdot 32r^2\sqrt{\frac{r}{s}}} = 8r\sqrt{2s\sqrt{sr}} \overset{(I)}{\geq} 8r\sqrt{2 \cdot 4r\sqrt{4r^2}} = 32r^2$

Therefore,

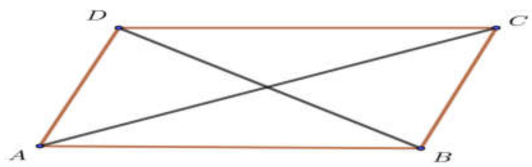
$$a^4 + b^4 + c^4 + d^4 \geq 8F^2\left(1-\sqrt{\frac{r}{R}}\right)$$

141. $ABCD$ **–parallelogram,** $AB + AD \geq n > 0$**. Prove that:**

$$max(AC^2, BD^2) \geq \frac{n^2}{2}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Adrian Popa-Romania



$$max(AC^2, AD^2) = AC^2$$

$\angle B$ –Optuse, then $cosB < 0$

$$AD \equiv BC \Rightarrow AD + BC > n$$

In ΔABC: $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot cosB$

We want to prove that:

$$AD^2 \geq \frac{(AB+BC)^2}{2} \Leftrightarrow AB^2 + BC^2 - 2AB \cdot BC \cdot cosB \geq \frac{AB^2 + 2AB \cdot BC + BC^2}{2} \Leftrightarrow$$

$$AB^2 + BC^2 \geq 2AB \cdot BC \cdot (1 + 2cosB)$$

But: $AB^2 + BC^2 \geq 2AB \cdot BC \geq 2AB \cdot BC(1 + 2cosB)$; $(\because 1 + 2cosB < 1)$

Therefore,

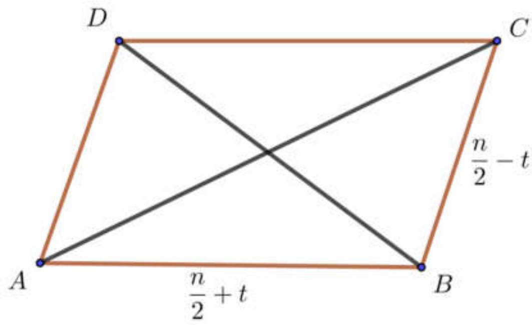
$$AC^2 > \frac{(AB + +AD)^2}{2} \geq \frac{n^2}{2}$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $AB = \frac{n}{2} + t, BC = \frac{n}{2} - t, 0 \leq |t| < \frac{n}{2}$

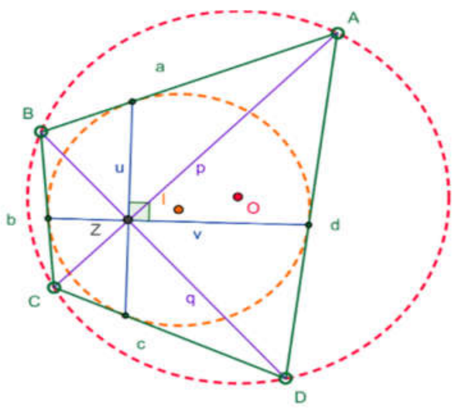
Let $\theta = max(\angle DAB, \angle CBA)$, then $\theta \geq \frac{\pi}{2}$

Assume $\theta = \angle CBA$, then



$$AC^2 = \left(\frac{n}{2} + t\right)^2 + \left(\frac{n}{2} - t\right)^2$$
$$- 2\left(\frac{n}{2} + t\right)\left(\frac{n}{2} - t\right) cos\theta \geq$$
$$2\left(\frac{n}{2}\right)^2 + 2t^2 \geq \frac{n^2}{2}; (\because cos\theta \leq)$$

142.



If e, f −diagonals, R −circumradii, O −circumcenter, I −incenter,

s −semiperimeter in a bicentric quadrilateral then:

$$\left(\frac{R - OI}{e}\right)^2 + \left(\frac{R + OI}{f}\right)^2 \geq \frac{s^2}{16R^2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that: $OI = \sqrt{R^2 + r^2 - r\sqrt{4R^2 + r^2}}(Fuss's\ theorem)$ and

$ef=2r\left(r+\sqrt{r^2+4R^2}\right)$. Hence,

$$\left(\frac{R-OI}{e}\right)^2+\left(\frac{R+OI}{f}\right)^2\overset{AM-GM}{\geq}2\cdot\frac{(R-OI)(R+OI)}{ef}=$$

$$=2\cdot\frac{R^2-\left(\sqrt{R^2+r^2-r\sqrt{4R^2+r^2}}\right)^2}{2r\left(r+\sqrt{r^2+4R^2}\right)}=\frac{\sqrt{r^2+4R^2}-r}{\sqrt{r^2+4R^2}+r}=$$

$$=\frac{\left(\sqrt{r^2+4R^2}-r\right)^2}{4R^2}\overset{(1)}{\geq}\frac{s^2}{16R^2}\Leftrightarrow s\leq 2\left(\sqrt{r^2+4R^2}-r\right)$$

By Blundon and Eddy's inequality:

$$s\leq\sqrt{r^2+4R^2}+r\overset{(2)}{\leq}2\left(\sqrt{r^2+4R^2}-r\right)\Leftrightarrow 3r\leq\sqrt{r^2+4R^2}\Leftrightarrow$$

$\sqrt{2}r\leq R$, which is true from Fejes Toth's inequality: $\sqrt{2}r\leq R$

Therefore,

$$\left(\frac{R-OI}{e}\right)^2+\left(\frac{R+OI}{f}\right)^2\geq\frac{s^2}{16R^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

If $m=$ product of the lengths of the diagonals, then : $\frac{m}{4r^2}-\frac{4R^2}{m}=1$

$$\Rightarrow m^2-4mr^2-16R^2r^2=0$$

$$\Rightarrow m=\frac{4r^2\pm\sqrt{16r^4+64R^2r^2}}{2}=2r^2\pm 2r\sqrt{4R^2+r^2}$$

$$=2r\left(r+\sqrt{4R^2+r^2}\right)\ (\because m>0)\ \therefore ef\overset{(a)}{=}2r\left(r+\sqrt{4R^2+r^2}\right)$$

Via Carlitz, $OI^2=R^2-\frac{2Rr}{s}\sqrt{\frac{(ab+cd)(ad+bc)}{ac+bd}}$ and consequently, $OI^2\overset{(i)}{<}R^2$

Via Nicolaus Fuss (1792), if $t=OI$, then : $\frac{1}{r^2}=\frac{1}{(R+t)^2}+\frac{1}{(R-t)^2}\Rightarrow\frac{1}{r^2}$

$$=\frac{2R^2+2t^2}{(R^2-t^2)^2}\Rightarrow t^4-2t^2(R^2+r^2)+R^4-2R^2r^2=0$$

$$\Rightarrow \mathrm{t}^2=\frac{2(\mathrm{R}^2+\mathrm{r}^2)\pm\sqrt{4(\mathrm{R}^2+\mathrm{r}^2)^2-4(\mathrm{R}^4-2\mathrm{R}^2\mathrm{r}^2)}}{2}=\mathrm{R}^2+\mathrm{r}^2\pm\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}\ \textit{and}$$

$$\because \mathrm{t}^2 \overset{\text{via (i)}}{<} \mathrm{R}^2 \therefore \mathrm{OI}^2 \overset{\text{(ii)}}{=} \mathrm{R}^2+\mathrm{r}^2-\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}$$

