$$M(m_1,m_2), m_1=\frac{c(sinA+cosA)}{2sinA}, m_2=-\frac{c}{2sinA}$$

$$L_e=(l_{l_1},l_{l_2}), l_{l_1}=\frac{b^2c}{a^2+b^2+c^2}, l_{l_2}=\frac{bc^2}{a^2+b^2+c^2}$$

$$\lambda_{AL_e}=\frac{l_{l_2}}{l_{l_1}}=\frac{c}{b}=\lambda_1; (1)$$

$$\lambda_{LM}=\frac{l_2-m_2}{l_1-m_1}=\frac{bsinA+bcosA+c}{-csinA-ccosA-b}=\lambda_2; (2)$$

$$AL_e\perp LM\to(\lambda_1+\lambda_2)cosA+\lambda_1\lambda_2+1=0\overset{(1,2)}{\to}$$

$$(b^2-c^2)[(sinA+cosA)cosA-1]=0; (b\neq a)\to$$

$$(sinA+cosA)cosA=1\to\angle A=45^\circ$$

$$a^2=b^2+c^2-2bc\cdot cosA\to a^2=b^2+c^2-2bc\sqrt{2}$$

$$[ABC]=\frac{bc}{2}\cdot sinA\overset{\angle A=45^\circ}{\Rightarrow}4[ABC]=b^2+c^2-a^2$$

$$\frac{1}{R}=\frac{4[ABC]}{abc}=\frac{b^2+c^2-a^2}{abc}$$

Therefore,

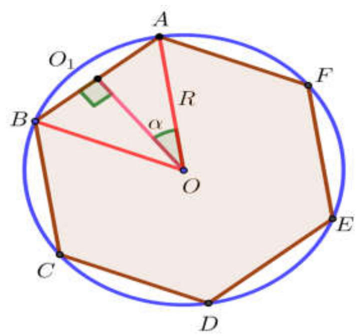
$$\frac{1}{R}+\frac{a}{bc}=\frac{b}{ca}+\frac{c}{ab}$$

160. a,b,c,d,e,f **−sides,** R **−circumradii in a bicentric hexagon. Prove that:**

$$abcdef\leq R^6$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let $a = AB, b = BC, c = CD, d = DE, e = EF, f = FA$ and O –be the circumcenter of $ABCDEF$.

$O_1, O_2, O_3, O_4, O_5, O_6$ –be the feet pf the perpendiculars from O to AB, BC, CD, DE, EF, FA respectively.

Let $\alpha_1 = \mu(O_1OA) = \mu(O_1OB), \alpha_2 = \mu(O_2OB) = \mu(O_2OC)$

We define $\alpha_3, \alpha_4, \alpha_5, \alpha_6$ as α_1 and α_2.

We have:

$$\sum_{i=1}^{6} \alpha_i = \pi, \alpha_i \in (0, \pi), \quad a = 2R\sin\alpha_1 (and\ analogs)$$

$$abcdef = \prod_{i=1}^{6} (2R\sin\alpha_i) = (2R)^6 \prod_{i=1}^{6} \sin\alpha_i \overset{AM-GM}{\leq} (2R)^6 \left(\frac{1}{6}\sum_{i=1}^{6} \sin\alpha_i\right)^6$$

$x \to \sin x$ –is concave on $(0, \pi)$, using Jensen, it follows that

$$abcdef \leq (2R)^6 \left(\sin\left(\frac{1}{6}\sum_{i=1}^{6} \alpha_i\right)\right)^6 = (2R)^6 \sin\left(\frac{\pi}{6}\right)$$

Therefore,

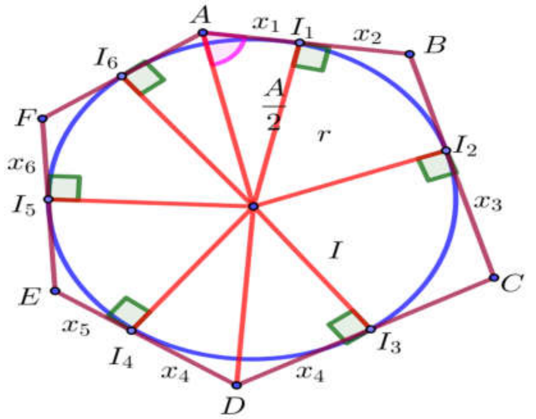
$$abcdef \leq R^6$$

161. a, b, c, d, e, f –sides, r –inradii in a bicentric hexagon. Prove that:

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \geq 8r^2$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let I –be the incenter of $ABCDEF$.

$I_1, I_2, I_3, I_4, I_5, I_6$ be the feet of the perpendiculars from I to AB, BC, CD, DE, EF, FA respectively.

Let $x_1 = AI_1 = AI_6, x_2 = BI_1 = BI_2, x_3 = CI_2 = CI_3, x_4 = DI_3 = DI_4$

$x_5 = EI_4 = EI_5, x_6 = FI_5 = FI_6.$

Let s –be the semiperimeter of $ABCDEF$, we have

$$s = \frac{1}{2}\sum_{cyc} a = \sum_{i=1}^{6} x_i$$

We have: $x_1 = \frac{r}{tan\frac{A}{2}}(and\ analogs) \rightarrow s = r\sum_{cyc}\frac{1}{tan\frac{A}{2}}.$

Let $f(x) = \frac{1}{tan\,x}, x \in (0,\pi), f'(x) = -\frac{1}{sin^2x}, f''(x) = \frac{2cosx}{sin^3x} \geq 0$. Using Jensen, it follows that:

$$\sum_{cyc}\frac{1}{tan\frac{A}{2}} = \sum_{cyc} f\left(\frac{A}{2}\right) \geq 6f\left(\frac{1}{6}\sum_{cyc}\frac{A}{2}\right) \overset{\sum A=\pi}{\cong} 6f\left(\frac{\pi}{6}\right) = 2\sqrt{3} \rightarrow s \geq 2\sqrt{3}r$$

$$\rightarrow \sum_{cyc} a^2 \overset{CBS}{\geq} \frac{1}{6}\left(\sum_{cyc} a\right)^2 = \frac{2}{3}s^2 \geq \frac{2}{3}\left(2\sqrt{3}r\right)^2 = 8r^2$$

162. N_a –Nagel's point of ΔABC

I –incenter of ΔABC

$BC = a = 16, BA = c = 10$

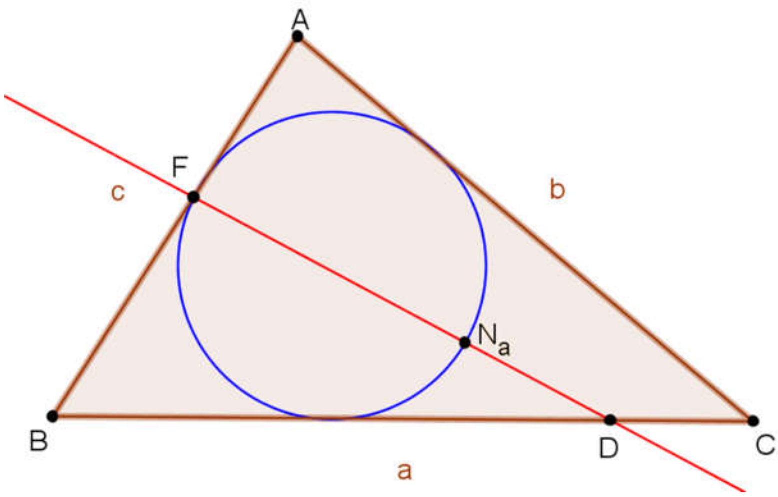
$BD = d = 12,$

$BF = f = 6$

$N_a \in DF, r$ –inradius of ΔABC.

Prove that:

$I \in DF, \quad N_a \in C(I,r)$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$N_a \in DF \to \frac{a}{d}\cdot\frac{s-c}{s}+\frac{c}{f}\cdot\frac{s-a}{s}=1; s=\frac{a+b+c}{2}$$

$$\frac{16}{12}\cdot\frac{16+b-10}{16+b+10}+\frac{10}{6}\cdot\frac{-16+b+10}{16+b+10}=1\to b=14.$$

$$\left(\frac{1}{d}-\frac{1}{a}\right)+\left(\frac{1}{f}-\frac{1}{c}\right)=\frac{b}{ac}\leftrightarrow\left(\frac{1}{12}-\frac{1}{16}\right)+\left(\frac{1}{6}-\frac{1}{10}\right)=\frac{14}{16\cdot 10}\leftrightarrow\frac{7}{80}=\frac{7}{80}$$

Hence, $I \in DF$.

$$cosB=\frac{a^2-b^2+c^2}{2ac}=\frac{1}{2}\to cosB=60^\circ;\ r=\frac{ac}{a+b+c}sinB\to r=2\sqrt{3}$$

Plagiogonal system: $BC\equiv Bx, BA\equiv By, B(0,0), I(i,i), N_a(n_1,n_2)$

$$i=\frac{ac}{a+b+c}=4, n_1=\frac{s-c}{s}\cdot a\to n_1=8, n_2=\frac{s-a}{s}\cdot c\to n_2=2$$

$$IN_a^2=(i-n_1)^2+(i-n_2)^2+2(i-n_1)(i-n_2)cosB\to$$

$$IN_a^2=12\to IN_a=2\sqrt{3}\to IN_a=r\to N_a\in C(I,r)$$

163.

In ΔABC :

p:" $bc\sqrt{4\cos^2 B+4\cos^2 C+1}=3\sqrt{3}R^2$, $3a=\pi$ " and q: " $a=\mu(A), b=\mu(B), c=\mu(C)$ "

Prove that : $p \leftrightarrow q$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we assume that p is true.

$$bc\sqrt{4\cos^2 B+4\cos^2 C+1}=3\sqrt{3}R^2 \leftrightarrow (2R\sin B).(2R\sin C)\sqrt{4\cos^2 B+4\cos^2 C+1}=3\sqrt{3}R^2$$

$$\leftrightarrow m=(4\sin^2 B)(4\sin^2 C)(4\cos^2 B+4\cos^2 C+1)=27$$

But we have, from AM − GM : m

$$\le\left(\frac{(4\sin^2 B)+(4\sin^2 C)+(4\cos^2 B+4\cos^2 C+1)}{3}\right)^3=27$$

Equality holds when : $4\sin^2 B=4\sin^2 C=4\cos^2 B+4\cos^2 C+1$

$\sin^2 B = \sin^2 C \leftrightarrow \sin B = \sin C$ ($\therefore \sin B, \sin C \geq 0$)

$\leftrightarrow \mu(B)=\mu(C)$ ($\textit{if not}$, $\mu(B)+\mu(C)=\pi$, contradiction)

$$\rightarrow 4(1-\cos^2 B) = 8\cos^2 B + 1 \leftrightarrow \cos^2 B = \frac{1}{4}$$

$$\leftrightarrow \mu(B)=\mu(C)=\frac{\pi}{3} \ or \ \left(\mu(B)=\mu(C)=\frac{2\pi}{3} \rightarrow \mu(B)+\mu(C)>\pi\right)$$

$$\rightarrow \mu(B)=\mu(C)=\frac{\pi}{3} \rightarrow \mu(A)=\frac{\pi}{3} \rightarrow \Delta ABC \text{ is equilateral}$$

$$\text{Since } a=\frac{\pi}{3} \rightarrow a=b=c=\mu(A)=\mu(B)=\mu(C)=\frac{\pi}{3} \rightarrow q \text{ is true.}$$

$$\text{Now, we assume that } q \text{ is true } \rightarrow \frac{a}{\sin a}=\frac{b}{\sin b}=\frac{c}{\sin c}=2R$$

$$\text{Let } f(x)=\frac{x}{\sin x}, x\in(0,\pi) \text{ and } g(x)=\sin x - x\cos x, x\in(0,\pi)$$

$$\text{We have: } f'(x)=\frac{g(x)}{\sin^2 x} \text{ and } g'(x)=x\sin x>0, \forall x\in(0,\pi) \rightarrow g\uparrow \text{ on } (0,\pi)$$

$$\rightarrow g(x)>g(0)=0, \forall x\in(0,\pi) \text{ so } f \text{ is strictly increasing on } (0,\pi)$$

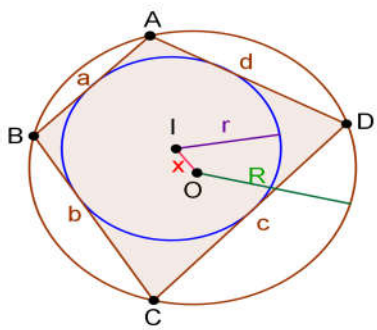
$$\rightarrow a=b=c \rightarrow a=b=c=\mu(A)=\mu(B)=\mu(C)=\frac{\pi}{3} \text{ and } R=\frac{\pi}{3\sqrt{3}}$$

$$\rightarrow bc\sqrt{4\cos^2 B+4\cos^2 C+1}=\left(\frac{\pi}{3}\right)^2\sqrt{4\cos^2\frac{\pi}{3}+4\cos^2\frac{\pi}{3}+1}=\frac{\pi^2\sqrt{3}}{9}=3\sqrt{3}\left(\frac{\pi}{3\sqrt{3}}\right)^2$$

$$=3\sqrt{3}R^2$$

$\textit{then } q \rightarrow p. \textit{ Therefore, } p \leftrightarrow q.$

164. O **–circumcenter,** I **–incenter,** R **–circumradii in a bicentric quadrilateral** $ABCD$**. If** $3\sin A\cdot\sin B=1$ **then find:** $\Omega=\frac{R}{OI}$



Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $e=AC, f=BD$. We have: $\sin A=\frac{f}{2R}$ and $\sin B=\frac{e}{2R}\rightarrow$

$$3\sin A\cdot\sin B=1\Leftrightarrow 3ef=4R^2$$

We know that: $ef = 2r\left(r+\sqrt{r^2+4R^2}\right) \to 3r\left(r+\sqrt{r^2+4R^2}\right) = 2R^2$

$$\Leftrightarrow 3\left(1+\sqrt{1+4x}\right) = 2x \text{, where } x = \left(\frac{R}{r}\right)^2 \Leftrightarrow 3\sqrt{1+4x} = 2x-3$$

$$\Leftrightarrow 9(1+4x) = 4x^2-12x+9 \text{ and } x > \frac{3}{2} \Leftrightarrow 4x(x-12) = 0 \Leftrightarrow x = 12 \to \frac{R}{r} = 2\sqrt{3}$$

From Fuss theorem, we have: $OI = \sqrt{R^2+r^2-r\sqrt{4R^2+r^2}}$

$$\to \Omega = \frac{R}{OI} = \left(1+\left(\frac{r}{R}\right)^2-\frac{r}{R}\sqrt{4+\left(\frac{r}{R}\right)^2}\right)^{-1} \overset{(1)}{=} \left(\sqrt{1+\left(\frac{1}{2\sqrt{3}}\right)^2-\frac{1}{2\sqrt{3}}\cdot\sqrt{4+\left(\frac{1}{2\sqrt{3}}\right)^2}}\right)^{-1}$$

Therefore,

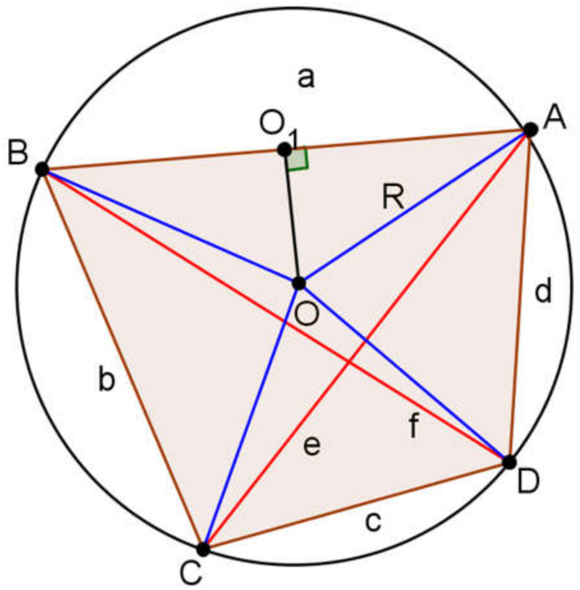
$$\Omega = \frac{R}{OI} = \sqrt{2}$$

165. **If a,b,c,d –sides, e,f –diagonals, R –circumradii in a cyclic quadrilateral then:**

$$R \geq \frac{2\sqrt{abcd}}{e+f}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let $a = AB, b = BC, c = CD, d = DA$ and O be the circumcenter of $ABCD$.

O_1, O_2, O_3, O_4 be the feet of the perpendiculars from O to AB, BC, CD, DA respectively.
Let $\alpha_1 = \mu(O_1OA) = \mu(O_1OB), \alpha_2 = \mu(O_2OB) = \mu(O_2OC)$
(we define α_3, α_4 as α_1 and α_2). We have:

$$\sum_{i=1}^{4} \alpha_i = \frac{2\pi}{2} = \pi, \alpha_i \in (0,\pi), a = 2R \cdot \sin\alpha_1 (and\ analogs)$$

$$\to abcd = \prod_{i=1}^{4} (2R \cdot \sin\alpha_i) = (2R)^4 \prod_{i=1}^{4} \sin\alpha_i \overset{AM-GM}{\leq} (2R)^4 \left(\frac{1}{4}\sum_{i=1}^{4} \alpha_i\right)^4$$

$x \to \sin x$ is concave on $(0,\pi)$, using Jensen's inequality:

$$\to abcd \leq (2R)^4 \left(\sin\left(\frac{1}{4}\sum_{i=1}^{4} \alpha_i\right)\right)^4 = (2R)^4 \left(\sin\left(\frac{\pi}{4}\right)\right)^4$$

$$\to abcd \leq (2R)^4 \left(\frac{1}{\sqrt{2}}\right)^4 = 4R^4 \to R \geq \frac{\sqrt{2}}{2} \cdot \sqrt[4]{abcd}; (1)$$

$ABCD$ is a cyclic quadrilateral, from Ptolemy's theorem it follows that:

$$ef = ac + bd \overset{AM-GM}{\geq} 2\sqrt{abcd}$$

$$e + f \overset{AM-GM}{\geq} 2\sqrt{ef} \geq 2\sqrt{2} \cdot \sqrt[4]{abcd}; (2)$$

From $(1), (2)$ it follows that:

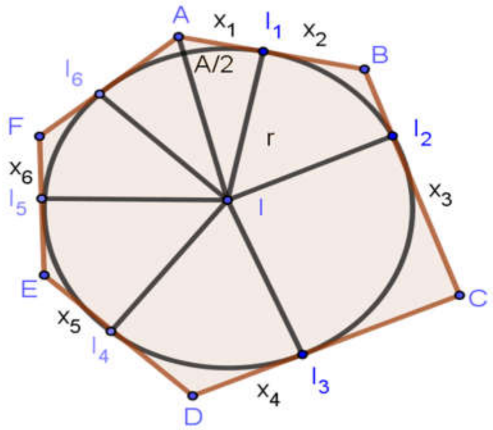
$$R(e+f) \geq 2\sqrt{abcd} \to R \geq \frac{2\sqrt{abcd}}{e+f}$$

166. a, b, c, d, e, f **−sides,** r **−inradii in a tangential hexagon. Prove that:**

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{e} + \frac{e^2}{f} + \frac{f^2}{a} \geq 4\sqrt{3}r$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$I_1, I_2, I_3, I_4, I_5, I_6$ be the feet of the perpendiculars from I to AB, BC, CD, DE, EF, FA respectively.

Let $x_1 = AI_1 = AI_6, x_2 = BI_1 = BI_2, x_3 = CI_2 = CI_3, x_4 = DI_3 = DI_4,$
$x_5 = EI_4 = EI_5, x_6 = FI_5 = FI_6$

Let s –be the semiperimeter of $ABCDEF$, we have

$$s = \frac{1}{2}\sum_{cyc} a = \sum_{i=1}^{6} x_i$$

We have: $x_1 = \frac{r}{\tan\frac{A}{2}}$ $(and\ analogs) \to s = r\sum\frac{1}{\tan\frac{A}{2}}$.

Let $f(x) = \frac{1}{\tan x}, x \in \left(0, \frac{\pi}{2}\right)$, we have: $f'(x) = -\frac{1}{\sin^2 x}$ and $f''(x) = \frac{2\cos x}{\sin^3 x} \geq 0$.