$$\text{Now, via (i), } \mathrm{R}-\mathrm{OI}>0 \therefore \left(\frac{\mathrm{R}-\mathrm{OI}}{\mathrm{e}}\right)^2+\left(\frac{\mathrm{R}+\mathrm{OI}}{\mathrm{f}}\right)^2 \overset{\text{A-G}}{\geq} 2\left(\frac{\mathrm{R}-\mathrm{OI}}{\mathrm{e}}\right)\left(\frac{\mathrm{R}+\mathrm{OI}}{\mathrm{f}}\right) \overset{\text{via }(a)}{=} \frac{2(\mathrm{R}^2-\mathrm{OI}^2)}{2\mathrm{r}\left(\mathrm{r}+\sqrt{4\mathrm{R}^2+\mathrm{r}^2}\right)} \overset{\text{via (ii)}}{=} \frac{\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}-\mathrm{r}^2}{\mathrm{r}\left(\mathrm{r}+\sqrt{4\mathrm{R}^2+\mathrm{r}^2}\right)}$$

$$=\frac{\left(\sqrt{4\mathrm{R}^2+\mathrm{r}^2}-\mathrm{r}\right)^2}{\left(\sqrt{4\mathrm{R}^2+\mathrm{r}^2}-\mathrm{r}\right)\left(\sqrt{4\mathrm{R}^2+\mathrm{r}^2}+\mathrm{r}\right)}=\frac{4\mathrm{R}^2+2\mathrm{r}^2-2\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}}{4\mathrm{R}^2+\mathrm{r}^2-\mathrm{r}^2}$$

$$\Rightarrow \left(\frac{\mathrm{R}-\mathrm{OI}}{\mathrm{e}}\right)^2+\left(\frac{\mathrm{R}+\mathrm{OI}}{\mathrm{f}}\right)^2 \overset{\text{(iii)}}{\geq} \frac{2\mathrm{R}^2+\mathrm{r}^2-\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}}{2\mathrm{R}^2}$$

$$\textit{Again}, \text{via Mirko Radic}, \frac{\mathrm{s}^2}{16\mathrm{R}^2}\leq\frac{\left(\sqrt{4\mathrm{R}^2+\mathrm{r}^2}+\mathrm{r}\right)^2}{16\mathrm{R}^2}$$

$$=\frac{2\mathrm{R}^2+\mathrm{r}^2+\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}}{8\mathrm{R}^2}\overset{?}{\leq}\frac{2\mathrm{R}^2+\mathrm{r}^2-\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}}{2\mathrm{R}^2}$$

$$\Leftrightarrow 8\mathrm{R}^2+4\mathrm{r}^2-4\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}\overset{?}{\geq}2\mathrm{R}^2+\mathrm{r}^2+\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}\Leftrightarrow 6\mathrm{R}^2+3\mathrm{r}^2\overset{?}{\geq}5\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}$$

$$\Leftrightarrow (6\mathrm{R}^2+3\mathrm{r}^2)^2-25\mathrm{r}^2(4\mathrm{R}^2+\mathrm{r}^2)\overset{?}{\geq}0$$

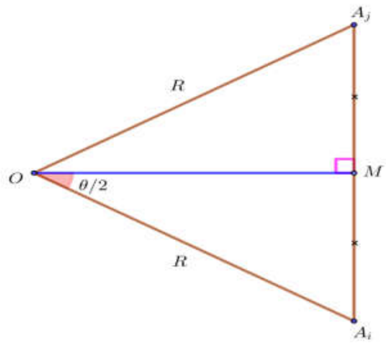
$$\Leftrightarrow 9x^2-16xy-4y^2\overset{?}{\geq}0\ (\text{where } x=\mathrm{R}^2 \textit{ and } y=\mathrm{r}^2)\Leftrightarrow (x-2y)(9x+2y)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because x\geq 2y \text{ via L. Fejes Toth}$$

$$\therefore \frac{\mathrm{s}^2}{16\mathrm{R}^2}\leq\frac{2\mathrm{R}^2+\mathrm{r}^2-\mathrm{r}\sqrt{4\mathrm{R}^2+\mathrm{r}^2}}{2\mathrm{R}^2}\overset{\text{via (iii)}}{\leq}\left(\frac{\mathrm{R}-\mathrm{OI}}{\mathrm{e}}\right)^2+\left(\frac{\mathrm{R}+\mathrm{OI}}{\mathrm{f}}\right)^2\ (\text{QED})$$

143. If $A_1A_2\ldots A_8$ – regular octagon then:

$$(A_1A_5+A_3A_5)(A_1A_7+A_3A_7)$$
$$=\left(2+\sqrt{2}\right)\cdot A_2A_5\cdot A_2A_7$$



Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let O and R be the center and the radius of the circle circumscribed to $A_1A_2\ldots A_8$.

In ΔOA_iA_j if $\mu\left(\widehat{A_iOA_j}\right)=\theta$

$$sin(MOA)=sin\frac{\theta}{2}=\frac{A_iA_j}{2R}\Rightarrow$$

$$A_iA_j = 2R \cdot sin\frac{\theta}{2}$$

We also know that: $\mu(A_\iota\widehat{O}A_{\iota+1}) = \frac{2\pi}{8} = \frac{\pi}{4} \Rightarrow A_1A_5 = 2R \cdot sin\frac{\pi}{2} = 2R;$

$$A_3A_5 = 2R \cdot sin\frac{\pi}{4} = \sqrt{2}R$$

$$A_1A_7 = 2R \cdot sin\frac{\pi}{4} = \sqrt{2}R;\ A_3A_7 = 2R \Rightarrow$$

$$(A_1A_5 + A_3A_5)(A_1A_7 + A_3A_7) = \left(2+\sqrt{2}\right)^2R^2;(1) \text{ and}$$

$$A_2A_5 = 2R \cdot sin\left(\frac{3\pi}{8}\right) = 2R \cdot cos\frac{\pi}{8} = \sqrt{2+\sqrt{2}}R$$

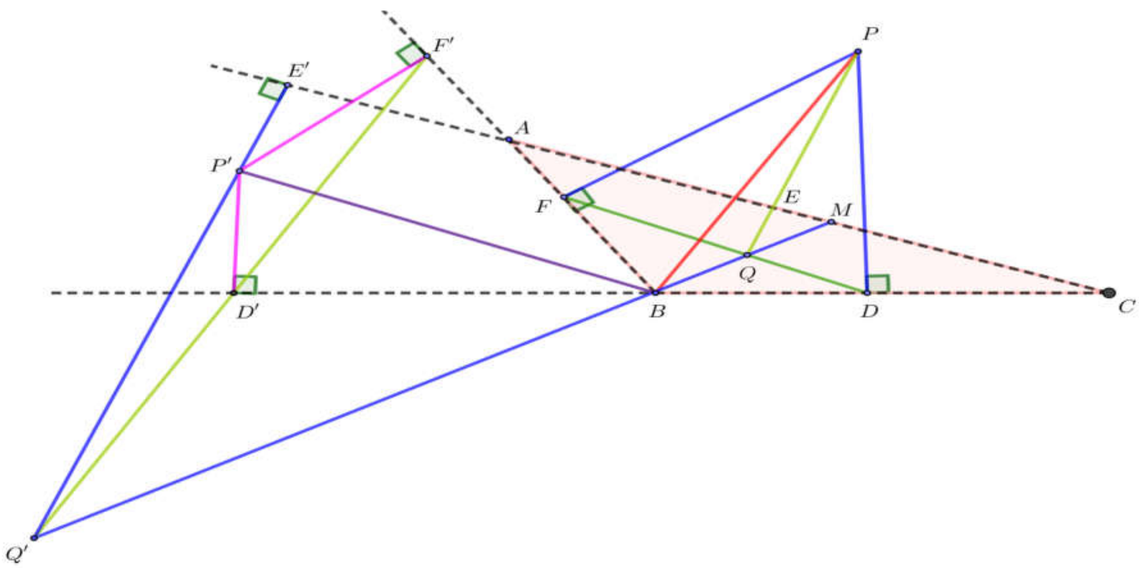
$$A_2A_7 = 2R \cdot sin\left(\frac{3\pi}{8}\right) = 2R \cdot cos\left(\frac{\pi}{8}\right) = \sqrt{2+\sqrt{2}}R$$

$$\Rightarrow \left(2+\sqrt{2}\right) \cdot A_2A_5 \cdot A_2A_7 = \left(2+\sqrt{2}\right)^2R^2;(2)$$

From (1),(2) it follows that:

$$(A_1A_5 + A_3A_5)(A_1A_7 + A_3A_7) = \left(2+\sqrt{2}\right) \cdot A_2A_5 \cdot A_2A_7$$

144.



ABC −random, BM −median; P $(or\ P')$ −random point on plane (ABC)

$PD\ (or\ P'D') \perp BC, PE\ (or\ P'E') \perp AC, PF\ (or\ P'F') \perp AB$

$PE \cap DF \cap BM = \{Q\}\ (or\ P'E' \cap D'F' \cap BM = \{Q'\})$

Prove that: $PD = DF$ $(or\ P'D' = P'F')$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal system: $BC \equiv Bx, BA \equiv By$

$$B(0,0), C(a,0), A(0,c), M\left(\frac{a}{2}, \frac{c}{2}\right), P(p_1, p_2), D(d,0), d = p_1 + p_2 \cos B$$

$$F(0,f), f = p_1 \cdot \cos B + p_2, \qquad \lambda_{PE} = \frac{c \cdot \cos B - a}{a \cdot \cos B - c}$$

$$DF: \frac{x}{d} + \frac{y}{f} = 1 \Rightarrow$$

$$(p_1 \cdot \cos B + p_2)x + (p_1 + p_2 \cdot \cos B)y = (p_1 \cdot \cos B + p_2)(p_1 + p_2 \cdot \cos B); (1)$$

$$PE: (y - p_2) = \lambda_{PE}(x - p_1)$$

$$(c \cdot \cos B - a)x + (c - a \cdot \cos B)y = p_1(c \cdot \cos B - a) + p_2(c - a \cdot \cos B); (2)$$

$$BM: \frac{y}{\frac{c}{2}} = \frac{x}{\frac{a}{2}} \Rightarrow cx - ay = 0; (3)$$

DF, PE, BM –concurrent

$$\begin{vmatrix} p_1 \cdot \cos B + p_2 & p_1 + p_2 \cdot \cos B & (p_1 \cdot \cos B + p_2)(p_1 + p_2 \cdot \cos B) \\ c \cdot \cos B - a & c - a \cdot \cos B & p_1(c \cdot \cos B - a) + p_2(c - a \cdot \cos B) \\ c & -a & 0 \end{vmatrix} = 0$$

$$ac \cdot \sin^2 B \cdot (x^2 - y^2) = 0 \Rightarrow x = y \text{ or } x = -y.$$

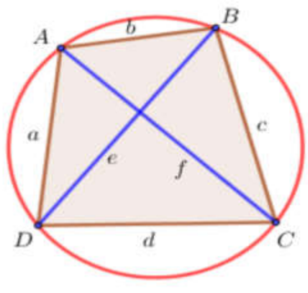
$P \in$internal bisector of $\angle ABC$.

145.

a, b, c, d –sides, e, f –diagonals in a cyclic quadrilateral.

If $\begin{cases} (ac)^5 - (bd)^5 = 242 \\ (ac)^3 + (bd)^3 = 28 \end{cases}$ **then:** $max(e^2, f^2) \geq 4$

Proposed by Radu Diaconu-Romania



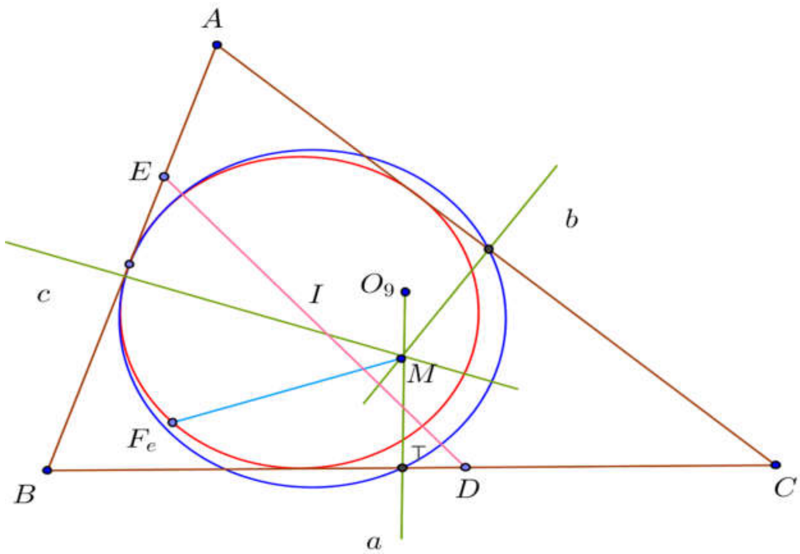
Solution by Adrian Popa-Romania

$$\begin{cases}(ac)^5-(bd)^5=242\\(ac)^3+(bd)^3=28\end{cases}\Rightarrow\begin{cases}ac=3\\bd=1\end{cases}$$

Using Ptolemy's theorem $ac+bd=ef\Rightarrow ef=4$

$$max(e^2,f^2)\geq\frac{e^2+f^2}{2}\overset{AM-GM}{\geq}ef=4$$

146.



$$\Delta ABC, BC=16, AB=10, BD=10, BE=\frac{20}{3}$$

I –incenter, $I\in DE$,F_e –Feuerbach point,M –Mittenpunkt

Find: MF_e

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Let $BC=a=16, BA=c=10, AC=b, BD=d=10, BE=e=\frac{20}{3}$

Is $b=\frac{a\cdot c}{d\cdot e}(d+e)-(a+c)=\frac{16\cdot10}{10\cdot\frac{20}{3}}\left(10+\frac{20}{3}\right)-(16+10)\Rightarrow b=14.$

$$\cos B=\frac{16^2+10^2-14^2}{2\cdot16\cdot10}=\frac{11}{2}\Rightarrow\mu(B)=60^\circ$$

Plagiogonal system: $BC\equiv BX; BA\equiv BD$

Let (c) the incircle, $(C): x^2+y^2+xy-12x-12y+36=0$

Let (ω) the N.P.C. circle, $(\omega): x^2+y^2+xy-13x-13y+40=0$

Hence, $F(2,2)$

Is $M\left(\frac{200}{31};\frac{80}{31}\right)$

$$FM^2=\left(\frac{80}{31}-2\right)^2+\left(\frac{200}{31}-2\right)^2+\left(\frac{80}{31}-2\right)\left(\frac{200}{31}-2\right)$$

Therefore,

$$MF_e=\frac{6}{31}\sqrt{607}$$

147. Let γ –be area of pedal triangle of Spieker's point in ΔABC. Find the value of γ in terms of s,r,R.

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Izumi Ainsworth-Lima-Peru

$\gamma=[DEF]$ =area of pedal triangle of Spieker's point in ΔABC, with $D,E,F\in BC,CA,AB$ respectively. Also, we know that:

$$S_pF=\frac{ab(a+b)}{8Rs},S_pE=\frac{ac(a+c)}{8Rs}\text{ and }\angle FS_pE=\pi-A$$

Applying the trigonometric area formula for 3 cyclic triangles, we have:

$$\gamma=\sum_{cyc}\frac{a^2bc(a+b)(a+c)\sin A}{2(8Rs)^2}=\frac{4RF}{2(8Rs)^2}\sum_{cyc}a(a+b)\cdot\frac{2F}{bc}=$$

$$=\frac{F}{(8Rs)^2}\sum_{cyc}a^2(a^2+ab+bc+ca)=\frac{rs}{(8Rs)^2}\left(\sum_{cyc}a^4+\sum_{cyc}ab\sum_{cyc}a^2\right)=$$

$$=\frac{rs}{(8Rs)^2}[2(s^2-r^2-4Rr)^2-8(rs)^2+(s^2+r^2+4Rr)\cdot 2(s^2-r^2-4Rr)]=$$