Using Jensen inequality, we get:

$$\sum_{cyc}\frac{1}{\tan\frac{A}{2}} = \sum_{cyc} f\left(\frac{A}{2}\right) \geq 6f\left(\frac{1}{6}\sum_{cyc}\frac{A}{2}\right) \overset{\sum A=\pi}{\gtrapprox} 6f\left(\frac{\pi}{3}\right) = 2\sqrt{3} \to s \geq 2\sqrt{3}r$$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{e} + \frac{e^2}{f} + \frac{f^2}{a} \overset{CBS}{\gtrapprox} \frac{(\sum a)^2}{\sum b} = \sum_{cyc} a = 2s = 2 \cdot 2\sqrt{3}r = 4\sqrt{3}r$$

Therefore,

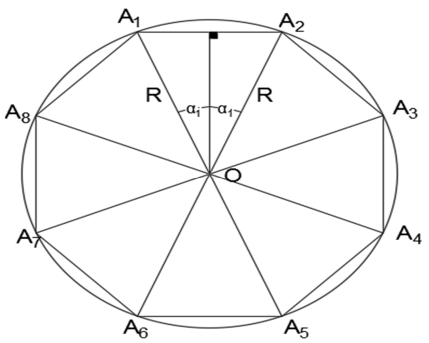
$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{e} + \frac{e^2}{f} + \frac{f^2}{a} \geq 4\sqrt{3}r$$

167. F –area, R –circumradii, r –inradii, s –semiperimeter in a bicentric octagon.Prove that:

$$\frac{r^2}{R \cdot \cos\frac{\pi}{8}} \leq \frac{F}{s} \leq \frac{R^2 \cdot \cos^2\frac{\pi}{8}}{r}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let O be the circumcenter of $A_1A_2A_3A_4A_5A_6A_7A_8$, and $O_i = pr_{A_iA_{i+1}}(O)$,

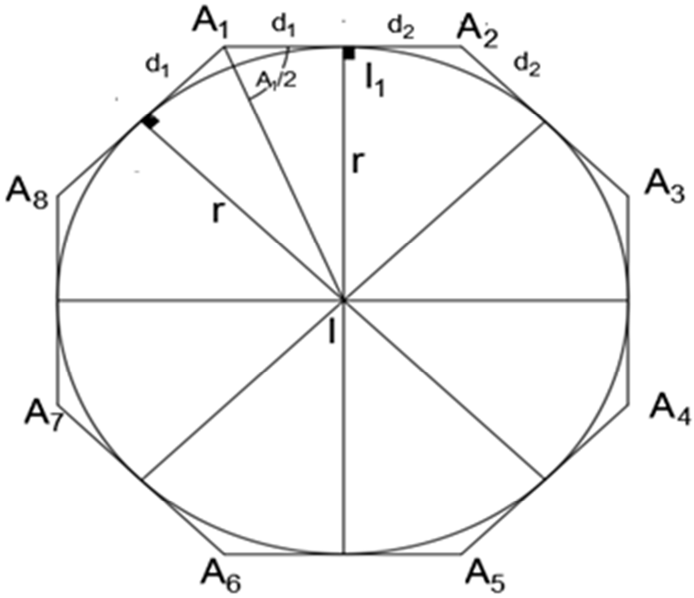
$\forall i\in\{1,2,\dots,8\}$ $(A_9=A_1)$.*Let* $a_i=A_iA_{i+1}$ *and* $\alpha_i=\mu(A_iOO_i)=\mu(O_iOA_{i+1})$,

$\forall i\in\{1,2,\dots,8\}$ $(A_9=A_1)$.*We have* : $a_i=2R\sin\alpha_i$, $\forall i\in\{1,2,\dots,8\}$ *and*

$$\sum_{i=1}^{8}\alpha_i=\frac{2\pi}{2}=\pi$$

We know that $x\to\sin x$ *is concave on* $\left(0,\frac{\pi}{2}\right)$

$$\to s=\frac{1}{2}\sum_{i=1}^{8}a_i=R\sum_{i=1}^{8}\sin\alpha_i\overset{Jensen}{\leq}8R.\sin\left(\frac{1}{8}\sum_{i=1}^{8}\alpha_i\right)=8R\sin\left(\frac{\pi}{8}\right)\to s\leq 8R\sin\left(\frac{\pi}{8}\right)\ \ (1)$$



Let I be the incenter of $A_1A_2A_3A_4A_5A_6A_7A_8$ *and* $I_i=pr_{A_iA_{i+1}}(I)$,
$\forall i\in\{1,2,\dots,8\}$ $(A_9=A_1)$.*Let* $d_i=A_iI_i=I_{i-1}A_i$,

$\forall i\in\{1,2,\dots,8\}$ $(I_0=I_8)$, *we have* : $d_i=\dfrac{r}{\tan\frac{A_i}{2}}$, $\forall i\in\{1,2,\dots,8\}$, *and* $s=\sum_{i=1}^{8}d_i$

Let $f(x)=\dfrac{1}{\tan x}$, $x\in\left(0,\frac{\pi}{2}\right)$, *we have* : $f'(x)=-\dfrac{1}{\sin^2 x}$ *and* $f''(x)=\dfrac{2\cos x}{\sin^3 x}\geq 0$

$$\to f-convex\ \to s=\sum_{i=1}^{8}d_i=r\sum_{i=1}^{8}\frac{1}{\tan\frac{A_i}{2}}$$

$$=r\sum_{i=1}^{8}f\left(\frac{A_i}{2}\right)\overset{Jensen}{\geq}8rf\left(\frac{1}{8}\sum_{i=1}^{8}\frac{A_i}{2}\right)\overset{\sum_{i=1}^{8}A_i=6\pi}{=}8rf\left(\frac{3\pi}{8}\right)$$

$$=\frac{8r}{\tan\left(\frac{3\pi}{8}\right)}=\frac{8r}{\tan\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}=8r\tan\left(\frac{\pi}{8}\right)\to s\geq 8r\tan\left(\frac{\pi}{8}\right)\ \ (2)$$

$$(1)\ and\ (2)\to r\leq R\cos\frac{\pi}{8}\to\frac{r^2}{R\cos\frac{\pi}{8}}\leq r=\frac{F}{s}\ and\ \frac{R^2\cos^2\frac{\pi}{8}}{r}\geq\frac{r^2}{r}=r=\frac{F}{s}$$

Therefore,

$$\frac{r^2}{R\cos\frac{\pi}{8}}\leq\frac{F}{s}\leq\frac{R^2\cos^2\frac{\pi}{8}}{r}.$$

168. If $A_1,A_2,\dots,A_n, n\geq 3$ is a convex polygon, and $M\in Int(A_1A_2\dots A_n)$, with

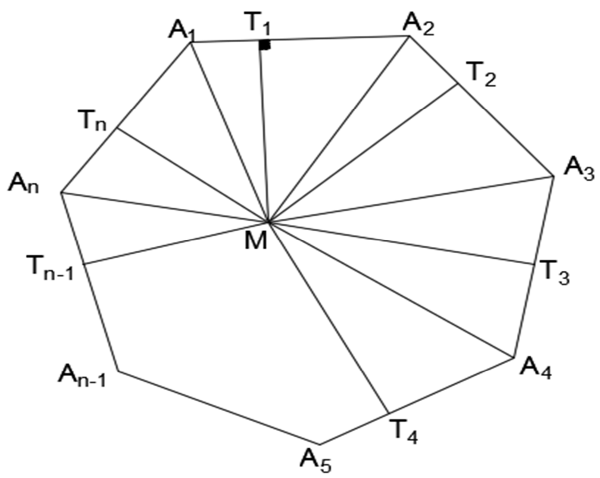
$pr_{A_kA_{k+1}}M=T_k\in[A_kA_{k+1}]$, for any $k\in\{1,2,\dots,n\}, A_{n+1}\equiv A_1$.

Prove that :

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}\geq 2n\tan\frac{\pi}{n}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let $\alpha_k=\mu(A_kMT_k),\beta_k=\mu(T_kMA_{k+1}),\forall k\in\{1,2,\dots,n\},A_{n+1}\equiv A_1,$

$$\text{It's clear that}:\ \alpha_k,\beta_k\in\left(0,\frac{\pi}{2}\right)\ \text{and}\ \sum_{k=1}^{n}(\alpha_k+\beta_k)=2\pi$$

We have :

$$\frac{A_kA_{k+1}}{MT_k}=\frac{A_kT_k}{MT_k}+\frac{T_kA_{k+1}}{MT_k}=\tan\alpha_k+\tan\beta_k,\forall k\in\{1,2,\dots,n\},A_{n+1}\equiv A_1$$

$$\text{Let } f(x)=\tan x, x\in\left(0,\frac{\pi}{2}\right), \text{we have :}$$

$$f'(x)=\frac{1}{\cos^2 x}, f''(x)=\frac{2\sin x}{\cos^3 x}>0\to f-\text{convex}$$

$$\overset{Using\ Jensen}{\Rightarrow}\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}=\sum_{k=1}^{n}(\tan\alpha_k+\tan\beta_k)\geq 2n\tan\left(\frac{1}{2n}\sum_{k=1}^{n}(\alpha_k+\beta_k)\right)$$

$$=2n\tan\left(\frac{2\pi}{2n}\right)=2n\tan\left(\frac{\pi}{n}\right)$$

Therefore,

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}\geq 2n\tan\frac{\pi}{n}$$

Solution 2 by Adrian Popa-Romania

$$\Delta T_kMA_k:\ \angle T_k=90^\circ\to\angle T_kMA_k<90^\circ;\quad \alpha_{k_1}=\angle T_kMA_k$$

$$\Delta T_kMA_{k+1}:\ \angle T_k=90^\circ\to\angle T_kMA_{k+1}<90^\circ;\quad \alpha_{k_2}=\angle T_kMA_{k+1}$$

$$\begin{cases}\Delta MT_kA_k:\ tan(\alpha_{k_1})=\dfrac{A_kT_k}{MT_k}\\ \Delta MT_kA_{k+1}:\ tan(\alpha_{k_2})=\dfrac{A_{k+1}T_k}{MT_k}\end{cases}\to$$

$$tan(\alpha_{k_1})+tan(\alpha_{k_2})=\frac{A_kT_k}{MT_k}+\frac{A_{k+1}T_k}{MT_k}=\frac{A_kA_{k+1}}{MT_k}$$

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}=\sum_{k=1}^{n}\left(tan(\alpha_{k_1})+tan(\alpha_{k_2})\right)$$

Let be the function: $f(x)=tan\,x, x\in\left(0,\frac{\pi}{2}\right), f'(x)=1+tan^2x$

$$f''(x)=1+2tan\,x\cdot(1+tan^2x)>0\to f\ \text{–convexe.}$$

From Jensen's inequality, we get:

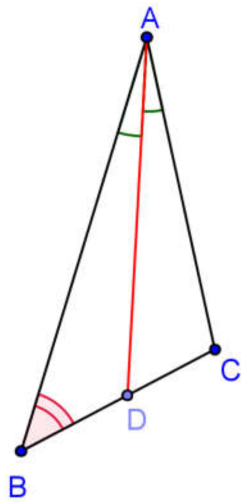
$$\frac{tan(\alpha_{11})+tan(\alpha_{12})+..+tan(\alpha_{n1})+tan(\alpha_{n2})}{2n}\geq$$

$$\geq tan\left(\frac{\alpha_{11}+\alpha_{12}+\dots+\alpha_{n1}+\alpha_{n2}}{2n}\right)=tan\left(\frac{2\pi}{2n}\right)=tan\left(\frac{\pi}{n}\right)$$

Therefore,

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}\geq 2n\tan\frac{\pi}{n}$$

169.



Given ΔABC.

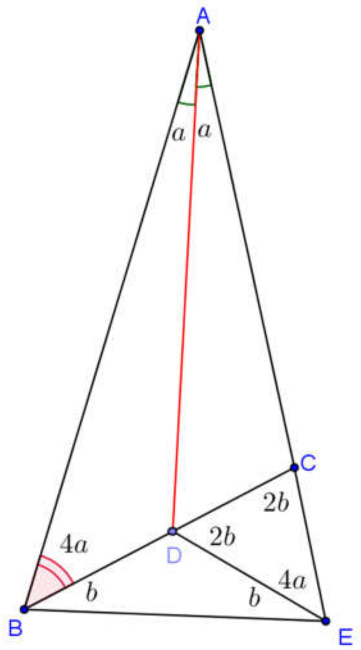
$$\angle CBA=2\cdot\angle BAC$$

AD is internal bisector of A.Prove that:

$$\frac{1}{AC}+\frac{1}{BD}=\frac{1}{DC}\Leftrightarrow\angle CBA=45^\circ$$

Proposed by Juan Jose Isach Mayo-Spain

Solution by Apostolis Manoloudis-Greece



$$\frac{1}{AC}+\frac{1}{BD}=\frac{1}{DC}\Leftrightarrow\angle CBA=45^\circ$$

$$\frac{DC}{AC}+\frac{DC}{BD}=1\Leftrightarrow\frac{BD}{AB}+\frac{AC}{AB}=1\Leftrightarrow AB=AC+BD$$

Let $AB=AE\Leftrightarrow BD=CE$.

But $\Delta ABD=\Delta AED\Leftrightarrow BD=DE$.

So, $\angle DCE=2b=90^\circ-2a\Leftrightarrow$

$$\angle ACB=90^\circ+2a\Leftrightarrow$$

$$90^\circ+2a+4a+2a=180^\circ\Leftrightarrow$$

$$4a=45^\circ\Leftrightarrow\angle ABC=45^\circ$$

170.

$BC = 16, BA = 10, BD = 10,$

$BF_1 = 5$

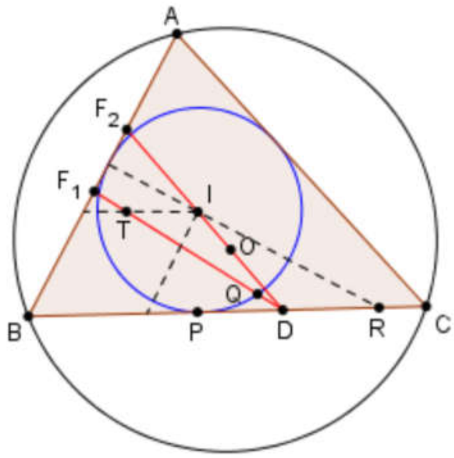
O –circumcenter

I –incenter

$O \in DF_1, I \in DF_2$

Find: F_1F_2

Find: exact value of area $[DPQ]$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\text{Is } D, O, F \text{ collinear} \to \left(\frac{BC}{BD} + \frac{BA}{BF_1}\right) - \left(\frac{BC}{BF_1} + \frac{BA}{BD}\right) \cdot \cos B = 2\sin^2 B$$

$$\to \left(\frac{16}{10} + \frac{10}{5}\right) - \left(\frac{16}{5} + \frac{10}{10}\right) \cdot \cos B = 2\sin^2 B$$

$$\to \frac{18}{5} - \frac{21}{5}\cos B = 2\sin^2 B$$

$$\to 18 - 21\cos B = 10 - 10\cos^2 B \to \cos B = \frac{1}{2} \to \angle B = 60^\circ.$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B = 16^2 + 10^2 - 2 \cdot 10 \cdot 16 \cdot \frac{1}{2} \to b^2 = 196 \to b = 14$$

$$\text{Is } D, I, F_2 \text{ collinear} \to \left(\frac{1}{BD} - \frac{1}{BC}\right) + \left(\frac{1}{BF_2} - \frac{1}{BA}\right) = \frac{AC}{AB \cdot BC}$$

$$\to \left(\frac{1}{10} - \frac{1}{16}\right) + \left(\frac{1}{BF_2} - \frac{1}{10}\right) = \frac{14}{16 \cdot 10} \to BF_2 = \frac{20}{3}$$

$$F_1F_2 = BF_2 - BF_1 = \frac{20}{3} - 5 \to F_1F_2 = \frac{5}{3}$$

$$\text{Plagiogonal System: } BC \equiv Bx, BA \equiv By, s = \frac{a+b+c}{2} = 20$$

$$I(i,i), i = \frac{ac}{a+b+c} = \frac{16 \cdot 10}{16 + 14 + 10} \to i = 4$$

$$r = i \cdot \sin B \to r = 4 \cdot \sin 60^\circ \to r = 2\sqrt{3}$$

$$BP = s - b = 20 - 14 \to BP = 6;\ DP = BD - BP = 10 - 6 \to DP = 4$$

$$BF_1 = 5, BD = 10, \angle B = 60^\circ \to \angle BF_1D = 90^\circ \to \angle BDF_1 = 30^\circ = \vartheta$$

$$D(10,0), F_1(0,5), \lambda_{DF_1} = \lambda = -\frac{1}{2}$$

Let $IR \parallel DF_1, IR: (y-4) = \lambda(x-4) \overset{y=0}{\Rightarrow} -4 = -\frac{1}{2}(x-4) \rightarrow x = BR = 12$

$$DR = BR - BD = 12 - 10 \rightarrow DR = 2$$

$$DF_1: \frac{x}{10} + \frac{y}{5} + 1 = 0, II_2: y = 4 \rightarrow x = 2$$

$$T(t_1, t_2), t_1 = 2, t_2 = 4$$

$$DT^2 = (10-2)^2 + (0-4)^2 + 2(10-2)(0-4)\cdot cos60^\circ \rightarrow DT = 4\sqrt{3}$$

Plagiogonal system: $DB \equiv Dx; DF_1 \equiv Dy, \vartheta = 30^\circ$

$$D(0,0), P(0,4), I(-2, 4\sqrt{3}), Q(0,q)$$

$$O(I,r): (x+2)^2 + \left(y - 4\sqrt{3}\right)^2 + 2(x+2)\left(y-4\sqrt{3}\right)^2 \cdot cos30^\circ \rightarrow r = 2\sqrt{3}$$

$$\rightarrow y = \frac{1}{2}\left(6\sqrt{3} - x\sqrt{3} — x^2 - 4x + 44\right) = f(x)$$

$$[DPQ] = sin\vartheta \cdot \int_0^4 f(x)dx = \frac{1}{2}\int_0^4 \frac{1}{2}\left(6\sqrt{3} - x\sqrt{3} — x^2 - 4x + 44\right)dx$$

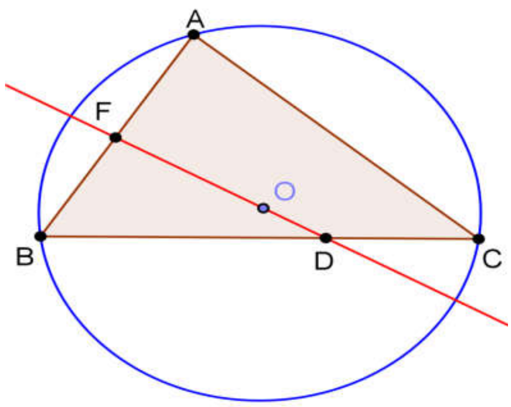
$$\rightarrow [DPQ] = -\sqrt{\frac{19}{2} - \frac{3\sqrt{13}}{2}} + 4\sqrt{3} - 20 + 6 \cdot tan^{-1}\left(\frac{1}{\sqrt{11}}\right) \approx 1,46231073729176$$

171.

***O* −circumcenter of ΔABC**

Prove that: D, O, F −collinear ⇔

$$\left(\frac{BC}{BD} + \frac{BA}{BF}\right) - \left(\frac{BC}{BF} + \frac{BA}{BD}\right) \cdot cosB = 2 \cdot sin^2B$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by proposer

Plagiogonal System: $BC \equiv Bx, BA \equiv By$

$$B(0,0), C(a,0), D(d,0), A(0,c), F(0,f), O(o_1, o_2)$$

$$o_1 = \frac{a - c\cdot cosB}{2sin^2B}, o_2 = \frac{c - a \cdot cosB}{2sin^2B}$$

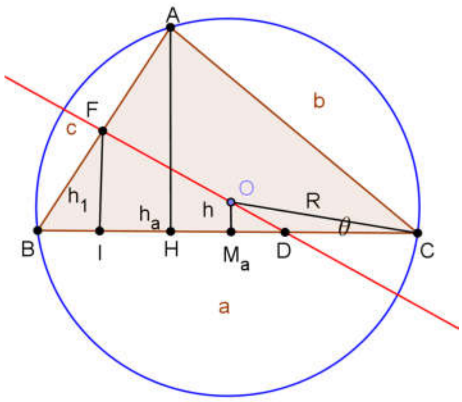
$$F, O, D \text{ −collinear} \Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & o_1 & d \\ f & o_2 & 0 \end{vmatrix} = 0$$

$$\Leftrightarrow a(f - d\cdot cosB) + c(d - f\cdot cosB) = 2df \cdot sin^2B$$

$$\left(\frac{a}{d}+\frac{c}{f}\right)-\left(\frac{a}{f}+\frac{c}{d}\right)\cdot cosB=2sin^2B$$

$$\left(\frac{BC}{BD}+\frac{BA}{BF}\right)-\left(\frac{BC}{BF}+\frac{BA}{BD}\right)\cdot cosB=2sin^2B$$

Solution 2 by Jose Ferreira Queiroz-Olinda-Brazil



$$BM_a=CM_a=\frac{a}{2},BD=x,AB=c,BC=a,CA=b,BF=y$$

$$FI=h_1=ysinB,AH=h_a=csinB,DC=a-x,OC=R$$

$$BJ=ysinB,HB=ccosB,DM_a=\frac{2x-a}{2}$$

In ΔOCM_a: $OC^2=OM_a^2+M_aC^2$

$$R^2=h^2+\frac{a^2}{4},b=2RsinB\to\ R=\frac{b}{2sinB}$$

$$\frac{b^2}{4sin^2B}=h^2+\frac{a^2}{4}\to h^2=\frac{b^2-a^2sin^2B}{4sin^2B};(I)$$