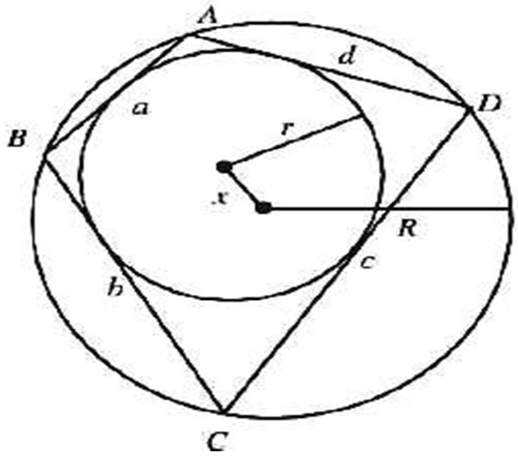
$$=\frac{rs}{(8Rs)^2}(4s^4-12(rs)^2-16s^2r)=\frac{rs(s^2-4Rr-3r^2)}{16R^2}$$

148. If e,f –diagonals, R –circumradii, r –inradii, s –semiperimeter in a bicentric quadrilateral, then:

$$2R\cdot\sqrt[3]{ef}\left(\sqrt[3]{e}+\sqrt[3]{f}\right)\le s\left(r+\sqrt{r^2+4R^2}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let $e = AC, f = BD$ and F –area of $ABCD$. We have:

$$2F = ad \cdot \sin A + bc \cdot \sin C = ab \cdot \sin B + cd \cdot \sin D$$

$$\sin A = \sin C = \frac{f}{2R}; \sin B = \sin D = \frac{e}{2R} \Rightarrow e = \frac{4RF}{ab + cd}; f = \frac{4RF}{ad + bc}; (1)$$

We also know that:

$$ef = 2r\left(2 + \sqrt{r^2 + 4R^2}\right); s = a + c = b + d; F = \sqrt{abcd}; r = \frac{\sqrt{abcd}}{a + c} = \frac{F}{s} \Rightarrow$$

$$\frac{s\left(r + \sqrt{r^2 + 4R^2}\right)}{2R} = \frac{s \cdot ef}{4Rr} \overset{(1)}{=} \frac{4Rs \cdot F^2}{r(ab + cd)(ad + bc)} = \frac{4RFs^2}{(ab + cd)(ad + bc)} =$$

$$= \frac{4RF(a + c)(b + d)}{(ab + cd)(ad + bc)} = \frac{4RF}{ab + cd} + \frac{4RF}{ad + bc} \overset{(1)}{=} e + f$$

So, we need to prove that:

$$\sqrt[3]{ef}\left(\sqrt[3]{e} + \sqrt[3]{f}\right) \le e + f \Leftrightarrow \left(\sqrt[3]{e} - \sqrt[3]{f}\right)^2\left(\sqrt[3]{e} + \sqrt[3]{f}\right) \ge 0, \text{ which is true.}$$

Therefore,

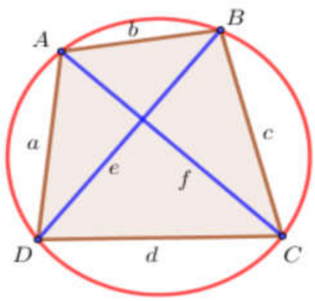
$$2R \cdot \sqrt[3]{ef}\left(\sqrt[3]{e} + \sqrt[3]{f}\right) \le s\left(r + \sqrt{r^2 + 4R^2}\right)$$

149. If a, b, c, d –sides, s –semiperimeter in a convex quadrilateral then:

$$\frac{(s - a)(s - b)(s - c)(s - d)}{(a + b)(b + c)(c + d)(d + a)} \le \frac{1}{16}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let e,f-diagonals. $a+b+c=(a+b)+c\overset{(\Delta)}{\geq}e+c\overset{(\Delta)}{\geq}d$.

So, let $s=x-a, y=s-b, z=s-c, t=s-d; x,y,z,t>0; x+y+z+t=2s$.

$$a+b=2s-(x+y)=z+t(and\ analogs)$$

So, we need to prove that: $16xyzt\leq(x+y)(y+z)(z+t)(t+x)$

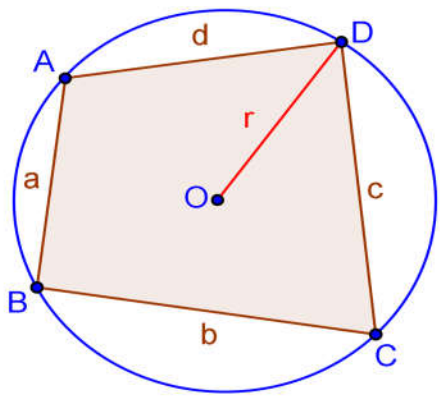
Which is true from $AM-GM$: $\prod(x+y)\geq\prod 2\sqrt{xy}=16xyzt$

Therefore,

$$\frac{(s-a)(s-b)(s-c)(s-d)}{(a+b)(b+c)(c+d)(d+a)}\leq\frac{1}{16}$$

150. a,b,c,d –sides, s –semiperimeter, r –inradii in a bicentric quadrilateral.Prove that:

$$\sum_{cyc}\frac{1}{a^2+b^2+c^2+r^2s^2}\leq\frac{1}{16r^4}$$



Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$s=a+c=b+d=\frac{(a+c)(b+d)}{s}\overset{AM-GM}{\geq}\frac{(2\sqrt{ac})(2\sqrt{bd})}{s}=4\cdot\frac{\sqrt{abcd}}{s}=4r\rightarrow$$

$$s \geq 4r; (1)$$

$$\sum_{cyc} \frac{1}{a^2+b^2+c^2+r^2s^2} \overset{AM-GM}{\leq} \sum_{cyc} \frac{1}{4\sqrt[4]{a^4b^4c^4s^2r^2}} = \frac{1}{4\sqrt{sr}} \sum_{cyc} \frac{1}{abc} =$$

$$= \frac{1}{4\sqrt{sr}} \cdot \frac{\sum a}{abcd} = \frac{1}{4\sqrt{sr}} \cdot \frac{2s}{(sr)^2} = \frac{1}{2\sqrt{r^5} \cdot \sqrt{s^3}} \overset{(1)}{\leq} \frac{1}{2\sqrt{r^5} \cdot \sqrt{4r^3}} = \frac{1}{16r^4}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$Via\ Beyer(1987), r = \frac{\sqrt{abcd}}{c+a} = \frac{\sqrt{abcd}}{b+d} = \frac{\sqrt{abcd}}{s} \Rightarrow abcd \overset{(i)}{=} r^2s^2$$

$$\sum_{cyc} \frac{1}{a^4+b^4+c^4+r^2s^2} \overset{A-G}{\leq} \sum_{cyc} \frac{1}{4\sqrt[4]{a^4b^4c^4r^2s^2}} = \frac{1}{4\sqrt{rs}} \left(\frac{1}{abc} + \frac{1}{bcd} + \frac{1}{cda} + \frac{1}{dab}\right)$$

$$= \frac{1}{4\sqrt{rs}} \left(\frac{a+b+c+d}{abcd}\right) \overset{via\ (i)}{=} \frac{2s}{4r^2s^2\sqrt{rs}} \overset{?}{\leq} \frac{1}{16r^4}$$

$$\Leftrightarrow (8s)^2 r^8 \overset{?}{\leq} r^4s^4.rs \Leftrightarrow 64r^3 \overset{?}{\leq} s^3 \Leftrightarrow s^2 \underset{(ii)}{\overset{?}{\geq}} 16r^2$$

$$Now, via\ Mirko\ Radic, s^2 \geq 8r\left(\sqrt{4R^2+r^2} - r\right) \overset{?}{\geq} 16r^2 \Leftrightarrow \sqrt{4R^2+r^2} - r \overset{?}{\geq} 2r$$

$$\Leftrightarrow \sqrt{4R^2+r^2} \overset{?}{\geq} 3r \Leftrightarrow 4R^2+r^2 \overset{?}{\geq} 9r^2 \Leftrightarrow R \overset{?}{\geq} \sqrt{2}r$$

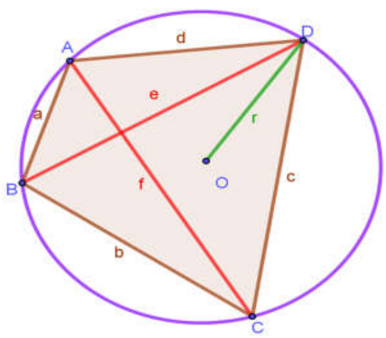
$$\rightarrow true\ via\ L.Fejes.Toth\ (1948) \therefore \sum_{cyc} \frac{1}{a^4+b^4+c^4+r^2s^2} \leq \frac{1}{16r^4}\ (QED)$$

151. a, b, c, d –sides, e, f –diagonals, s –semiperimeter, r –inradii in a bicentric quadrilateral. Prove that:

$$\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} + \frac{d}{s-d} \geq 2 + \frac{16r^2}{ef}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$\sum_{cyc}\frac{a}{s-a}\overset{Chebyshev's}{\geq}\frac{1}{4}\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{s-a}\right)\overset{CBS}{\geq}\frac{1}{4}\left(\sum_{cyc}a\right)\left(\frac{16}{\sum(s-a)}=4;\quad(1)\right.$$

We know that: $ef=2r\left(r+\sqrt{4R^2+r^2}\right)\overset{R\geq\sqrt{2}r}{\geq}8R^2$

$$\rightarrow 2+\frac{16r^2}{ef}\leq 2+\frac{16r^2}{8r^2}=4\overset{(1)}{\leq}\sum_{cyc}\frac{a}{s-a}$$

Therefore,

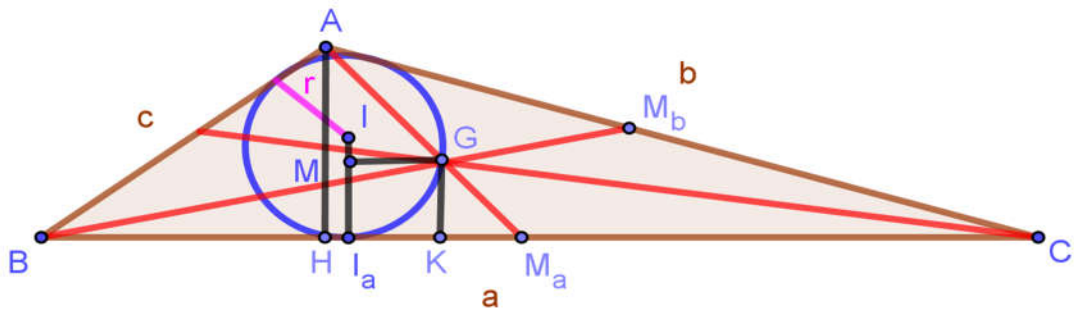
$$\sum_{cyc}\frac{a}{s-a}\geq 2+\frac{16r^2}{ef}$$

152. If $G\in C(I,r)$, then $\frac{8(a^2+b^2+c^2)}{3(a+b+c)^2}=1$

$$\nexists\Delta ABC:\ G\in NPC$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



G –centroid, $NH_a=CH_a=\frac{a}{2}, CH=\frac{b^2+a^2-c^2}{2a}, HM_a=CH-CH_a=\frac{b^2-c^2}{2a}$

$$AH=h_a, BI_a=s-b, CI_a=s-c, r=\frac{ah_a}{2s}; h_a=\frac{2F}{a}$$

ΔAHM_a is similar ΔGKM_a, so

$$\frac{AH}{GK}=\frac{AM_a}{GM_a}=\frac{HM_a}{KM_a}\rightarrow GK=\frac{h_a}{3}, KM_a=\frac{HM_a}{3}$$

If $G\in C(I,r)\rightarrow d(I,G)=r$

InΔIMG: $IM=II_a-MI_a=II_a-GK=r-\frac{h_a}{3}$

$$I_aK=\frac{b-c}{2}-\frac{b^2-c^2}{6a}=\frac{b-c}{6a}(4a-2s)$$

Now, $IG^2=IM^2+I_aK^2$

$$r^2=\left(r-\frac{h_a}{3}\right)^2+\left[\frac{b-c}{6a}(4a-2s)\right]^2\to$$

$$r^2=r^2-\frac{2}{3}rh_a+\frac{(b-c)^2(4a-2s)^2}{36a^2}+\frac{h_a^2}{9}$$

$$\frac{h_a^2}{9}-\frac{2}{3}rh_a+\frac{(b-c)^2(4a-2s)^2}{36a^2}=0$$

$$\frac{h_a^2}{9}-\frac{2}{3}h_a\cdot\frac{ah_a}{2s}+\frac{(b-c)^2(4a-2s)^2}{36a^2}=0$$

$$\frac{h_a^2}{9}-\frac{ah_a^2}{3s}+\frac{(b-c)^2(4a-2s)^2}{36a^2}=0$$

$$\frac{1}{9}\left(\frac{2F}{a}\right)^2-\frac{a}{3s}\left(\frac{2F}{a}\right)^2+\frac{(b-c)^2(4a-2s)^2}{36a^2}=0$$

$$16F^2s-48F^2a+s(b-c)^2(4a-2s)^2=0$$

$$16T^2(s-3a)+s(b-c)^2(4a-2s)^2=0$$

$$\because F^2=s(s-a)(s-b)(s-c)$$

$$16s(s-a)(s-b)(s-c)(s-3a)+s(b-c)^2(4a-2s)^2=0$$

$$(b+c-a)(a+c-b)(a+b-c)(b+c-5a)+(b-c)^2(3a-b-c)^2=0$$

Developing the expression, we have:

$$a^2(5a^2-6ab-6ac+5b^2-6bc+5c^2)=0, a\neq 0$$

$$5a^2+5b^2+5c^2-6ab-6bc-6ca=0$$

$$8(a^2+b^2+c^2)-3(a^2+b^2+c^2+2ab+2bc+2ca)=0$$

$$8(a^2+b^2+c^2)-3(a+b+c)^2=0$$

Therefore,

$$\frac{8(a^2+b^2+c^2)}{3(a+b+c)^2}=1$$

The radius of the nine point circle is half the radius of the circumference circumscribed to ΔABC, so if $G\in NPC\to NG=\frac{R}{2}$, N –nine point center, R –circumradius.

On the Euler line, $GH = 2GO, HN = 3NG$ and $HN = ON = \frac{OH}{2}$

O –circumcenter, G –centroid, H –orthocenter.

So, $NG = \frac{R}{2} \to HN = \frac{3R}{2} \to OH = 3R$

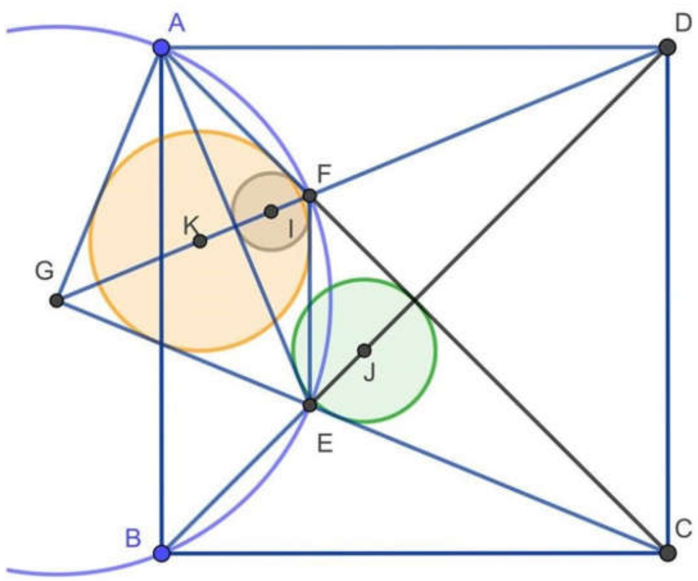
We know that:

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2)$$

$(3R)^2 = 9R^2 - (a^2 + b^2 + c^2) \to a^2 + b^2 + c^2 = 0$ impossible.

$\nexists \Delta ABC$.

153.



$ABCD$ –a square. Let E, F be the intersects of ΔABC and ΔABD.

Let G –be the circumcenter of ΔAFB, I, J, K be the incenters of $\Delta AEF, \Delta ACE$ and ΔACG respectively. Prove that: $r_{(K)} = r_{(J)} + r_{(I)}$

Proposed by Juan Jose Isach Mayo-Spain

Solution by proposer

$$AB = 1, AC = BD = \sqrt{2}, BH = \frac{\sqrt{2}}{2}, r_{(E)} = r_{(F)} = \frac{F_{\Delta ABC}}{s_{\Delta ABC}} = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2}$$

By symmetry $AF = BE = BH - EH = \sqrt{2} - 1 = EF$.

In ΔAEB by law of cosines: $AE=\sqrt{AB^2+BE^2-\sqrt{2}\cdot AB\cdot BE}=\sqrt{2-\sqrt{2}}=BF=CE$.

The radius of the circumcircle of ΔAFB is:

$$GE=GA=GB=\frac{AF\cdot FB\cdot AB}{4F_{\Delta ABF}}=\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}=\frac{\sqrt{2}}{2}\cdot\sqrt{2-\sqrt{2}}.$$

Let K be the incenter of ΔAGC rectangle in G. How $AG=\frac{\sqrt{2}}{2}\cdot\sqrt{2-\sqrt{2}}, AC=\sqrt{2}$

$$GC=GE+EC=\left(\frac{\sqrt{2}}{2}+1\right)\cdot\sqrt{2-\sqrt{2}}$$

$$r_{(K)}=\frac{F_{\Delta AGC}}{s_{\Delta AGC}}=\frac{AG\cdot GC}{AG+AC+GC}=\frac{\sqrt{2}}{(2\sqrt{2}+2)\cdot\sqrt{2-\sqrt{2}}+2\sqrt{2}}=$$

$$=\frac{(\sqrt{2}+1)\sqrt{2-\sqrt{2}}-\sqrt{2}}{2}$$

Let I be the incenter of ΔAFE,

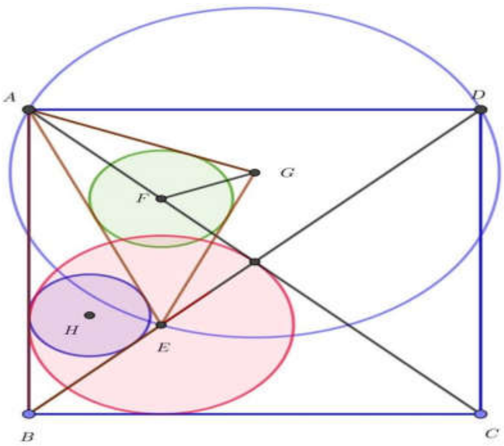
$$r_{(I)}=\frac{F_{\Delta AFE}}{s_{\Delta AFE}}=\frac{AF^2\cdot\sin 135^\circ}{2AF+AE}=\frac{\frac{3\sqrt{2}-4}{2}}{2\sqrt{2}-2+\sqrt{2-\sqrt{2}}}=1-\left(\frac{\sqrt{2}+1}{2}\cdot\sqrt{2-\sqrt{2}}\right)$$

$\Delta AFE\sim\Delta AEC$ and how $\frac{AC}{AE}=\frac{2}{\sqrt{2}\cdot\sqrt{2-\sqrt{2}}}\to$

$$r_{(J)}=r_{(I)}\cdot\frac{2}{\sqrt{2}\cdot\sqrt{2-\sqrt{2}}}=(\sqrt{2}+1)\sqrt{2-\sqrt{2}}-\frac{\sqrt{2}}{2}-1$$

How $r_{(I)}+r_{(J)}=\frac{(\sqrt{2}+1)\sqrt{2-\sqrt{2}}-\sqrt{2}}{2}$ then $r_{(K)}=r_{(J)}+r_{(I)}$.

154.



$ABCD$ – square

E – incenter of ΔABC, F – incenter of ΔABD.

G – circumcenter of ΔAED, H – incenter

of ΔABE.Prove that: $r_{(E)} = r_{(H)} + r_{(F)}$

Proposed by Juan Jose Isach Mayo-Spain

Solution by proposer

$$AB = 1; AC = BD = \sqrt{2}; BW = \frac{\sqrt{2}}{2}$$

$$E \text{ −incenter of } \Delta ABC, r_{(E)} = \frac{F_{\Delta ABC}}{s_{\Delta ABC}} = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2}$$

$$F \text{ −incenter of } \Delta ABD \text{ by symmetry: } FW = EW = r_{(E)} \text{ and}$$

$$BE = AF = BW - EW = \sqrt{2} - 1$$

$$\text{In } \Delta ABE\text{: } AE = \sqrt{AB^2 + BE^2 - 2AB \cdot BE \cdot \frac{\sqrt{2}}{2}} = \sqrt{2 - \sqrt{2}}$$

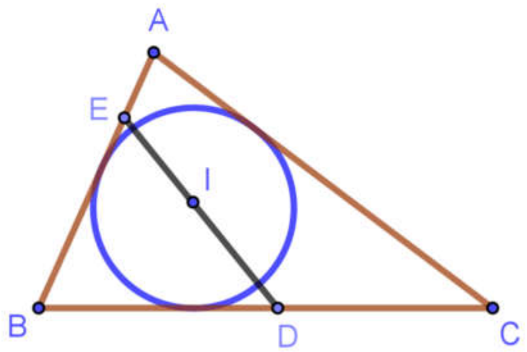
$$\Delta ABC \sim \Delta AEC \text{ and } \frac{AE}{AC} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}} \text{ then:}$$

$$r_{(F)} = \frac{AE}{AC} \cdot r_{(E)} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2} = \frac{1}{2}(\sqrt{2}-1)\sqrt{2-\sqrt{2}}$$

Let H −be the incenter of ΔABE, $r_{(H)} = \frac{F_{\Delta ABE}}{s_{\Delta ABE}} = \frac{\frac{2-\sqrt{2}}{4}}{\frac{\sqrt{2}+\sqrt{2-\sqrt{2}}}{2}} = \frac{1}{2}(1-\sqrt{2})\sqrt{2-\sqrt{2}} - \frac{\sqrt{2}}{2} + 1$

$$r_{(H)} + r_{(F)} = \frac{1}{2}(\sqrt{2}-1)\sqrt{2-\sqrt{2}} + \frac{1}{2}(1-\sqrt{2})\sqrt{2-\sqrt{2}} - \frac{\sqrt{2}}{2} + 1 = \frac{2-\sqrt{2}}{2} = r_{(E)}$$

155.

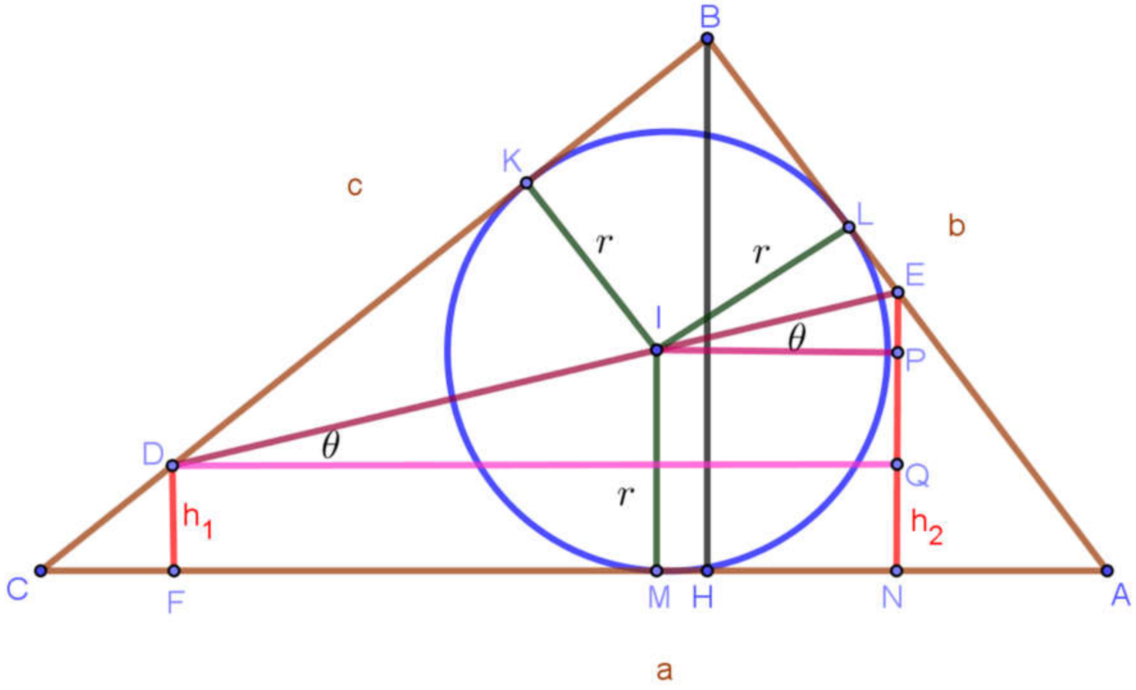


$\Delta ABC, D \in BC, E \in BA, E, I, D$ −collinear iff

$$\left(\frac{1}{BD} - \frac{1}{BC}\right) + \left(\frac{1}{BE} - \frac{1}{BA}\right) = \frac{AC}{AB \cdot BC}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil



$$2s = a + b + c; F = \frac{1}{2}bh_b = sr; r = \frac{bh_b}{2s}; BD = x, BE = y; BH = h_b, IM = r$$

$$DF = h_1, EN = h_2, ID = z, IE = l - z$$

Applying angle bisectors theorem, we have:

$$\frac{ID}{BD} = \frac{IC}{BC} \to \frac{z}{x} = \frac{l-z}{y} \to z = \frac{xl}{x+y} \to \frac{l}{z} = \frac{x+y}{x}; (I)$$

ΔBHC is similar ΔDFC, so

$$\frac{BH}{EN} = \frac{AB}{AE} \to \frac{h_b}{h_2} = \frac{c}{c-y} \to h_2 = \frac{h_b}{c}(c-y)$$

$$EQ = h_2 - h_1 = \frac{h_b}{ac}(cx - ay) \to EP = h_2 - r = \frac{h_b}{c}(c-y) - \frac{bh_b}{2s}$$

$$= \frac{h_b}{2sc}[2s(c-y) - bc]$$

ΔEQD is similar ΔEPI, so

$$\frac{EQ}{EP} = \frac{ED}{IE} \to \frac{\frac{h_b}{ac}(cx-ay)}{\frac{h_b}{2sc}[2s(c-y)-bc]} = \frac{l}{l-z}$$

Of expression (I): $\frac{2s(cx-ay)}{a[2s(c-y)-bc]} = \frac{l}{l-z}$

$$1-\frac{x}{x+y}=\frac{a[2s(c-y)-bc]}{2s(cx-ay)}\rightarrow b=\frac{ac}{xy}(x+y)-(a+c)$$

$$\frac{b}{ac}=\frac{x+y}{xy}-\frac{a+c}{ac}=\frac{1}{x}+\frac{1}{y}-\frac{1}{a}-\frac{1}{c}$$
$$\frac{AC}{AB\cdot BC}=\frac{1}{BD}-\frac{1}{BC}+\frac{1}{BE}-\frac{1}{AB}$$

Therefore,
$$\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(\frac{1}{BE}-\frac{1}{BA}\right)=\frac{AC}{AB\cdot BC}$$

Solution 2 by proposer

Plagiogonal system: $BC\equiv Bx, BA\equiv By$.

Let $BD=d, BE=e, B(0,0), C(a,0), D(d,0), A(0,c), E(o,e), I(i,i), i=\frac{ac}{a+b+c}$

$$E,I,D-\text{collinears}\leftrightarrow\begin{vmatrix}1&1&1\\0&i&d\\e&i&0\end{vmatrix}=0\leftrightarrow i=\frac{de}{d+e}\leftrightarrow\frac{1}{i}=\frac{d+e}{de}\leftrightarrow\frac{a+b+c}{ac}=\frac{d+e}{de}$$

$$\frac{1}{c}+\frac{b}{ac}+\frac{1}{a}=\frac{1}{d}+\frac{1}{e}\leftrightarrow\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(\frac{1}{BE}-\frac{1}{BA}\right)=\frac{AC}{AB\cdot BC}$$

156. If $A_1A_2\ldots A_n$ −convexe polygon $(n\in\mathbb{N}, n\geq 3)$ then:

$$\sum_{i=1}^{n}\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi-\mu(A_i)}\geq\frac{n(n-1)}{n+1}$$

Proposed by Radu Diaconu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{i=1}^{n}\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi-\mu(A_i)}\overset{Chebychev's}{\geq}\frac{1}{n}\left(\sum_{i=1}^{n}[(n-2)\pi-\mu(A_i)]\right)\left(\sum_{i=1}^{n}\frac{1}{(n-2)\pi++\mu(A_i)}\right)\geq$$

$$\overset{CBS}{\geq}\frac{1}{n}\left(n(n-2)\pi-\sum_{i=1}^{n}\mu(A_i)\right)\left(\frac{n^2}{\sum_{i=1}^{n}[(n-2)\pi+\mu(A_i)]}\right)=$$

$$=n\left(\frac{n(n-2)\pi-(n-2)\pi}{n(n-2)\pi+(n-2)\pi}\right)=\frac{n(n-1)}{n+1}$$

Therefore,

$$\sum_{i=1}^{n}\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi-\mu(A_i)}\geq\frac{n(n-1)}{n+1}$$

Solution 2 by George Florin Şerban-Romania

Let $f(x)=\frac{a-x}{a+x}, f'(x)=\frac{-2a}{(a+x)^2}=-2a(a+x)^{-2}$

$f''(x)=\frac{4a}{(a+x)^3}>0, \forall a=(n-2)\pi, a>0, \forall n\geq 3, x>0\to f$ −convexe, applying Jensen inequality, it follows that:

$$f\left(\frac{\sum_{i=1}^{n}\mu(A_i)}{n}\right)\leq\frac{1}{n}\sum_{i=1}^{n}f(\mu(A_i))$$

$$f\left(\frac{(n-2)\pi}{n}\right)\leq\frac{1}{n}\sum_{i=1}^{n}f(\mu(A_i))=\frac{1}{n}\sum_{i=1}^{n}\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi-\mu(A_i)}$$

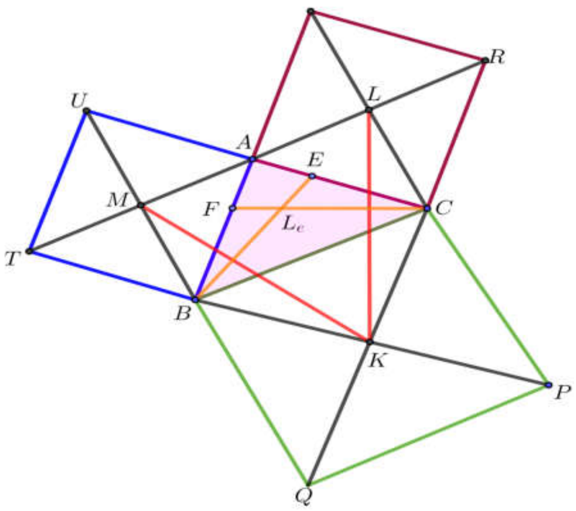
$$\sum_{i=1}^{n}\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi-\mu(A_i)}\geq nf\left(\frac{(n-2)\pi}{n}\right)=n\left(\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi+\frac{(n-2)\pi}{n}}\right)=\frac{n(n-1)}{n+1}$$

Therefore,

$$\sum_{i=1}^{n}\frac{(n-2)\pi-\mu(A_i)}{(n-2)\pi-\mu(A_i)}\geq\frac{n(n-1)}{n+1}$$

157.

In $\Delta ABC: BE, CF$ −symmedians, K, L, M −centers of squares, $KL\perp CF$, $KM\perp BE$. Find the angles of ΔABC.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Plagiogonal system: $BC\equiv Bx, BA\equiv By$

$$B(0,0), L_e(l_{l_1}, l_{l_2}), l_{l_1} = \frac{ac^2}{a^2+b^2+c^2}, l_{l_2} = \frac{a^2c}{a^2+b^2+c^2}$$

$$K(k_1,k_2), k_1 = \frac{a(sinB+cosB)}{2sinA}, k_2 = -\frac{a}{2sinA}$$

$$M(m_1,m_2); m_1 = -\frac{c}{2sinB}, m_2 = \frac{c(sinB+cosB)}{2sinB}$$

$$\lambda_{BL_e} = \lambda_1 = \frac{a}{c};(1);\quad \lambda_{KL} = \lambda_2 = \frac{l_{l_2}-m_2}{l_{l_1}-m_1} = \frac{c\cdot sinB + c\cdot cosB}{-a\cdot sinB - a\cdot sinB - c};(2)$$

$$BL_e \perp KM \to (\lambda_1+\lambda_2)cosB + \lambda_1\lambda_2 + 1 = 0 \to$$

$$(a^2-c^2)[(sinB+cosB)cosB - 1] = 0$$

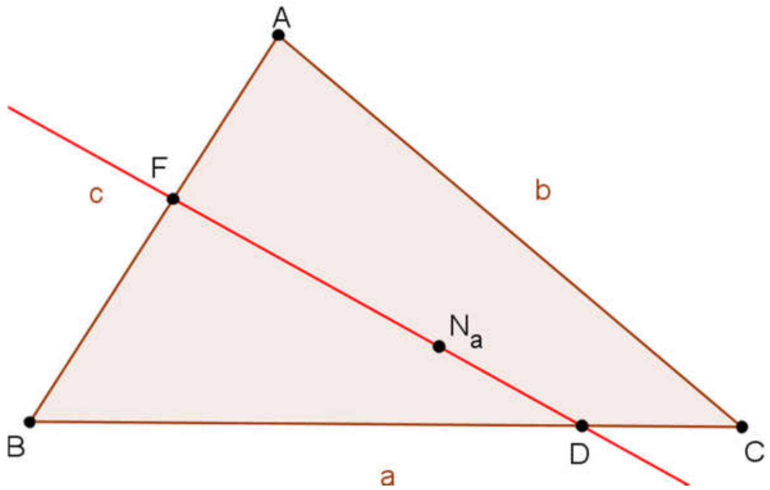
$$a = 0 \text{ or } (sinB+cosB)cosB - 1 = 0 \to a = c \text{ or } \angle B = 45^\circ$$

Similarly: $CL_e \perp KL \to a = b$ **or** $C = 45^\circ$.

Therefore, ΔABC **is right and isosceles or** ΔABC **is equilateral.**

Thus, $(A,B,C) \in \{(60^\circ,60^\circ,60^\circ)(90^\circ,45^\circ,45^\circ)\}$

158.



N_a **—Nagel's point of** ΔABC

$$s = \frac{a+b+c}{2}$$

Prove that:

D, F, N_a **—collinear**

$$\Leftrightarrow \frac{BC}{BD}\cdot\frac{s-c}{s} + \frac{BA}{BF}\cdot\frac{s-a}{s} = 1$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil

Applying the Menelaus theorem, we have:

$$\frac{AF}{BF}\cdot\frac{BD}{PD} - \frac{N_aP}{AN_a} = 1$$

$$\frac{c-BF}{BF}\cdot\frac{BD}{BD-(s-c)}\cdot\frac{s-a}{a}=1$$

$$BD\cdot(c-BF)(s-a)=a\cdot BF\cdot[BD-(s-c)]$$

$$BD\cdot(cs-ac-s\cdot BF+a\cdot BF)=a\cdot BF\cdot BD=a\cdot BC\cdot(s-c)$$

$$BD\cdot cs-ac\cdot BD-s\cdot BD\cdot BF+a\cdot BF\cdot BD=a\cdot BF\cdot BF-a\cdot BF\cdot(s-c)$$

$$a\cdot BF\cdot(s-c)+c\cdot BD\cdot(s-a)=s\cdot BD\cdot BF$$

$$\frac{a}{BD}\cdot\frac{s-c}{s}+\frac{c}{BF}\cdot\frac{s-a}{s}=1$$

Therefore,

$$\frac{BC}{BD}\cdot\frac{s-c}{s}+\frac{BA}{BF}\cdot\frac{s-a}{s}=1$$

159.

$\Delta ABC, b\neq c$

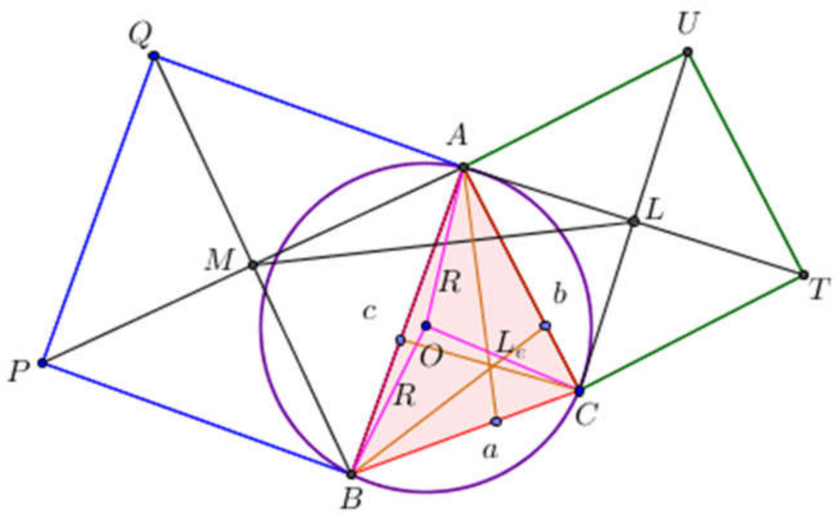
R —circumradius of ΔABC

L, M —centers of squares

L_e —Lemoine's point of ΔABC

$LM\perp AL_e$

Prove that:



$$\frac{1}{R}+\frac{a}{bc}=\frac{b}{ca}+\frac{c}{ab}$$

Proposed by Thanasis Gakopulos-Farsala-Greece

Solution by proposer

Plagiogonal system: $AB\equiv Ax, AC\equiv Ay$

$$A(0,0), L(l_1,l_2), l_1=-\frac{b}{2sinA}, l_2=\frac{b(sinA+cosA)}{2sinA}$$