ΔFJD is similar ΔAM_aD: $tan\theta=\frac{FJ}{JD}=\frac{OM_a}{M_aD}\to\frac{h_1}{x-ycosB}=\frac{h}{\frac{2x-a}{2}}$

$$\frac{h_1}{x-ycosB}=\frac{2h}{2x-a}\to h=\frac{h_1(2x-a)}{2(x-ycosB)}$$

Now, ΔAHB is similar ΔFJB: $\frac{AH}{FJ}=\frac{AB}{FB}\to\frac{h_a}{h_1}=\frac{c}{y}\to h_1=\frac{y}{c}h_a$

$$h=\frac{\frac{y}{c}h_a(2x-a)}{2(x-ycosB)}=\frac{yh_a(2x-a)}{2c(x-ycosB)}=\frac{cysinB(2x-a)}{2c(x-ycosB)}$$

$$\rightarrow h=\frac{y(2x-a)sinB}{2(x-ycosB)};(II)$$

From $(I),(II)$ we have:

$$h^2=\frac{y^2(2x-a)^2sin^2B}{4(x-ycosB)^2}=\frac{b^2-a^2sin^2B}{4sin^2B}$$

$$b^2=a^2+c^2-2accosB$$

$$\frac{y^2(2x-a)^2sin^2B}{(x-ycosB)^2}=\frac{a^2+c^2-2accosB-a^2(1-cos^2B)}{sin^2B}$$

$$y^2(2x-a)^2sin^4B=(x-ycosB)^2(c-acosB)^2$$

$$y^2(2x-a)^2sin^4B-(x-ycosB)^2(c-acosB)^2=0$$

$$[y(2x-a)sin^2B-(x-ycosB)(c-acosB)][y(2x-a)sin^2B$$
$$+(x-ycosB)(c-acosB)]=0$$

So,

$$(i)\quad y(2x-a)sin^2B-(x-ycosB)(c-acosB)=0$$

$$y(2x-a)sin^2B-cx+axcosB+cycosB-aycos^2B=0$$

$$2xysin^2B=(cx+ay)-(cy+ax)cosB$$

$$\left(\frac{BC}{BD}+\frac{BA}{BF}\right)-\left(\frac{BC}{BF}+\frac{BA}{BD}\right)\cdot cosB=2sin^2B$$

$$(ii)\quad y(2x-a)sin^2B+(x-ycosB)(c-acosB)=0$$

$\left(\frac{c}{x}+\frac{a}{y}\right)cosB-\left(\frac{a}{x}+\frac{c}{y}\right)=2\left(1-\frac{a}{x}\right)sin^2B$, does not satisfy the problem.

Therefore,

$$\left(\frac{BC}{BD}+\frac{BA}{BF}\right)-\left(\frac{BC}{BF}+\frac{BA}{BD}\right)\cdot cosB=2sin^2B$$

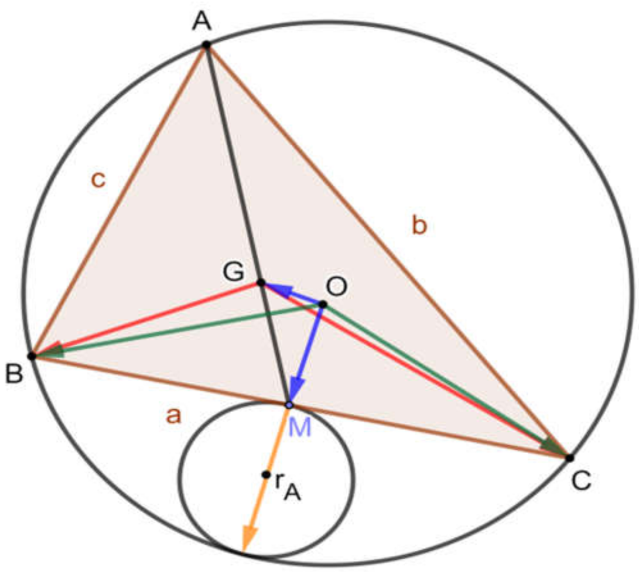
172.

In acute $\Delta ABC, r_A$ —radii of circle tangent simultaneous to BC in the middle of BC and circumcircle of ΔABC(internal tangent). If r_B, r_C —are similarly defined then:

$$R_A+R_B+R_C=R-\frac{r}{2}$$

Mehmet Şahin-Ankara-Turkey

Solution by Adrian Popa-Romania



Let P be tangent point of these two circle.

$$OP = OM + MP \rightarrow R = OM + 2r_A$$

$$\begin{cases} \angle BOC = \widehat{BC} \\ \angle A = \frac{\widehat{BC}}{2} \end{cases} \rightarrow \angle BOC = 2\angle A, OB \equiv OC = R, \Delta OBC \text{ −isoscelles, then } \angle BOM = \angle A.$$

$$\Delta BOM(\angle M = 90^\circ):\ cosA = \frac{OM}{R} \rightarrow OM = R \cdot cosA$$

$$R = R \cdot cosA + 2r_A \rightarrow 2r_A = R(1 - cosA) = R \cdot 2sin^2\frac{A}{2}$$

$$r_A = R \cdot sin^2\frac{A}{2}$$

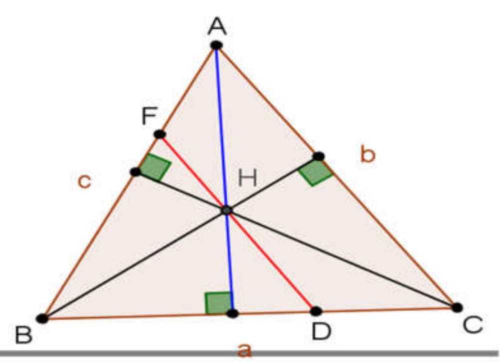
$$r_A + r_B + r_C = R\left(sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2}\right) = R\left(1 - \frac{r}{2R}\right) = R - \frac{r}{2}$$

173.

H −orthocenter of ΔABC

Prove:

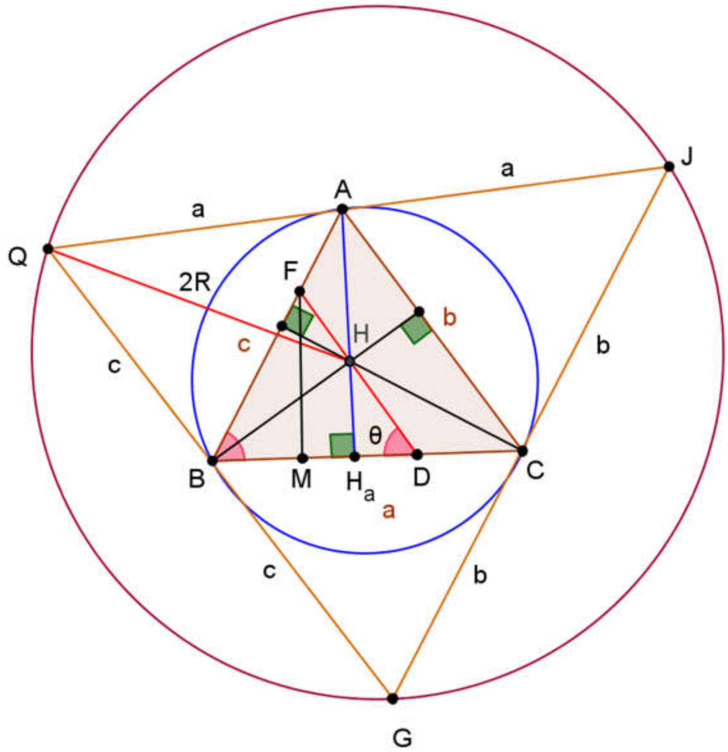
D, H, F −collinear⇔



$$\left(\frac{BC}{BF}+\frac{BA}{BD}\right)-\left(\frac{BC}{BD}+\frac{BA}{BF}\right)\cdot cosB = sinB\cdot tanB$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$b = 2R\cdot sinB, BD = x, BF = y, AB = c, BC = a, C = b$$

$$AH_a = h_a = c\cdot sinB, FM = h_1 = y\cdot sinB, HH_a = h$$

$$BM = y\cdot cosB, QA = AJ = BC = a, AH = AH_a - HH_a = h_a - h$$

$$HQ = 2R, BH_a = c\cdot cosB$$

The orthocentre of ΔABC is the circumcenter of ΔGJQ, so

$$(2R)^2 = a^2 + (h_a - h)^2; (I)$$

ΔAH_aB is similar to ΔFMB, so $\frac{AH_a}{FM} = \frac{AB}{BF} \to \frac{h_a}{h_1} = \frac{c}{y}$

ΔFMD is similar to ΔHH_aD, so

$$\frac{FM}{HH_a} = \frac{MD}{H_aD} \to \frac{h_1}{h} = \frac{x - BM}{x - H_aB} = \frac{x - y\cdot cosB}{x - c\cdot cosB}$$

$$h = h_1 \cdot \frac{x - c \cdot cosB}{x - y \cdot cosB} = y \cdot sinB \cdot \frac{x - c \cdot cosB}{x - y \cdot cosB} \to h = \frac{y \cdot sinB \cdot (x - c \cdot cosB)}{x - y \cdot cosB}$$

Now,

$$h_a - h = c \cdot sinB - h = c \cdot sinB - \frac{y \cdot sinB \cdot (x - c \cdot cosB)}{x - y \cdot cosB}$$

$$h_a - h = \frac{x \cdot sinB \cdot (c - y)}{x - y \cdot cosB}; (II)$$

From $(I), (II)$ we have:

$$\frac{b^2}{sin^2B} = a^2 + \frac{x^2 \cdot sin^2B \cdot (c - y)^2}{(x - y \cdot cosB)^2}$$

$$\frac{b^2 - a^2 \cdot sin^2B}{sin^2B} = \frac{x^2 \cdot sin^2B \cdot (c - y)^2}{(x - y \cdot cosB)^2}$$

$$b^2 = a^2 + c^2 - 2ac \cdot cosB$$

$$\frac{a^2 + c^2 - 2ac \cdot cosB - a^2 \cdot sin^2B}{sin^2B} = \frac{x^2 \cdot sin^2B \cdot (c - y)^2}{(x - y \cdot cosB)^2}$$

$$\frac{a^2 + c^2 - 2ac \cdot cosB - a^2(1 - cos^2B)}{sin^2B} = \frac{x^2 \cdot sin^2B(c - y)^2}{(x - y \cdot cosB)^2}$$

$$\frac{(c - a \cdot cosB)^2}{sin^2B} = \frac{x^2 \cdot sin^2B \cdot (x - y)^2}{(x - y \cdot cosB)^2}$$

$$(c - a \cdot cosB)^2 \cdot (x - y \cdot cosB)^2 - x^2 \cdot sinB \cdot (c - y)^2 = 0$$

$$[(c - a \cdot cosB) \cdot (x - y \cdot cosB) - x \cdot sin^2B \cdot (c - y)] \cdot [(c - a \cdot cosB) \cdot (x - y \cdot cosB) + x \cdot sin^2B \cdot (c - y)] = 0$$

So,

i) $(c - a \cdot cosB) \cdot (x - y \cdot cosB) - x \cdot sin^2B \cdot (c - y) = 0$

$$cx - (cy + ax) \cdot sinB + ay \cdot cos^2B - (cx - xy) \cdot sin^2B = 0$$

$$cx - (cy + ax) \cdot cosB + ay \cdot cos^2B - cx \cdot sin^2B + xy \cdot sin^2B = 0$$

$$(ax + cy) \cdot cos^2B - (ay - cx) \cdot cosB + xy \cdot sin^2B = 0$$

$$(ax + cy) - (ay + cx) \cdot cosB = xy \cdot sinB \cdot tanB$$

Or,

ii) $(c - a \cdot cosB) \cdot (x - y \cdot cosB) + x \cdot sin^2B \cdot (c - y) = 0$

$$cx - (cy + ax) \cdot cosB + ay \cdot cos^2B + (cx - ay) \cdot sin^2B = 0$$

$$cx - (cy + ax) \cdot cosB + ax \cdot cos^2B + cx \cdot sin^2B - xy \cdot sin^2B = 0$$

$$2cx - (cy + ax) \cdot cosB + (ay - cx) \cdot cos^2B = xy \cdot sin^2B$$

$\frac{2c}{y} - \left(\frac{c}{x} + \frac{a}{y}\right) \cdot cosB + \left(\frac{a}{x} - \frac{c}{y}\right) \cdot cos^2B = sin^2B$, **does not satisfy the problem.**

So,

$$(ax+cy)-(ay+cx)\cdot cosB = xy\cdot sinB\cdot tanB$$

$$\left(\frac{a}{y}+\frac{c}{x}\right)-\left(\frac{a}{x}+\frac{c}{y}\right)\cdot cosB = sinB\cdot tanB$$

Therefore,

$$\left(\frac{BC}{BF}+\frac{BA}{BD}\right)-\left(\frac{BC}{BD}+\frac{BA}{BF}\right)\cdot cosB = sinB\cdot tanB$$

174.

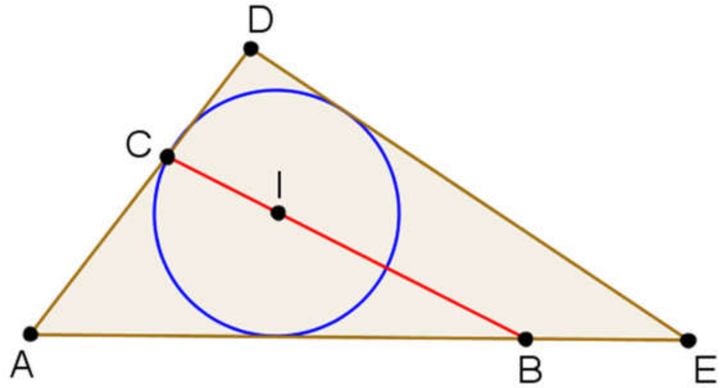
$A_{ABC} = A_{BCDE}$

$P_{ABC} = P_{BCDE}$

Prove: $I \in BC$

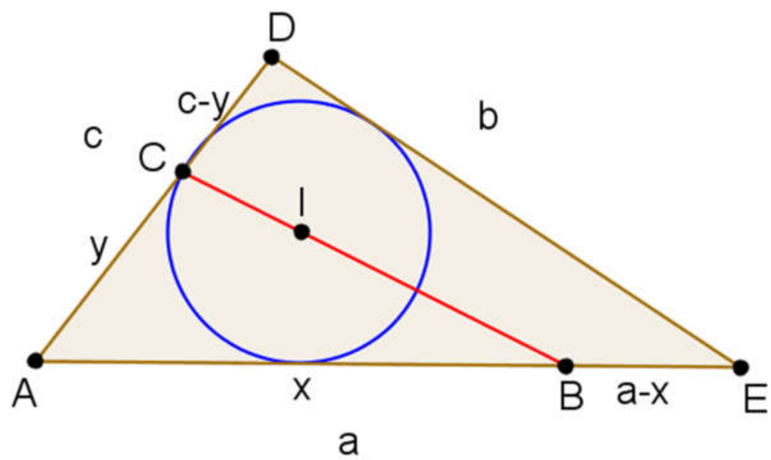
A_{ABC} –area of ΔABC.

P_{ABC} –perimeter of ΔABC.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil



$$BC = l, AB = x, AC = y, DE = b, AE = a, AD = c, BE = a-x, CD = c-y$$

If $I \in BC$, so i) $l^2 = \frac{1}{xy}\cdot(x+y)^2(ac+xy)-2(a+c)(x+y)$

ii) $\frac{b}{ac} = \left(\frac{1}{y}-\frac{1}{c}\right)+\left(\frac{1}{x}-\frac{1}{a}\right)$

Now, $A_{ABC} = A_{BCDE}; A_{ABC} = A_{ADE} - A_{ABC} \to 2A_{ABC} = A_{ADE}$

$$2\cdot\frac{1}{2}\cdot xy\cdot sinA=\frac{1}{2}\cdot ac\cdot sinA$$

$$ac=2xy;(1)\to\frac{xy}{ac}=\frac{1}{2}$$

$$P_{ABC}=P_{BCDE}\to x+y+l=l+(c-y)+(a-x)+b$$

$$2(x+y)=a+b+c;(2)\to b=2(x+y)-(a+c)$$

Replacing (1)in (i), we have: $l^2=\frac{1}{xy}(x+y)^2(2xy+xy)-2(a+c)(x+y)$

$$l^2=3(x+y)^2-2(a+c)(x+y);(3)$$

Being, $l^2=x^2+y^2-2xy\cdot cosB$and $b^2=a^2+c^2-2ac\cdot cosB$, so

$$l^2=x^2+y^2-2xy\cdot\frac{a^2+c^2-b^2}{2ac}$$

$$l^2=x^2+y^2-\frac{xy}{ac}(a^2+c^2-b^2)\to l^2=x^2+y^2-\frac{1}{2}(a^2+c^2-b^2)$$

$$\to l^2=x^2+y^2-\frac{1}{2}\left[a^2+c^2-\left(2(x+y)-(a+c)\right)^2\right]$$

$$l^2=(x+y)^2-2xy-\frac{1}{2}[a^2+c^2-4(x+y)^2+4(x+y)(a+c)-(a+c)^2]$$

$$2l^2=2(x+y)^2-2ac-a^2-c^2+4(x+y)^2-4(x+y)(a+c)+(a+c)^2$$

$$2l^2=2(x+y)^2-(a+c)^2+4(x+y)^2-4(x+y)(a+c)+(a+c)^2$$

$$2l^2=6(x+y)^2-4(x+y)(a+c)$$

$$l^2=3(x+y)^2-2(a+c)(x+y);(4)$$

So, $(3)=(4)$

Now, using the expression (2) it follows that: $b=2x+2y-a-c$

$$\frac{b}{ac}=2\cdot\frac{x}{ac}+2\cdot\frac{y}{ac}-\frac{a}{ac}-\frac{c}{ac}$$

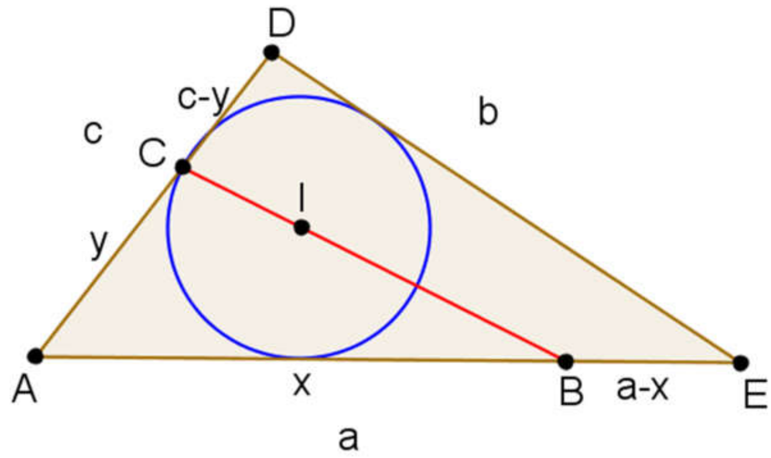
$$\frac{b}{ac}=2\cdot\frac{x}{2xy}+2\cdot\frac{y}{2xy}-\frac{1}{c}-\frac{1}{a}$$

$$\frac{b}{ac}=\left(\frac{1}{y}-\frac{1}{c}\right)+\left(\frac{1}{x}-\frac{1}{a}\right);(5)$$

We conclude that conditions (i) and (ii) are satisfied.

Therefore, $I\in BC$

Solution 2 by Gheorghe Oniciuc-Romania



$$F_{ADE} = F_{BCDE} \to F_{ADE} = \frac{1}{2}F_{ABC} \to \frac{xy \cdot \sin A}{2} = \frac{1}{2} \cdot \frac{bc \cdot \sin A}{2} \to xy = \frac{bc}{2}; (1)$$

$$P_{ADE} = P_{BCDE} \to x + y = a + b + c - (x + y) \to x + y = s; (2s = a + b + c)$$

$$I \in \overrightarrow{BC} \Leftrightarrow \overrightarrow{AI} = \lambda\overrightarrow{AC} + (1 - \lambda)\overrightarrow{AB}, \qquad \vec{r}_A = \vec{O}$$

$$\vec{r}_I = \frac{a\vec{r}_A + b\vec{r}_E + c\vec{r}_D}{2s} = \frac{b\vec{r}_E + c\vec{r}_D}{2s}$$

$$\vec{r}_E = \frac{x}{b}\vec{r}_C, \qquad \vec{r}_D = \frac{y}{c}\vec{r}_B$$

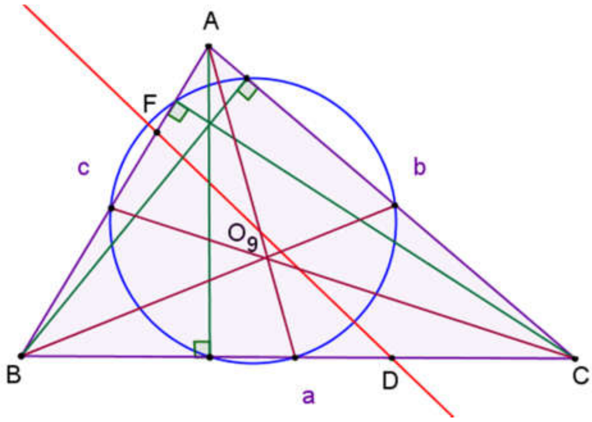
So, $I \in (BC) \Leftrightarrow \exists\lambda \in R$ such that:

$$\frac{\lambda x}{b} \cdot \vec{r}_C + \frac{(1-\lambda)y}{c} \cdot \vec{r}_B = \frac{c}{2s} \cdot \vec{r}_C + \frac{b}{2s} \cdot \vec{r}_B$$

$$\Leftrightarrow \begin{cases} \dfrac{\lambda x}{b} = \dfrac{c}{2s} \\ \dfrac{(1-\lambda)y}{c} = \dfrac{b}{2s} \end{cases} \Leftrightarrow \begin{cases} \lambda = \dfrac{bc}{2sx} \\ 1 - \lambda = \dfrac{bc}{2sy} \end{cases} \Leftrightarrow \frac{bc}{2sx} + \frac{bc}{2sy} = 1 \Leftrightarrow$$

$$\frac{1}{x} + \frac{1}{y} = \frac{2s}{bc} \Leftrightarrow \frac{x+y}{xy} = \frac{2s}{bc} \Leftrightarrow \frac{s}{\frac{bc}{2}} = \frac{2s}{bc}$$

175.



In $\Delta ABC, D \in BC, F \in BA,$

$$BD = d, BF = f$$

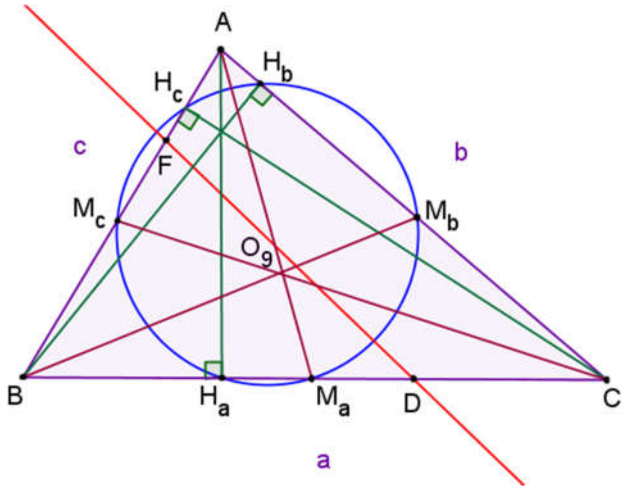
O_9 **−N.P.C.**

$$\frac{1}{d} + \frac{1}{f} = \frac{6}{a+c}$$

Prove that: $\angle ABC = 60^\circ$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



We know that D, O and F are collinear. So,

$$\left(\frac{BC}{BF} + \frac{BA}{BD}\right) cosB - \left(\frac{BC}{BD} + \frac{BA}{BF} - 2\right) cos2B = 2, \left(\frac{a}{f} + \frac{c}{d}\right) cosB - \left(\frac{a}{d} + \frac{c}{f} - 2\right) cos2B = 2$$

$$\left(6 - \frac{a}{d} - \frac{c}{f}\right) cosB - \left(\frac{a}{d} + \frac{c}{f} - 2\right)(2cos^2B - 1) = 2$$

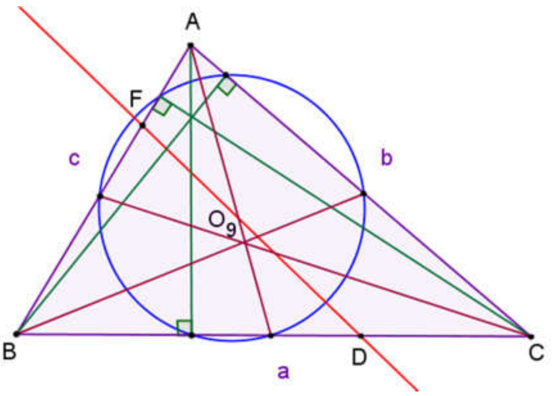
Replacing $\frac{a}{d} + \frac{c}{f} = y$, **so** $(6 - y)cosB - (y - 2)(2cos^2B - 1) = 2$

$$(6 - y)cosB - 2(y - 2)cos^2B + (y - 2) = 0 \quad 2(y - 2)cos^2B - (6 - y)cosB + (4 - y) = 0$$

$$cosB_1 = \frac{1}{2} \to B_1 = 60^\circ, cosB_2 = \frac{4 - y}{y - 2} = \frac{4 - \left(\frac{a}{d} + \frac{c}{f}\right)}{\left(\frac{a}{d} + \frac{c}{f}\right) - 2}$$

Does not satisfy the problem.

176.

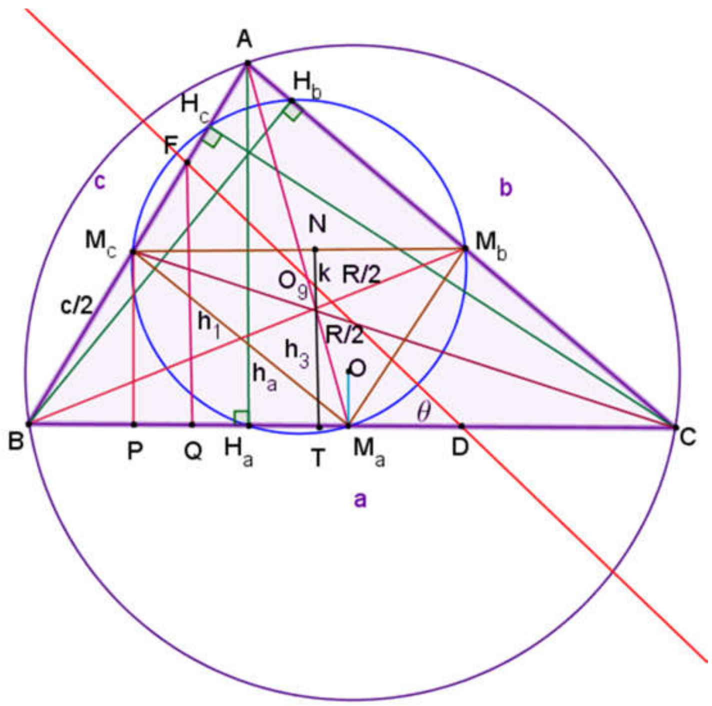


O_9 −N.P.C. of ΔABC. **Prove that:**

$$D, F, O_9 \text{ −collinear} \Leftrightarrow \left(\frac{BC}{BF}+\frac{BA}{BD}\right)\cos B-\left(\frac{BC}{BD}+\frac{BA}{BF}-2\right)\cos 2B=2$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Jose Ferreira Queiroz-Olinda-Brazil



$$BM_a=CM_a=\frac{a}{2}, BD=d, BF=f, BC=a, AB=c, AC=b$$

$$NM_b=NM_c=\frac{a}{4}, FQ=h_1, AH_a=h_a, O_9N=k, O_9T=h_3$$

$$O_9M_a = O_9M_b = \frac{R}{2}, O_9N + O_9T = \frac{c}{2}\cdot sinB, k = \frac{c}{2}\cdot sinB - h_3$$

In ΔO_9NM_b: $O_9M_b^2 = O_9N^2 + M_bN^2$

$$\left(\frac{R}{r}\right)^2 = k^2 + \left(\frac{a}{4}\right)^2 \to \left(\frac{R}{2}\right)^2 = \left(\frac{c}{2}sinB - h_3\right)^2 + \left(\frac{a}{4}\right)^2 ;(I)$$

In ΔO_9TM_a: $O_9M_a^2 = O_9T^2 + TM_a^2$

$$TM_a = BM_a - PT - BP = \frac{a}{2} - \frac{a}{4} - \frac{c}{2}\cdot cosB$$

$$TM_a = \frac{a}{4} - \frac{c}{2}\cdot cosB \to \left(\frac{R}{2}\right)^2 = h_3^2 + \left(\frac{a}{4} - \frac{c}{2}cosB\right)^2 ;(II)$$

So, $(I) = (II)$

$$h_3^2 + \left(\frac{a}{4} - \frac{c}{2}cosB\right)^2 = \left(\frac{a}{4}\right)^2 + \left(\frac{c}{2}sinB - h_3\right)^2 \to$$

$$h_3 = \frac{c\cdot sin^2B + a\cdot cosB - c\cdot cos^2B}{4sinB}$$

Now,

$$tan\theta = \frac{FQ}{QD} = \frac{O_9T}{QD} \to D, F, O_9 \text{ —are collinear.}$$

$$\frac{h_1}{d - fcosB} = \frac{c\cdot sin^2B + a\cdot cosB - c\cdot cos^2B}{(4d - a - 2c\cdot cosB)sinB}$$

$$f\cdot sin^2B\cdot(4d - a - 2c\cdot cosB) - (d - f\cdot cosB)\left(c\cdot sin^2B + a\cdot cos^2B - c\cdot cos^2B\right) = 0$$

$$sin^2B\cdot(4df - af) - 2fc\cdot cosBsin^2B - dc\cdot sin^2B - ad\cdot cosB + dc\cdot cos^2B +$$

$$+fc\cdot sin^2BcosB + af\cdot cos^2B - fc\cdot cosBcos^2B = 0$$

$$sin^2B\cdot(4df - af - dc) - fc\cdot cosB\cdot\left(sin^2B + cos^2B\right) - ad\cdot cosB + (af + dc)cos^2B = 0$$

$$sin^2B\cdot(4df - af - dc) - (fc + ad)cosB + (af + dc)cos^2B = 0$$

$$(ad + fc)cosB - (af + dc - 2fd)cos2B = 2df$$

Therefore,

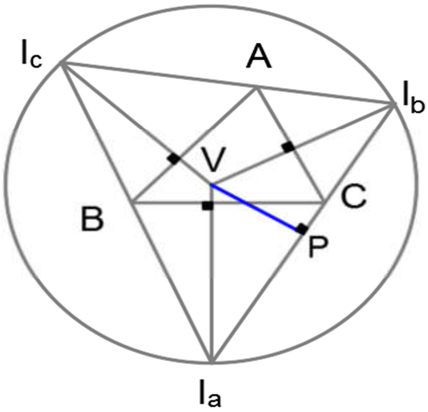
$$\left(\frac{BC}{BF} + \frac{BA}{BD}\right)cosB - \left(\frac{BC}{BD} + \frac{BA}{BF} - 2\right)cos2B = 2$$

177. In $\Delta ABC, V-$ Bevan's point, I_a, I_b, I_c- excenters, R_a, R_b, R_c- circumradii in $\Delta VI_aI_b, \Delta VI_bI_c, \Delta VI_cI_a$.

Prove that : $rR_aR_bR_c = 4R^4$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$Let\ P\ be\ the\ feet\ of\ the\ perpendicular\ from\ V\ to\ I_aI_b.$$

$$We\ have:\ \mu(I_aCB) = \frac{\pi - C}{2} \to \mu(CI_aV) = \frac{\pi}{2} - \mu(I_aCB) = \frac{C}{2}$$

$$\Delta VI_aI_b\ is\ isosceles \to \mu(I_aVI_b) = \pi - 2\mu(CI_aV) = \pi - C\ and\ I_aP = \frac{I_aI_b}{2}$$

$$\to \cos VI_aI_b = \frac{I_aP}{I_aV} = \frac{I_aI_b}{2.2R} \to I_aI_b = 4R\cos\frac{C}{2} \qquad (I_aV = 2R)$$

$$In\ \Delta VI_aI_b, we\ have:\ \sin I_aVI_b = \frac{I_aI_b}{2R_a} \to R_a = \frac{I_aI_b}{2\sin I_aVI_b} = \frac{4R\cos\frac{C}{2}}{2\sin(\pi - C)} = \frac{2R\cos\frac{C}{2}}{2\sin C}$$

$$= R\csc\frac{C}{2}$$

$$\to R_a = R\csc\frac{C}{2}\ (and\ analogs) \to R_aR_bR_c = R^3\prod\csc\frac{C}{2} = R^3.\frac{4R}{r} = \frac{4R^4}{r}$$

$$Therefore, \qquad rR_aR_bR_c = 4R^4$$

178. In $\Delta ABC, n_a$ −Nagel's cevian, the following relationship holds:

$$2\left(\frac{R}{r}-1\right)\sum_{cyc} h_a = 4R + r + \sum_{cyc} \frac{n_a^2}{r_a}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\because n_a^2 = s^2 - 2r_a h_a; (and\ analogs)$$

$$\to \sum_{cyc} \frac{n_a^2}{r_a} = s^2 \sum_{cyc} \frac{1}{r_a} - \sum_{cyc} 2h_a = \frac{s^2}{r} - 2\sum_{cyc} h_a$$

$$\to 4R + r + \sum_{cyc} \frac{n_a^2}{r_a} = \frac{(4R+r)r + s^2}{r} - 2\sum_{cyc} h_a = \frac{s^2 + r^2 + 4Rr}{r} - 2\sum_{cyc} h_a =$$

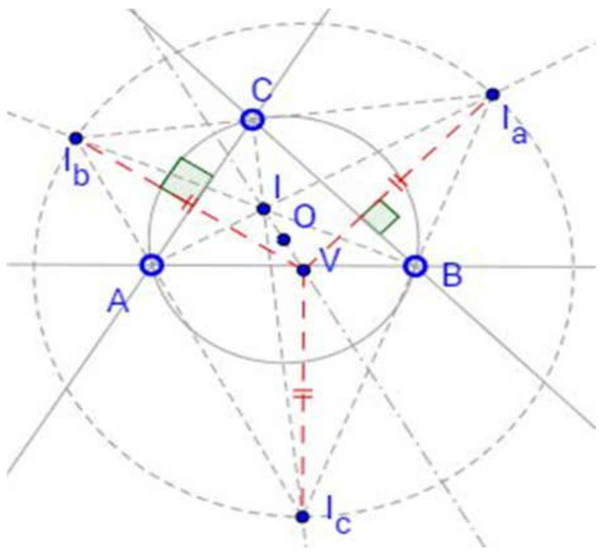
$$= \frac{\sum ab}{r} - 2\sum_{cyc} h_a = \frac{\sum 2Rh_c}{r} - 2\sum_{cyc} h_a = 2\left(\frac{R}{r}-1\right)\sum_{cyc} h_a$$

Therefore,

$$2\left(\frac{R}{r}-1\right)\sum_{cyc} h_a = 4R + r + \sum_{cyc} \frac{n_a^2}{r_a}$$

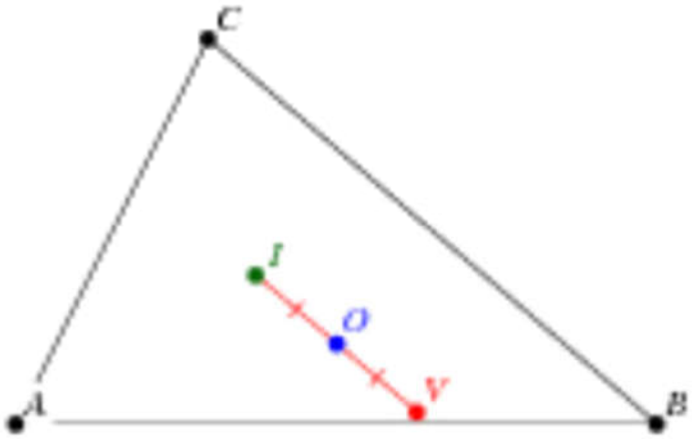
179. If V −Bevan' s point in ΔABC then :

$$\sum AV^2 = 12R^2 - s^2 - r^2 - 4Rr$$



Proposed by Mehmet Şahin-Ankara-Turkey

Solution 1 by Tran Hong-DongThap-Vietnam



In ΔAIV we have:

$$AO^2 = \frac{2AI^2 + 2AV^2 - IV^2}{4} = \frac{2AI^2 + 2AV^2 - (2OI)^2}{4} = \frac{AI^2 + AV^2}{2} - OI^2$$

$$\Rightarrow AV^2 = 2AO^2 + 2OI^2 - AI^2 = 2R^2 + 2(R^2 - 2Rr) - (r^2 + (p-a)^2)$$

Similarly: $BV^2 = 4R^2 - 4Rr - (r^2 + (p-b)^2)$

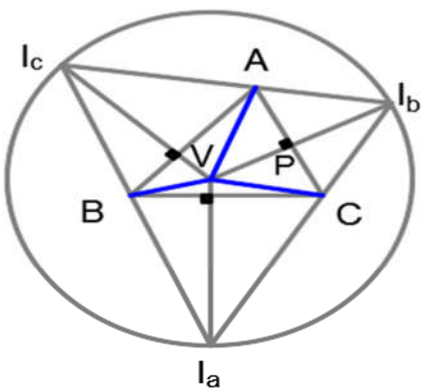
$$CV^2 = 4R^2 - 4Rr - (r^2 + (p-c)^2)$$

$$\Rightarrow AV^2 + BV^2 + CV^2 = 12R^2 - 12Rr - 3r^2 - (3p^2 - 2(a+b+c)p + a^2 + b^2 + c^2)$$

$$= 12R^2 - 12Rr - 3r^2 - (3p^2 - 4p^2 + 2p^2 - 8Rr - 2r^2)$$

$$= 12R^2 - 4Rr - p^2 - r^2$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco



$Let\ P\ be\ the\ feet\ of\ the\ perpendicular\ from\ I_b\ to\ AC.$

$We\ know\ that: P \in VI_b, AP = s - c, PI_b = r_b\ and\ VI_a = VI_b = VI_c = 2R$

$$\to AV^2 = AP^2 + PV^2 = (s-c)^2 + (2R - r_b)^2 \ \ (and\ analogs)$$

$$\to \sum AV^2 = \sum (s-c)^2 + \sum (2R - r_b)^2$$

$$= 3s^2 - 2s\sum c + \sum c^2 + 12R^2 - 4R\sum r_b + \sum r_b{}^2$$

$$= 3s^2 - 4s^2 + 2(s^2 - r^2 - 4Rr) + 12R^2 - 4R(4R + r) + (4R + r)^2 - 2s^2$$

$$Therefore, \quad \sum AV^2 = 12R^2 - s^2 - r^2 - 4Rr$$

180.

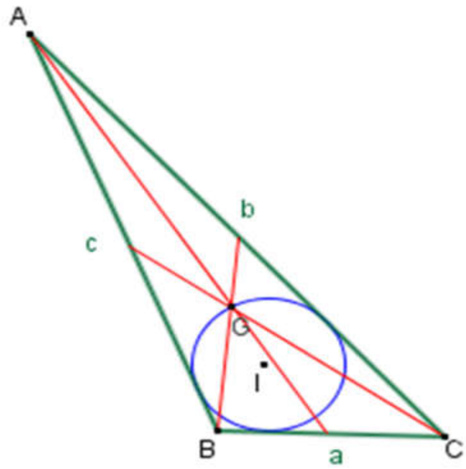
$G \in C(I, r)$

a, b, c —are natural numbers

$a \neq b \neq c \neq a$

$F_{\Delta ABC}$ —area of ΔABC.

Find: $min\{F_{\Delta ABC}\}$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

Let $a < b < c$.

$$G \in C(I, r) \to 8(a^2 + b^2 + c^2) = 3(a + b + c)^2$$

From wolframalfa we have:

If $a = 1, 2, 3, 4$ then, the result is not acceptable.

If $a = 5 \to (a, b, c) = (5, 10, 13)$

$$F_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}; s = \frac{a + b + c}{2} \to F_{\Delta ABC} = 6\sqrt{14}$$

181.

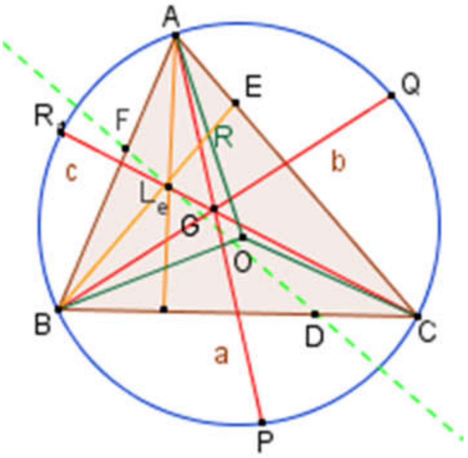
L_e –Lemoine's point of ΔABC

$BD = d, BF = f$

R –circumradius of ΔABC

Prove that:

$$\frac{a}{f} + \frac{c}{d} \ge \frac{b}{R}\sqrt{3}$$



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by proposer

$$\text{Is: } \frac{a}{f} + \frac{c}{d} = \frac{a^2+b^2+c^2}{ac} \text{ and } \frac{a^2+b^2+c^2}{ac} \ge \frac{4\sqrt{3}F}{ac} = \frac{4\sqrt{3}\cdot\frac{1}{2}ac\cdot sinB}{ac} \to$$

$$\frac{a}{f} + \frac{c}{d} \ge 2\sqrt{3}\cdot sinB \to \frac{a}{f} + \frac{c}{d} \ge 2\sqrt{3}\cdot\frac{b}{2R}$$

$$\to \frac{a}{f} + \frac{c}{d} \ge \frac{b}{R}\sqrt{3} \text{ or } \frac{a}{f} + \frac{c}{d} \ge \frac{F}{ac}\cdot 4\sqrt{3}$$

182. V –Bevan's point in $\Delta ABC, I_a, I_b, I_c$ –excenters, $VK \perp (I_bI_c)$, $K \in (I_bI_c), VL \perp (I_cI_a), L \in (I_cI_a), VM \perp (I_aI_b), M \in (I_aI_b)$. Prove that :

$$\frac{a}{bc.VK^2} + \frac{b}{ca.VL^2} + \frac{c}{ab.VM^2} = \frac{r_a + r_b + r_c}{2R^2F}$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco