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2301. In ΔABC the following relationship holds:

$$\left(1+\frac{1}{a}\tan\frac{A}{2}\right)\left(1+\frac{1}{b}\tan\frac{B}{2}\right)\left(1+\frac{1}{c}\tan\frac{C}{2}\right)\geq\left(1+\frac{9}{2}\cdot\frac{r}{s^2}\right)^3$$

Proposed by Florică Anastase-Romania

Solution 1 by Probal Chakraborty-India

$$\left(1+\frac{1}{a}\tan\frac{A}{2}\right)\left(1+\frac{1}{b}\tan\frac{B}{2}\right)\left(1+\frac{1}{c}\tan\frac{C}{2}\right)=$$

$$=\left(1+\frac{r}{a(s-a)}\right)\left(1+\frac{r}{b(s-b)}\right)\left(1+\frac{r}{s(s-c)}\right)\geq$$

$$\geq\left(1+\frac{9r}{2s^2}\right)^3$$

Note: $-\frac{b+c-a}{2}=s-a,\ a+b+c=2s,\ s\geq s-a$

$$a+b+c\geq 3\sqrt[3]{abc}\to s\geq\frac{3}{2}\sqrt[3]{abc};\ \sqrt[3]{abc}\geq 3a,\ b+c>a$$

$$s\geq\frac{9}{2}a\to\frac{2s^2}{9}\geq a(s-a)\to\frac{1}{a(s-a)}\geq\frac{9}{2s^2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Holder inequality:

$$\prod_{cyc}\left(1+\frac{1}{a}tan\frac{A}{2}\right)\geq\left(1+\sqrt[3]{\frac{1}{abc}\prod_{cyc}tan\frac{A}{2}}\right)^3=$$

$$=\left(1+\sqrt[3]{\frac{1}{4Rrs}\cdot\frac{r}{s}}\right)^3=\left(1+\frac{1}{\sqrt[3]{4Rs^2}}\right)^3\overset{(*)}{\geq}\left(1+\frac{9r}{2s^2}\right)^3$$

$(*)\Leftrightarrow 2s^2\geq 9r\cdot\sqrt[3]{4Rs^2}\Leftrightarrow 2s^4\geq 9^3Rr^3$, which is true, because $2s^2\geq 27Rr$ and $s^2\geq 27r^2(Mitrinovic)$.

Therefore,

$$\left(1+\tfrac{1}{a}\tan\tfrac{A}{2}\right)\left(1+\tfrac{1}{b}\tan\tfrac{B}{2}\right)\left(1+\tfrac{1}{c}\tan\tfrac{C}{2}\right)\geq\left(1+\tfrac{9}{2}\cdot\tfrac{r}{s^2}\right)^3$$

Solution 3 by proposer

$$a\cot\frac{A}{2}+b\cot\frac{B}{2}+c\cot\frac{C}{2}\leq\frac{2s^2}{3r};(1)$$

$$a\cot\frac{A}{2}=a\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{as(s-a)}{F}\Rightarrow\sum_{cyc}\frac{as(s-a)}{F}\le\frac{2s^2}{3r}\Leftrightarrow$$

$$\sum_{cyc}\frac{a(b+c-a)}{2}\le\frac{2(a+b+c)^2}{3\cdot 4}\Leftrightarrow\sum_{cyc}ab\le\sum_{cyc}a^2;(true)$$

$$\sum_{cyc}\frac{1}{x+a\cot\frac{A}{2}}\overset{BCS}{\ge}\frac{9}{3x+\sum a\cot\frac{A}{2}}\overset{(1)}{\ge}\frac{9}{3x+\frac{2s^2}{3r}}=\frac{3}{x+\frac{2s^2}{9r}}\Leftrightarrow$$

$$\int_0^1\sum_{cyc}\frac{1}{x+a\cot\frac{A}{2}}dx\ge 3\int_0^1\frac{dx}{x+\frac{2s^2}{9r}}\Leftrightarrow$$

$$\sum_{cyc}\log\left(x+a\cot\frac{A}{2}\right)\Bigg|_0^1\ge 3\log\left(x+\frac{2s^2}{9r}\right)\Bigg|_0^1\Leftrightarrow$$

$$\sum_{cyc}\log\left(1+\frac{1}{a\cot\frac{A}{2}}\right)\ge 3\log\left(1+\frac{9r}{2s^2}\right)\Leftrightarrow$$

$$\log\left(\prod_{cyc}\left(1+\frac{1}{a}\tan\frac{A}{2}\right)\right)\ge\log\left(1+\frac{9r}{2s^2}\right)^3\Leftrightarrow$$

$$\prod_{cyc}\left(1+\frac{1}{a}\tan\frac{A}{2}\right)\ge\left(1+\frac{9r}{2s^2}\right)^3$$

2302. In ΔABC, n_a –Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{b^2n_b+c^2n_c-a^2n_a}{bc\sqrt{n_bn_c}}\le 3$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let }\boxed{a\sqrt{n_a}=x,b\sqrt{n_b}=y\text{ and }c\sqrt{n_c}=z}\text{ and }\because a\sqrt{n_a},b\sqrt{n_b},c\sqrt{n_c}>0$$

$$\therefore\boxed{x,y,z>0}\text{ and }\sum_{cyc}\frac{b^2n_b+c^2n_c-a^2n_a}{bc\sqrt{n_bn_c}}-3$$

$$=\sum_{cyc}\frac{y^2+z^2-x^2}{yz}-3=\frac{x(y^2+z^2-x^2)+y(z^2+x^2-y^2)+z(x^2+y^2-z^2)-3xyz}{xyz}$$

$$=\frac{-1}{xyz}\left(\sum_{cyc}x^3+3xyz-\sum_{cyc}x^2y-\sum_{cyc}xy^2\right)$$

$$\leq 0\left(\because\sum_{cyc}x^3+3xyz-\sum_{cyc}x^2y-\sum_{cyc}xy^2\overset{\text{Schur}}{\geq}0\ \forall x,y,z>0 \text{ and } \because xyz>0\right)$$

$$\Rightarrow\sum_{cyc}\frac{b^2n_b+c^2n_c-a^2n_a}{bc\sqrt{n_bn_c}}-3\leq 0\Rightarrow\sum_{cyc}\frac{b^2n_b+c^2n_c-a^2n_a}{bc\sqrt{n_bn_c}}\leq 3(\text{QED})$$

2303. In ΔABC the following relationship holds:

$$(s_a+s_b)c^3+(s_b+s_c)a^3+(s_c+s_a)b^3\geq 16\sqrt{3}F^2$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that: $s_a\geq h_a(and\ analogs)$

$$s_a+s_b\geq h_a+h_b=2F\left(\frac{1}{a}+\frac{1}{b}\right)\overset{CBS}{\geq}2F\cdot\frac{4}{a+b}=\frac{8F}{a+b}$$

Hence,

$$\sum_{cyc}(s_a+s_b)c^3\geq 8F\sum_{cyc}\frac{c^3}{a+b}\overset{Holder}{\geq}8F\cdot\frac{(\sum c)^3}{3\sum(a+b)}=\frac{4F}{3}(\sum a)^2=$$

$$=\frac{16F}{3}s^2\overset{Mitrinovic}{\geq}\frac{16F}{3}\cdot 3\sqrt{3}sr=16\sqrt{3}F^2$$

Therefore,

$$(s_a+s_b)c^3+(s_b+s_c)a^3+(s_c+s_a)b^3\geq 16\sqrt{3}F^2$$

Solution 2 by Marian Ursărescu-Romania

Because $s_a\geq h_a(and\ analogs)$ we must to show that:

$$\sum_{cyc}(h_a+h_b)c^3\geq 16\sqrt{3}F^2;(1)$$

$$\sum_{cyc}(h_a+h_b)c^3\geq\sum_{cyc}2\sqrt{h_ah_b}c^3\geq 3\sqrt[3]{8h_ah_bh_c\cdot a^3b^3c^3}=6abc\sqrt[3]{h_ah_bh_c};(2)$$

$$h_a h_b h_c = \frac{2s^2r^2}{R} \overset{Coşniţă-Turtoiu}{\gtreqless} \frac{27Rr\cdot r^2}{R} = 27r^3; (3)$$

From $(2),(3)$ we must to prove that:

$$\sum_{cyc}(h_a+h_b)c^3 \ge 18r\cdot abc = 18\cdot 4Rr^2s; (4)$$

From $(1),(4)$ we must show that: $18\cdot 4Rr^2s \ge 16\sqrt{3}r^2s^2 \Leftrightarrow$

$$9R \ge 2\sqrt{3}s \Leftrightarrow s \le \frac{3\sqrt{3}}{2}R\ (Mitrinovic)$$

2304. In ΔABC the following relationship holds:

$$\frac{9r}{2} \le \sum_{cyc}\frac{a}{b+c}h_a \le \frac{9R}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Mohammed Diai-Rabat-Morocco

$$\frac{9r}{2} \le \sum_{cyc}\frac{a}{b+c}h_a \le \frac{9R}{4}; (1)$$

$$\sum_{cyc}\frac{a}{b+c}h_a = \sum_{cyc}\frac{a}{b+c}\cdot\frac{2F}{a} = 2F\sum_{cyc}\frac{1}{b+c} = 2F\sum_{cyc}\frac{1}{2s-a}$$

By Cauchy-Schwarz inequality:

$$\sum_{cyc}\frac{1}{2s-a} \ge \frac{9}{\sum(2s-a)} = \frac{9}{4s} \to 4F\sum_{cyc}\frac{1}{2s-a} \ge \frac{18sr}{4s} = \frac{9r}{2}; (2)$$

By AM-GM inequality:

$$\sum_{cyc}\frac{1}{b+c} \le \frac{1}{2}\sum_{cyc}\frac{1}{\sqrt{bc}} = \frac{\sum\sqrt{a}}{2\sqrt{4RF}}; \sum_{cyc}\sqrt{a} \le \sqrt{3\sum_{cyc}a} = \sqrt{6s}$$

$$2F\sum_{cyc}\frac{1}{b+c} \le \frac{s}{2}\sqrt{\frac{6r}{R}} \le \frac{\sqrt{2}}{2}s\ (Euler)$$

$$2F\sum_{cyc}\frac{1}{b+c} \le \frac{9R}{4}(Mitrinovic); (3)$$

From $(1),(2),(3)$ it follows that:

$$\frac{9r}{2}\le\sum_{cyc}\frac{a}{b+c}h_a\le\frac{9R}{4}$$

2305. In ΔABC the following relationship holds:

$$\frac{(a+b)^3}{(a^3+b^3)c^2}+\frac{(b+c)^3}{(b^3+c^3)a^2}+\frac{(c+a)^3}{(c^3+a^3)b^2}\le\frac{1}{r^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$\frac{(a+b)^3}{a^3+b^3}\le 4\Leftrightarrow\frac{(a+b)^3}{(a+b)(a^2-ab+b^2)}\le 4\Leftrightarrow\frac{(a+b)^2}{a^2-ab+b^2}\le 4$$

$\Leftrightarrow 3a^2-6ab+3b^2\ge 0\Leftrightarrow 3(a-b)^2\ge 0$, which is true. Hence,

$$\sum_{cyc}\frac{(a+b)^3}{(a^3+b^3)c^2}\le 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right);(1)$$

From (1) we must show that: $4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\le\frac{1}{r^2}\Leftrightarrow$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4r^2};(2)\Leftrightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\le\frac{1}{4}\cdot\frac{s}{(s-a)(s-b)(s-c)};(3)$$

Now, let $s-a=x, s-b=y, s-c=z\Rightarrow x+y+z=s$ and

$$a=y+z, b=z+x, c=x+y;(4)$$

From (3),(4) we must show that:

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\le\frac{1}{4}\cdot\frac{x+yz}{xyz}\Leftrightarrow$$

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\le\frac{1}{4}\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right);(5)$$

But: $(x+y)^2\ge 4xy\Rightarrow\frac{1}{(x+y)^2}\le\frac{1}{4xy}\Rightarrow\sum_{cyc}\frac{1}{(x+y)^2}\le\frac{1}{4}\sum_{cyc}\frac{1}{xy}\Rightarrow(5)$ is true.

2306. In ΔABC the following relationship holds:

$$s^2\ge 3\sqrt{3}F+\frac{1}{6}(|a-b|^2+|b-c|^2+|c-a|^2)$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Adrian Popa-Romania

Applying Hadwinger-Finsler: $a^2+b^2+c^2\geq 4\sqrt{3}F+(a-b)^2+(b-c)^2+(c-a)^2$

$$\rightarrow a^2+b^2+c^2\geq 4\sqrt{3}F+2a^2+2b^2+2c^2-2ab-2bc-2ca$$

$$\rightarrow 2ab+2bc+2ca-a^2-b^2-c^2\geq 4\sqrt{3}F|:\frac{3}{4}$$

$$\rightarrow \frac{6ab+6bc+6ca-3a^2-3b^2-3c^2}{4}\geq 3\sqrt{3}F$$

Now, we must to prove that:

$$\frac{a^2+b^2+c^2+2ab+2bc+2ca}{4}\geq\frac{6ab+6bc+6ca-3a^2-3b^2-3c^2}{4}+$$

$$+\frac{2a^2+2b^2+2c^2-2ab-2bc-2ca}{6}$$

$$\leftrightarrow 8(a^2+b^2+c^2)\geq 8(ab+bc+ca)$$

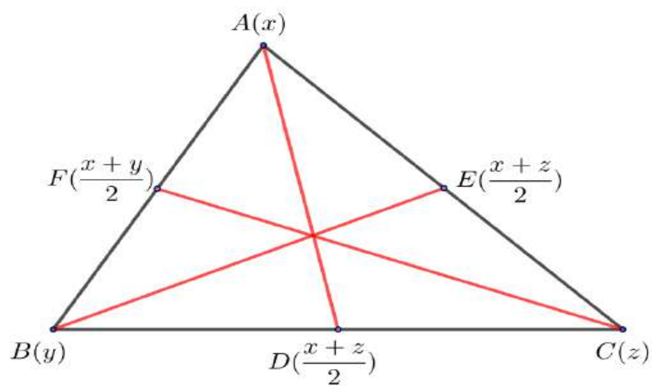
$$\leftrightarrow a^2+b^2+c^2\geq ab+bc+ca \text{ true } \forall a,b,c>0.$$

2307. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{bm_b+cm_c-am_a}{\sqrt{bc\cdot m_bm_c}}\leq 3$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let x,y,z be the complexe coordinates of points A,B and C respectively.

We have: $am_a=|y-z|\left|\frac{y+z}{2}-x\right|\rightarrow am_a=\frac{1}{2}|y^2-z^2-2xy+2xz|$

Similarly, $bm_b=\frac{1}{2}|z^2-x^2-2zy+2yx|$ **and** $cm_c=\frac{1}{2}|y^2-x^2-2zy+2xz|\rightarrow$

$$am_a+bm_b=\frac{1}{2}|y^2-z^2-2xy+2xz|+\frac{1}{2}|z^2-x^2-2zy+2yx|\ge$$

$$\ge\frac{1}{2}|y^2-x^2-2zy+2xz|=cm_c\to$$

$$am_a+bm_b\ge cm_c(and\ analogs)$$

am_a, bm_b, cm_c –can be the sides of a triangle. So, it is suffices to prove that:

$$\sum_{cyc}\frac{b+c-a}{\sqrt{bc}}\le 3;(*)$$

$$(*)\Leftrightarrow 2\sum_{cyc}\frac{s-a}{\sqrt{bc}}\le 3\Leftrightarrow\sum_{cyc}\sqrt{\frac{s-a}{bc}}\cdot\sqrt{s-a}\le\frac{3}{2}$$

By CBS inequality:

$$\sum_{cyc}\sqrt{\frac{s-a}{bc}}\cdot\sqrt{s-a}\le\sqrt{\left(\sum_{cyc}\frac{s-a}{bc}\right)\left(\sum_{cyc}(s-a)\right)}=$$

$$=\sqrt{\frac{1}{4Rrs}\left(s\sum_{cyc}a-\sum_{cyc}a^2\right)\left(3s-\sum_{cyc}a\right)}=\sqrt{\frac{4R+r}{2R}}\overset{(1)}{\le}\frac{3}{2}$$

$$(1)\Leftrightarrow R\ge 2r(Euler).$$

Therefore,

$$\sum_{cyc}\frac{bm_b+cm_c-am_a}{\sqrt{bc\cdot m_bm_c}}\le 3$$

2308. **In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:**

$$6\sqrt{3}\sum_{cyc}\frac{a\cdot n_a}{bc}\le\sum_{cyc}\frac{h_a}{n_a}\cdot\sum_{cyc}cot^2\frac{A}{2}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let's prove that: $\sum_{cyc}cot^2\frac{A}{2}\ge\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2$

$$\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2\overset{CBS}{\leq}\left(\sum_{cyc}\frac{n_a^2}{h_a}\right)\left(\sum_{cyc}\frac{1}{h_a}\right)=\frac{1}{r}\sum_{cyc}\frac{s^2-2r_ah_a}{h_a}=\frac{s^2}{r}\sum_{cyc}\frac{1}{h_a}-\frac{2}{r}\sum_{cyc}r_a$$

$$\rightarrow\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2\leq\frac{s^2-8Rr-2r^2}{r^2};(1)$$

$$\sum_{cyc}cot^2\frac{A}{2}=\sum_{cyc}\frac{s(s-a)}{(s-b)(s-c)}=\sum_{cyc}\frac{(s-a)^2}{r^2}=\frac{3s^2-2s\sum a+\sum a^2}{r^2}=$$

$$=\frac{3s^2-4s^2+2(s^2-5Rr-r^2)}{r^2}=\frac{s^2-8Rr-2r^2}{r^2}$$

$$\rightarrow\sum_{cyc}cot^2\frac{A}{2}\overset{(1)}{\geq}\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2$$

$$\sum_{cyc}\frac{h_a}{n_a}\overset{CBS}{\geq}\frac{9}{\sum\frac{n_a}{h_a}}\rightarrow\left(\sum_{cyc}\frac{h_a}{n_a}\right)\left(\sum_{cyc}cot^2\frac{A}{2}\right)\geq 9\sum_{cyc}\frac{n_a}{h_a};(i)$$

If $a\geq b\geq c$: we know that: $n_a^2=s(s-a)+\frac{(b-c)^2}{a}\cdot s\rightarrow\frac{2}{s}\left(a^2n_a^2-b^2n_b^2\right)=$

$$=b^3-a^3+ab^2-a^2b+a^2c+2ac^2-b^2c-2bc^2=$$

$$=(b-a)[a(a-c)+b(b-c)+2(ab-c^2)]\leq 0$$

$\rightarrow an_a\leq bn_b\leq cn_c$ **and** $\frac{1}{bc}\geq\frac{1}{ca}\geq\frac{1}{ab}$. **By Chebyshev's inequality:**

$$\sum_{cyc}\frac{an_a}{bc}\leq\frac{1}{3}\left(\sum_{cyc}an_a\right)\left(\sum_{cyc}\frac{1}{bc}\right)$$

$$\rightarrow 6\sqrt{3}\sum_{cyc}\frac{an_a}{bc}\leq 2\sqrt{3}\left(\sum_{cyc}\frac{n_a}{h_a}\cdot 2sr\right)\cdot\frac{\sum a}{abc}=2\sqrt{3}\cdot\frac{s}{R}\sum_{cyc}\frac{n_a}{h_a}\overset{Mitrinovic}{\leq}9\sum_{cyc}\frac{n_a}{h_a}$$

$$6\sqrt{3}\sum_{cyc}\frac{an_a}{bc}\overset{(i)}{\leq}\left(\sum_{cyc}\frac{h_a}{n_a}\right)\left(\sum_{cyc}cot^2\frac{A}{2}\right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum cot^2\frac{A}{2}=\sum\frac{s^2}{r_a^2}=\frac{s^2}{s^4r^2}\left(\left(\sum r_ar_b\right)^2-2r_ar_br_c(r_a+r_b+r_c)\right)$$

$$=\frac{s^4-2rs^2(4R+r)}{s^2r^2}\Rightarrow\sum cot^2\frac{A}{2}\overset{(i)}{=}\frac{s^2-8Rr-2r^2}{r^2}$$

$$Stewart's\ theorem\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\ \therefore\ \sum\frac{n_a}{h_a}$$

$$=\sum\left(\frac{n_a}{\sqrt{h_a}}\right)\left(\frac{1}{\sqrt{h_a}}\right)\overset{CBS}{\leq}\sqrt{\sum\frac{n_a^2}{h_a}}\sqrt{\sum\frac{1}{h_a}}=\sqrt{\sum\frac{s^2-2h_ar_a}{h_a}}\sqrt{\frac{1}{r}}$$

$$=\sqrt{\frac{\frac{s^2(2s)}{2rs}-2(4R+r)}{r}}\Rightarrow\sum\frac{n_a}{h_a}\leq\frac{\sqrt{s^2-8Rr-2r^2}}{r}\overset{via\ (i)}{=}\sqrt{\sum\cot^2\frac{A}{2}}$$

$$\Rightarrow\left(\sum\frac{h_a}{n_a}\right)\sum\cot^2\frac{A}{2}\overset{(ii)}{\geq}\left(\sum\frac{h_a}{n_a}\right)\left(\sum\frac{n_a}{h_a}\right)^2$$

$$6\sqrt{3}\sum\frac{an_a}{bc}=6\sqrt{3}\sum\frac{n_a\sin A}{h_a}\overset{CBS}{\leq}6\sqrt{3}\sqrt{\sum\left(\frac{n_a}{h_a}\right)^2}\sqrt{\sum\sin^2A}$$

$$=6\sqrt{3}\sqrt{\sum\left(\frac{n_a}{h_a}\right)^2}\sqrt{\frac{\sum a^2}{4R^2}}\overset{Leibnitz}{\leq}6\sqrt{3}\sqrt{\sum\left(\frac{n_a}{h_a}\right)^2}\sqrt{\frac{9R^2}{4R^2}}$$

$$\therefore 6\sqrt{3}\sum\frac{an_a}{bc}\overset{(iii)}{\leq}9\sqrt{3}\sqrt{\sum\left(\frac{n_a}{h_a}\right)^2}\ \therefore (i),(ii)\Rightarrow it\ suffices\ to\ prove$$

$$:\left(\sum\frac{n_a}{h_a}\right)^2\left(\sum\frac{h_a}{n_a}\right)\geq 9\sqrt{3}\sqrt{\sum\left(\frac{n_a}{h_a}\right)^2}$$

$$\Leftrightarrow\left(\sum x\right)^4\left(\sum\frac{1}{x}\right)^2\geq 243\left(\sum x^2\right)\left(where\ x=\frac{n_a}{h_a},y=\frac{n_b}{h_b},z=\frac{n_c}{h_c}\right)$$

$$\Leftrightarrow\left(\sum x\right)^4\left(\sum xy\right)^2\overset{(a)}{\geq}243x^2y^2z^2\left(\sum x^2\right)and\ it\ suffices\ to\ prove\ (a)\forall\ x,y,z>0$$

$Let\ y+z=a',z+x=b',x+y=c'\ and\ \because a'+b'>c',b'+c'>a',a'+b'>c'$

$\therefore a',b',c'$ *are sides of a triangle*

$$with\ semiperimeter, circumradius\ and\ inradius = p, m, n\ respectively\ (say)$$

$$\therefore 2\sum x = 2p \therefore x = p - a', y = p - b', z = p - c'$$

$$and \sum xy = \sum (p-a')(p-b') = 4mn + n^2 and \sum x^2 = (p^2 - 2pa' + (a')^2)$$

$$= 3p^2 - 4p^2 + 2(p^2 - 4mn - n^2)$$

$$= p^2 - 8mn - 2n^2 \therefore (a) \Leftrightarrow p^4n^2(4m+n)^2 \ge 243n^4p^2(p^2 - 8mn - 2n^2)$$

$$\Leftrightarrow p^2(4m+n)^2 \overset{(1)}{\geq} 243n^2(p^2 - 8mn - 2n^2)$$

$$Now, p^2(4m+n)^2 \overset{Trucht}{\geq} 3p^4 \overset{Gerretsen}{\geq} 3(16mn - 5n^2)p^2 \overset{?}{\geq} 243n^2(p^2 - 8mn - 2n^2)$$

$$\Leftrightarrow (8m - 43n)p^2 + 81n^2(4m+n) \underset{(2)}{\overset{?}{\geq}} 0$$

$\boxed{Case\ 1}$ $8m - 43n \ge 0$ *and then*, *LHS of* $(2) > 0 \Rightarrow (2) \Rightarrow (1)$ *is true*

$$\therefore \left(\sum x\right)^4 \left(\sum xy\right)^2 > 243x^2y^2z^2\left(\sum x^2\right)$$

$\boxed{Case\ 2}$ $8m - 43n < 0$ *and then*, $43n - 8m > 0$ *and* $\because (2) \Leftrightarrow (43n - 8m)p^2$

$\le 81n^2(4m+n) \therefore$ *it suffices to prove* :

$$(43n - 8m)(4m^2 + 4mn + 3n^2)$$

$$\le 81n^2(4m+n)\left(\because (43n - 8m)p^2 \overset{Gerretsen}{\leq} (43n - 8m)(4m^2 + 4mn + 3n^2)\right)$$

$$\Leftrightarrow 8t^3 - 35t^2 + 44t - 12 \ge 0 \left(where\ t = \frac{m}{n} \overset{Euler}{\geq} 2\right) \Leftrightarrow (8t-3)(t-2)^2 \ge 0 \to true$$

$\because t \ge 2 \Rightarrow (2) \Rightarrow (1)$ *is true*

$$\therefore \left(\sum x\right)^4 \left(\sum xy\right)^2$$

$$\ge 243x^2y^2z^2\left(\sum x^2\right) and\ combining\ cases\ (1), (2), \left(\sum x\right)^4 \left(\sum xy\right)^2$$

$$\ge 243x^2y^2z^2\left(\sum x^2\right) \forall\, x, y, z > 0$$

$$\Rightarrow (a)\ is\ true\ \forall\, x, y, z > 0 \therefore 6\sqrt{3}\sum \frac{an_a}{bc} \le \left(\sum \frac{h_a}{n_a}\right)\sum cot^2\frac{A}{2}\ (QED)$$

2309. In ΔABC the following relationship holds:

$$\frac{27r^2}{2} \le \sum_{cyc} \frac{a}{b+c} r_b r_c \le \frac{27Rr}{4}$$

Proposed by Marin Chiciu-Romania

Solution by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{a}{b+c} r_b r_c = \sum_{cyc} \frac{a}{b+c} \cdot \frac{s^2}{(s-b)(s-c)} = s \sum_{cyc} \frac{a(s-a)}{b+c}$$

$$= s^2 \sum_{cyc} \frac{a}{b+c} - s \sum_{cyc} \frac{a^2}{b+c}; (1)$$

$$\sum_{cyc} \frac{a}{b+c} = \frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr}; \quad \sum_{cyc} \frac{a^2}{b+c} = \frac{2s(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr}; (2)$$

From (1), (2) it follows that:

$$\sum_{cyc} \frac{a}{b+c} r_b r_c = \frac{2s^2(2r^2 + 3Rr)}{s^2 + r^2 + 2Rr}$$

For LHS, we must to show that:

$$\frac{27r^2}{2} \le \frac{2s^2(2r^2 + 3Rr)}{s^2 + r^2 + 2Rr} \Leftrightarrow 4s^2(3R + 2r) \ge 27r(s^2 + r^2 + 2Rr) \Leftrightarrow$$

$$s^2(12R + 8r - 27r) \ge 27r(r^2 + 2Rr) \Leftrightarrow$$

$s^2(12R - 19r) \ge 27r^2(2R + r)$, but $s^2 \ge 27r^2 (Mitrinovic) \rightarrow$

$12R - 19r \ge 2R + r \Leftrightarrow R \ge 2r (Euler)$. For RHS, we must to show that:

$$\frac{2s^2(2r^2 + 3Rr)}{s^2 + r^2 + 2Rr} \le \frac{7Rr}{4} \Leftrightarrow 8s^2(3R + 2r) \le 27R(s^2 + r^2 + 2Rr)$$

$\Leftrightarrow s^2(16r - 3R) \le 27Rr(2R + r)$ but $s^2 \le \frac{27R^2}{4} (Mitrinovic)$

$$16Rr - 3R^2 \le 8Rr + 4r^2 \Leftrightarrow 3R^2 - 8Rr + 4r^2 \ge 0 \Leftrightarrow$$

$(R - 2r)(3R - 2r) \ge 0$, true from $R \ge 2r (Euler)$.

2310. In ΔABC the following relationship holds:

$$\frac{12r^2}{R^2} \le \frac{h_a^2}{r_b r_c} + \frac{h_b^2}{r_c r_a} + \frac{h_c^2}{r_a r_b} \le 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

For the left side, applying AM-GM inequality:

$$\frac{h_a^2}{r_br_c}+\frac{h_b^2}{r_cr_a}+\frac{h_c^2}{r_ar_b}\geq 3\sqrt[3]{\frac{h_a^2}{r_br_c}\cdot\frac{h_b^2}{r_cr_a}\cdot\frac{h_c^2}{r_ar_b}}$$

We must to prove that:

$$3\sqrt[3]{\left(\frac{h_ah_bh_c}{r_ar_br_c}\right)^2}\geq\frac{12r^2}{R^2}\Leftrightarrow\sqrt[3]{\frac{h_ah_bh_c}{r_ar_br_c}}\geq\frac{2r}{R}\Leftrightarrow\frac{h_ah_bh_c}{r_ar_br_c}\geq\frac{8r^3}{R^3};(1)$$

But: $h_ah_bh_c=\frac{2s^2r^2}{R};(2)$ **and** $r_ar_br_c=s^2r;(3)$

From (1),(2),(3) we must show that:

$$\frac{2r}{R}\geq\frac{8r^3}{R^3}\Leftrightarrow 1\geq\frac{4r^2}{R^2}\Leftrightarrow R^2\geq 4r^2\Leftrightarrow R\geq 2r(Euler).$$

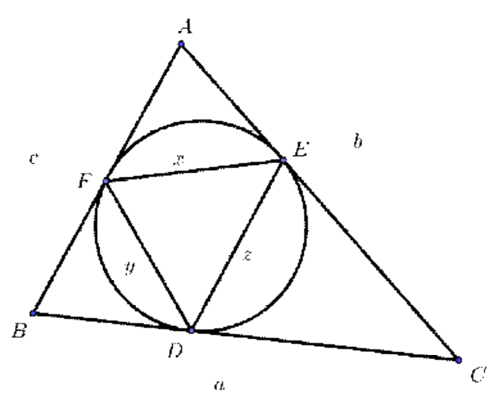
For the right side:

$$\frac{h_a^2}{r_br_c}=\frac{\frac{4F^2}{a^2}}{\frac{F^2}{(s-b)(s-c)}}=\frac{4(s-b)(s-c)}{a^2};(4)$$

But: $\sqrt{(s-b)(s-c)}\leq\frac{s-b+s-c}{2}=\frac{a}{2}\Rightarrow(s-b)(s-c)\leq\frac{a^2}{4};(5)$

From (4),(5) it follows that: $\frac{h_a^2}{r_br_c}\leq 1\Rightarrow\sum_{cyc}\frac{h_a^2}{r_br_c}\leq 3.$

2311.



$$\left(\frac{a}{x}\right)^{n+1}+\left(\frac{b}{y}\right)^{n+1}+\left(\frac{c}{z}\right)^{n+1}\geq 3\cdot 2^{n+1},\qquad n\in\mathbb{N}$$

Proposed by Florică Anastase-Romania

Solution 1 by Marian Ursărescu-Romania

$$\left(\frac{a}{x}\right)^{n+1}+\left(\frac{b}{y}\right)^{n+1}+\left(\frac{c}{z}\right)^{n+1}\ge 3\cdot\sqrt[3]{\left(\frac{abc}{xyz}\right)^{n+1}}$$

We must to prove that:

$$3\cdot\sqrt[3]{\left(\frac{abc}{xyz}\right)^{n+1}}\ge 3\cdot 2^{n+1}\Leftrightarrow\left(\frac{abc}{xyz}\right)^{n+1}\ge 8^{n+1}\Leftrightarrow\frac{abc}{xyz}\ge 8;\quad(1)$$

$$\mu(\angle EDF)=\frac{\pi}{2}-\frac{A}{2}$$

In ΔDEF **we have:** $x=2r\cdot sin\left(\frac{\pi}{2}-\frac{A}{2}\right)=\frac{2r}{cdot}cos\frac{A}{2}$ **and** $r=4R\cdot sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$

$$x=8R\cdot sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}cos\frac{A}{2}=4R\cdot sinAsin\frac{B}{2}sin\frac{C}{2}=2a\cdot sin\frac{B}{2}sin\frac{C}{2}\to$$

$$\frac{x}{a}=\frac{a}{2a\cdot sin\frac{B}{2}sin\frac{C}{2}}=\frac{1}{2sin\frac{B}{2}sin\frac{C}{2}};(and\ analogs);(2)$$

From $(1),(2)$ **we have:**

$$\frac{1}{8sin^2\frac{A}{2}sin^2\frac{B}{2}sin^2\frac{C}{2}}\ge 8\Leftrightarrow sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\le\frac{1}{8}$$

Which is true because:

$$sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}\le\frac{2}{2\sqrt{bc}}\to sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\le\frac{abc}{8abc}=\frac{1}{8}$$

Solution 2 by Marian Dincă-Romania

$$FE=x=\sqrt{2r^2-2r^2cos(\pi-A)}=\sqrt{2r^2(1+cosA)}=2rcos\frac{A}{2}$$

$$FD=2rcos\frac{B}{2},DE=2rcos\frac{C}{2}$$

$$\frac{a}{x}=\frac{2RsinA}{2rcos\frac{A}{2}}=2\frac{R}{r}sin\frac{A}{2}=\frac{2Rsin\frac{A}{2}}{4Rsin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}}=\frac{1}{2sin\frac{B}{2}sin\frac{C}{2}}$$

$$\sum_{cyc}\left(\frac{a}{x}\right)^n=\sum_{cyc}\left(\frac{1}{2\sin\frac{B}{2}\sin\frac{C}{2}}\right)^n=\frac{1}{2^n}\sum_{cyc}\left(\frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}\right)^n\geq$$

$$\geq\frac{1}{2^n}\cdot\sqrt[3]{\frac{1}{\prod\left(\sin\frac{B}{2}\sin\frac{C}{2}\right)^n}}=\frac{3}{2^n}\cdot\frac{1}{\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\frac{2n}{3}}}\geq\frac{3}{2^n}\cdot\frac{1}{\left(\frac{1}{8}\right)^{\frac{2n}{3}}}=3\cdot 2^n$$

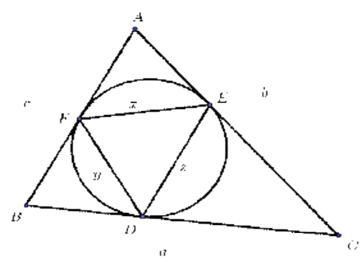
Because: $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$

Solution 3 by proposer

We have: $\mu(\angle FBD)=\frac{\pi}{2}-\frac{B}{2},\mu(\angle EDC)=\frac{\pi}{2}-\frac{C}{2}$, so

$$\mu(\angle EDF)=\frac{\pi}{2}-\frac{A}{2}$$

$$EF=2r\cdot\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)$$



Let us denote: $\alpha=\frac{\pi}{2}-\frac{A}{2},\beta=\frac{\pi}{2}-\frac{B}{2},\gamma=\frac{\pi}{2}-\frac{C}{2}$ and we have: $\alpha+\beta+\gamma=\pi;\ \alpha,\beta,\gamma\in\left(0,\frac{\pi}{2}\right)$

Hence,

$$\frac{a}{x}=\frac{2R\sin A}{2r\cos\frac{A}{2}}=\frac{2R\sin\frac{A}{2}\cos\frac{A}{2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}}=\frac{1}{2ain\frac{B}{2}\sin\frac{C}{2}}$$

$$\sum_{cyc}\frac{a}{x}=\sum_{cyc}\frac{1}{2\sin\frac{B}{2}\sin\frac{C}{2}}\geq\frac{3}{2}\cdot\sqrt{\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}}$$

$$\because\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}\Leftrightarrow 2r\leq R(Euler)\Rightarrow\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\geq 8\Rightarrow$$

$$\sum_{cyc}\frac{a}{x}\geq 6;(1)$$

Now, we want to prove that:

$$\sum_{cyc}\frac{x}{a}\leq\frac{3}{2};(2)\Leftrightarrow\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{3}{4}$$

$$IA=\frac{r}{\sin\frac{A}{2}}=\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}=4R\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{3}\left(\sum_{cyc}\sin\frac{A}{2}\right)^2$$

It is enough to prove that: $\sum_{cyc}\sin\frac{A}{2}\le\frac{3}{2};(3)$

$$\sum_{cyc}\sin\frac{A}{2}=\sum_{cyc}\cos\alpha=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+1-2\sin^2\frac{\gamma}{2}=$$

$$=\frac{3}{2}+2\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}-2\sin^2\frac{\gamma}{2}-\frac{1}{2}=\frac{3}{2}-\frac{1}{2}\left(4\sin^2\frac{\gamma}{2}-4\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}+1\right)=$$

$$=\frac{3}{2}-\frac{1}{2}\left[\left(2\sin\frac{\gamma}{2}-\cos\frac{\alpha-\beta}{2}\right)^2+\sin^2\frac{\alpha-\beta}{2}\right]\le\frac{3}{2}\Rightarrow$$

$$\sum_{cyc}\sin\frac{A}{2}\le\frac{3}{2};(3)\Rightarrow\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{3}{4}\Rightarrow\sum_{cyc}\frac{x}{a}\le\frac{3}{2};(2)$$

From (1),(2) and Holder inequality, it follows that:

$$\sum_{cyc}\left(\frac{a}{x}\right)^{n+1}=\sum_{cyc}\frac{\left(\frac{a}{x}\right)^n}{\frac{x}{a}}\overset{Holder}{\ge}\frac{\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)^n}{3^{n-2}\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)}\ge\frac{6^n}{3^{n-2}\cdot\frac{3}{2}}=\frac{2\cdot2^n\cdot3^n}{3^{n-1}}=3\cdot2^{n+1}$$

2312. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\frac{h_a}{r}|r_b-r_c|\ge2\cos\frac{A}{2}\sqrt{n_a^2-h_a^2}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$r_b-r_c=\frac{sr}{s-b}-\frac{sr}{s-c}=\frac{sr(b-c)}{(s-b)(s-c)}=\frac{sr(b-c)(s-a)}{sr^2}=$$

$$=\frac{(b-c)4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{(b-c)\cos\frac{A}{2}}{\sin\frac{A}{2}}$$

$$\frac{h_a}{r}|r_b - r_c| \ge 2\cos\frac{A}{2}\sqrt{n_a^2 - h_a^2}; (*) \Leftrightarrow \frac{|b-c|}{\sin\frac{A}{2}} \ge \frac{a}{s}\sqrt{n_a^2 - h_a^2} \Leftrightarrow$$

$$\frac{(b-c)^2}{\sin^2\frac{A}{2}} \ge \frac{a^2}{s^2(n_a^2 - h_a^2)}$$

$$n_a^2 = s^2 - 2r_a h_a \to (*) \Leftrightarrow \frac{(b-c)^2}{\sin^2\frac{A}{2}} \ge a^2 - \frac{4sr^2}{s-a} \Leftrightarrow$$

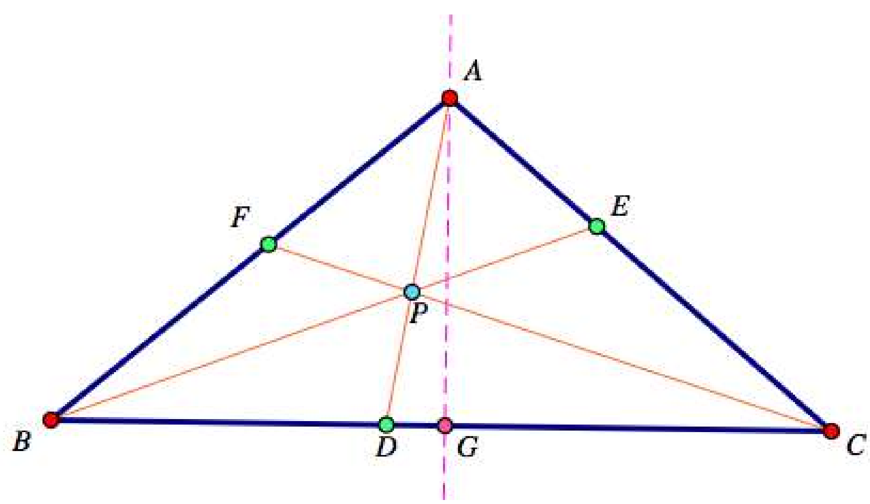
$$\frac{(b-c)^2}{\sin^2\frac{A}{2}} \ge a^2 - \frac{2ra}{s}\left(\frac{2sr}{s-a} + \frac{2sr}{a}\right) = a^2 - \frac{4sr^2}{s-a} \Leftrightarrow$$

$$\frac{(b-c)^2}{\sin^2\frac{A}{2}} \ge a^2 - 4rr_a = a^2 + 4r_b r_c - 4bc \Leftrightarrow$$

$$\frac{(b-c)^2}{\sin^2\frac{A}{2}} + 4bc \ge a^2 + 4s(s-a) = a^2 + (a+b+c)(-a+b+c) = (b+c)^2 \Leftrightarrow$$

$$\frac{(b-c)^2}{\sin^2\frac{A}{2}} \ge (b-c)^2, \sin\frac{A}{2} \le 1$$

Solution 2 by Soumava Chakraborty-Kolkata-India



$$Let\ AD, BE, CF\ be\ the\ Nagel\ cevians\ in\ \Delta\ ABC and\ AG = h_a$$

$$BD = s - c\ and\ BG = c\cos B \Rightarrow DG = c\cos B - s + c\ and\ CD = s - b\ and\ CG = b\cos C \Rightarrow DG = s - b - b\cos C$$

$$\therefore 2DG = c\cos B - s + c + s - b - b\cos C = c - b + c\cos B - b\cos C$$

Here $c > b$ *and proceeding in a similar manner when* $b > c$, *one would obtain* : $2DG = b - c + b\cos C - c\cos B$

$$\therefore 2DG = |b - c + b\cos C - c\cos B| = \left|b - c + b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - c\left(\frac{c^2 + a^2 - b^2}{2ca}\right)\right|$$

$$= \left|b - c + \frac{2(b^2 - c^2)}{2a}\right| = \left|\frac{2s}{a}(b - c)\right|$$

$$\Rightarrow DG^2 = \frac{s^2(b-c)^2}{a^2} \Rightarrow AD^2 - AG^2 = \frac{s^2(b-c)^2}{a^2} \Rightarrow n_a^2 - h_a^2 = \frac{s^2(b-c)^2}{a^2}$$

$$\Rightarrow 2\cos\frac{A}{2}\sqrt{n_a^2 - h_a^2} \overset{(i)}{=} \frac{2s}{a}\cos\frac{A}{2}|b - c|$$

$$\frac{h_a}{r}|r_b - r_c| = \frac{2r^2s^2|b-c|}{ar(s-b)(s-c)} = \frac{2s(s-a)(s-b)(s-c)|b-c|}{ar(s-b)(s-c)}$$

$$= \frac{2s}{a}|b - c|\frac{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{2s}{a}\cos\frac{A}{2}|b - c|.\operatorname{cosec}\frac{A}{2}$$

$$\overset{\operatorname{cosec}\frac{A}{2} > 1}{\geq} \frac{2s}{a}\cos\frac{A}{2}|b - c| \overset{via\ (i)}{=} 2\cos\frac{A}{2}\sqrt{n_a^2 - h_a^2}\ (QED)$$

2313. In ΔABC the following relationship holds:

$$\frac{a^4}{r_a r_b} + \frac{b^4}{r_b r_c} + \frac{c^4}{r_c r_a} \geq \frac{16F}{\sqrt{3}}$$

Proposed by D.M.Bătineţu-Giurgiu, Flaviu Cristian Verde-Romania

Solution 1 by Marian Ursărescu-Romania

We mut show that:

$$\frac{1}{r_a r_b r_c}(a^4 r_c + b^4 r_a + c^4 r_b) \geq \frac{16F}{\sqrt{3}} \Leftrightarrow$$

$$\frac{1}{rs^2}\left(\frac{a^4}{\frac{1}{r_c}} + \frac{b^4}{\frac{1}{r_a}} + \frac{c^4}{\frac{1}{r_b}}\right) \geq \frac{16F}{\sqrt{3}}; (1)$$

$$\frac{a^4}{\frac{1}{r_c}}+\frac{b^4}{\frac{1}{r_a}}+\frac{c^4}{\frac{1}{r_b}}\overset{Holder}{\geq}\frac{(a+b+c)^4}{9\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}=\frac{16s^4}{9\cdot\frac{1}{r}};(2)$$

From $(1),(2)$ we must show:

$$\frac{1}{s^2r}\cdot\frac{16s^4}{9\cdot\frac{1}{r}}\geq\frac{16rs}{\sqrt{3}}\Leftrightarrow s\geq 3\sqrt{3}r(Mitrinovic)$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{a^4}{r_ar_b}\overset{AM-GM}{\geq}3\left(\frac{a^4b^4c^4}{r_a^2r_b^2r_c^2}\right)^{\frac{1}{3}}=3\left(\frac{(4Rrs)^4}{(rs^2)^2}\right)^{\frac{1}{3}}=3\left(\frac{256R^4r^4s^4}{s^4r^2}\right)^{\frac{1}{3}}=3(256R^4r^2)^{\frac{1}{3}}$$

Need to show: $3(256R^4r^2)^{\frac{1}{3}}\geq\frac{16F}{\sqrt{3}}$

$$\rightarrow 27\cdot 256R^4r^2\geq\frac{256\cdot 16\cdot r^3s^3}{3\sqrt{3}}\rightarrow 81\sqrt{3}R^4\geq 16rs^3(true)$$

$$\because R\geq 2r\rightarrow 81\sqrt{3}R^3\cdot R\geq 2r\cdot 8s^3$$

$$\because 3\sqrt{3}R\geq 2s\rightarrow 81\sqrt{3}R^4\geq 16rs^3\rightarrow 81\sqrt{3}R^3\geq 8s^3$$

Solution 3 by Alex Szoros-Romania

$$\sum_{cyc}r_ar_b=\sum_{cyc}\frac{s^2}{(s-a)(s-b)}=\sum_{cyc}s(s-c)=s\sum_{cyc}(s-c)=s^2$$

$$\sum_{cyc}a^2\geq 4F\sqrt{3}(Ionescu-Weitzenbock)$$

$$3\sum_{cyc}a^2\geq\left(\sum_{cyc}a\right)^2=4s^2\rightarrow\sum_{cyc}a^2\geq\frac{4}{3}s^2$$

Applying Bergstrom inequality, we get:

$$\sum_{cyc}\frac{a^4}{r_ar_b}=\sum_{cyc}\frac{(a^2)^2}{r_ar_b}\geq\frac{(\sum a^2)^2}{\sum r_br_c}\geq\frac{4F\sqrt{3}\cdot 4s^2}{3s^2}=\frac{16F}{\sqrt{3}}$$

2314. In acute ΔABC the following relationship holds:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{a^2+b^2+c^2}{2abc}+\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}$$

Proposed by Alex Szoros-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{a}{b^2+c^2}=\sum_{cyc}\frac{a^2}{a(b^2+c^2)}\overset{CBS}{\geq}\frac{(\sum a)^2}{\sum a(b^2+c^2)}=\frac{4s^2}{(\sum a)(\sum ab)-3abc}=$$

$$=\frac{4s^2}{2s(s^2+r^2+4Rr)-3\cdot 4Rrs}$$

$$\rightarrow\sum_{cyc}\frac{a}{b^2+c^2}\geq\frac{2s}{s^2+r^2-2Rr};(1)$$

$$\sum_{cyc}\frac{1}{a}-\frac{\sum a^2}{2abc}=\frac{2(\sum ab)-(\sum a)^2}{2abc}=\frac{2(s^2+r^2+4Rr)-2(s^2-r^2-4Rr)}{8Rrs}=\frac{4R+r}{2Rs}$$

$$\leftrightarrow s^2\leq 8R^2-2Rr-r^2$$

By Gerretsen, we have: $s^2\leq 4R^2+4Rr+3r^2\overset{(*)}{\leq}8R^2-2Rr-r^2\leftrightarrow$

$(R-2r)(4R+2r)\geq 0$, which is true from $R\geq 2r(Euler)$.

Therefore,

$$\sum_{cyc}\frac{1}{a}-\frac{\sum a^2}{2abc}\leq\frac{2s}{s^2+r^2-2Rr}\overset{(1)}{\leq}\sum_{cyc}\frac{a}{b^2+c^2}\rightarrow$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{a^2+b^2+c^2}{2abc}+\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}$$

2315. In ΔABC the following relationship holds:

$$\frac{m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2}{m_am_b+m_bm_c+m_cm_a}\leq\frac{9R^2}{4}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}m_a^2m_b^2=\sum_{cyc}\frac{2b^2+2c^2-a^2}{4}\cdot\frac{2a^2+2c^2-b^2}{4}=$$

$$=\frac{1}{16}\sum_{cyc}(4a^2b^2+4b^2c^2-2b^4+4c^2a^2+4c^4-2b^2c^2-2a^4-2c^2a^2+b^2a^2)$$

$$\rightarrow \sum_{cyc} m_a^2 m_b^2 = \frac{9}{16}\sum_{cyc} a^2b^2 \,;(1)$$

$$\sum_{cyc} m_a m_b \overset{Tereshin}{\geq} \sum_{cyc} \frac{b^2+c^2}{4R}\cdot\frac{c^2+a^2}{4R} = \frac{1}{16R^2}\sum_{cyc}(a^2b^2+b^2c^2+c^2a^2+c^4) =$$

$$= \frac{1}{16}\left(\sum_{cyc} a^4 + 3\sum_{cyc} a^2b^2\right) \overset{\sum x^2 \geq \sum xy}{\geq} \frac{1}{16R^2}\left(\sum_{cyc} a^2b^2 + 3\sum_{cyc} a^2b^2\right) =$$

$$= \frac{1}{4R^2}\sum_{cyc} a^2b^2$$

$$\rightarrow \sum_{cyc} m_a m_b \geq \frac{1}{4R^2}\sum_{cyc} a^2b^2 \overset{(1)}{=} \frac{4}{9R^2}\sum_{cyc} m_a^2 m_b^2$$

Therefore, $\frac{\sum m_a^2 m_b^2}{\sum m_a m_b} \leq \frac{9R^2}{4}$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$m_a \geq \frac{b^2+c^2}{4R}(and\ analogs) \rightarrow \sum_{cyc} m_a m_b \geq \sum_{cyc}\frac{(b^2+c^2)(a^2+c^2)}{16R^2} =$$

$$= \frac{3(a^2b^2+b^2c^2+c^2a^2)+a^4+b^4+c^4)}{16R^2}$$

$$\sum_{cyc} m_a^2 m_b^2 = \sum_{cyc}\left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2a^2+2c^2-b^2}{4}\right) =$$

$$= \frac{1}{16}\sum_{cyc}\left((2b^2+2c^2-a^2)(2a^2+2c^2-b^2)\right) = \frac{9}{16}(a^2b^2+b^2c^2+c^2a^2)$$

$$\frac{\sum m_a^2 m_b^2}{\sum m_a m_b} \leq \frac{\frac{9}{16}(a^2b^2+b^2c^2+c^2a^2)}{\frac{3(a^2b^2+b^2c^2+c^2a^2)+a^4+b^4+c^4)}{16R^2}} = \frac{9R^2\sum a^2b^2}{3\sum a^2b^2+\sum a^4}$$

We need to prove that:

$$\frac{9R^2\sum a^2b^2}{3\sum a^2b^2+\sum a^4} \leq \frac{9R^2}{4} \leftrightarrow \sum a^4 + 3\sum a^2b^2 \geq 4\sum a^2b^2$$

$\leftrightarrow \sum a^4 \geq \sum a^2b^2$, **which is true because:** $\sum x^2 \geq \sum xy, x = a^2, y = b^2, z = c^2$

$$\rightarrow \sum a^4 \geq \sum a^2b^2$$

2316. In ΔABC the following relationship holds:

$$24Rr \le \frac{a^4}{h_b h_c} + \frac{b^4}{h_c h_a} + \frac{c^4}{h_a h_b} \le 4R^2\left(\frac{2R}{r} - 1\right)$$

Proposed by Marin Chirciu-Romania

Solution by Alex Szoros-Romania

$$\frac{a^4}{h_b h_c} = \frac{a^4 bc}{4F^2} = \frac{abc \cdot a^3}{4r^2 s^2} = \frac{4Rrs}{4r^2 s^2} \cdot a^3 \to$$

$$\frac{a^4}{h_b h_c} = \frac{R}{sr} \cdot a^3 \to \sum_{cyc} \frac{a^4}{h_b h_c} = \frac{R}{rs} \sum_{cyc} a^3$$

$$\to \sum_{cyc} \frac{a^4}{h_b h_c} = \frac{2Rs(s^2 - 3r^2 - 6Rr)}{sr} = \frac{2R}{r}(s^2 - 3r^2 - 6Rr)$$

$$24Rr \le \sum_{cyc} \frac{a^4}{h_b h_c} \Leftrightarrow 24Rr \le \frac{2R}{r}(s^2 - 3r^2 - 6Rr) \Leftrightarrow$$

$$12r^2 \le s^2 - 3r^2 - 6Rr \Leftrightarrow 6Rr + 15r^2 \le s^2; (1)$$

$$R \ge 2r(Euler) \to 20r^2 \le 10Rr \to 6Rr + 15r^2 \le 16Rr - 5r^2 \overset{Gerretsen}{\le} s^2$$

$\to 6Rr + 15r^2 \le s^2 \to (1)$ **is true.**

$$\sum_{cyc} \frac{a^4}{h_b h_c} \le 4R^2\left(\frac{2R}{r} - 1\right) \Leftrightarrow \frac{2R}{r}(s^2 - 3r^2 - 6Rr) \le \frac{4R^2}{r}(2R - r)$$

$$\Leftrightarrow s^2 - 3r^2 - 6Rr \le 2R(2R - r) \Leftrightarrow s^2 \le 4R^2 - 2Rr + 6Rr + 3r^2 \Leftrightarrow$$

$$s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen).$$

2317. In ΔABC the following relationship holds:

$$\frac{m_a^2}{b} + \frac{m_b^2}{c} + \frac{m_c^2}{a} \ge 6s\left(\frac{r}{R}\right)^2$$

Proposed by D.M. Bătineţu-Giurgiu, Flaviu Cristian Verde-Romania

Solution by Marian Ursărescu-Romania

$$\frac{m_a^2}{b} + \frac{m_b^2}{c} + \frac{m_c^2}{a} \ge 3\sqrt[3]{\frac{(m_a^2 m_b^2 m_c^2)^2}{abc}}$$

We must show:

$$3\sqrt[3]{\frac{(m_a^2m_b^2m_c^2)^2}{abc}} \ge 6s\left(\frac{r}{R}\right)^2 \Leftrightarrow \frac{(m_a^2m_b^2m_c^2)^2}{abc} \ge 8s^3\left(\frac{r}{R}\right)^6; (1)$$

But $m_a \ge \sqrt{s(s-a)} \to (m_am_bm_c)^2 \ge (sF)^2 = (s^2r)^2 = s^3r^2; (2)$

$$abc = 4Rrs; (3)$$

From $(1), (2), (3)$ we must show that:

$$\frac{s^2r^2}{4Rrs} \ge 8s^3\left(\frac{r}{R}\right)^6 \Leftrightarrow R^5 \ge 32r^5 \Leftrightarrow R \ge 2r(Euler).$$

2318. In ΔABC the following relationship holds:

$$\frac{16}{9}\sum_{cyc} m_bm_c \le \frac{\sum a^4 + 3\sum b^2c^2}{\sum a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{16}{9}\sum_{cyc} m_bm_c \le \frac{\sum a^4 + 3\sum b^2c^2}{\sum a^2}; (*)$$

$$\because \sum_{cyc} a^2 = \frac{4}{3}\sum_{cyc} m_a^2; 16\sum_{cyc} m_a^4 = \sum_{cyc}(2b^2+2c^2-a^2)^2 = 9\sum_{cyc} a^4$$

$$16\sum_{cyc} m_a^2m_b^2 = \sum_{cyc}(2b^2+2c^2-a^2)(2b^2+2c^2-a^2) = 9\sum_{cyc} b^2c^2$$

$$(*) \Leftrightarrow \frac{16}{9}\left(\sum_{cyc} m_bm_c\right)\left(\frac{4}{3}\sum_{cyc} m_a^2\right) \le \frac{16}{9}\left(\sum_{cyc} m_a^4 + 3\sum_{cyc} m_a^2m_b^2\right)$$

$$\Leftrightarrow 4\left(\sum_{cyc} m_am_b\right)\left(\sum_{cyc} m_a^2\right) \le 3\sum_{cyc} m_a^4 + 9\sum_{cyc} m_a^2m_b^2$$

m_a, m_b, m_c –can be the sides of a triangle, so it is suffices to prove that:

$$4\sum_{cyc} ab \cdot \sum_{cyc} a^2 \overset{(**)}{\le} 3\sum_{cyc} a^4 + 9\sum_{cyc} a^2b^2; \forall\Delta$$

$$(**)\Leftrightarrow 4\sum_{cyc}ab\cdot\sum_{cyc}a^2\le 3\left(\sum_{cyc}a^4-2\sum_{cyc}a^2b^2\right)+15\left[\left(\sum_{cyc}ab\right)^2-2abc\sum_{cyc}a\right]$$

$$\Leftrightarrow 30abc\sum_{cyc}a\le 3(-16F^2)+\left(\sum_{cyc}ab\right)\left(15\sum_{cyc}ab-4\sum_{cyc}a^2\right)$$

$$\Leftrightarrow 240Rrs^2+48r^2s^2\le(s^2+r^2+4Rr)(7s^2+23r^2+92Rr)$$

$$f(s)=s^2(120Rr+18r^2-7s^2)\le 23r^2(4R+r)^2$$

$$f'(s)=2s(120Rr+18r^2-14s^2)\overset{Gerretsen}{\lesseqqgtr}2s(-104Rr+88r^2)\overset{Euler}{\lesseqqgtr}0$$

$$\to s^2\overset{Gerretsen}{\lesseqqgtr}r(16R-5r)\to f(s)\le r^2(16R-5r)(8R+53r)\overset{?}{\lesseqqgtr}23r^2(4R+r)^2$$

$$\Leftrightarrow 15R^2-39Rr+18r^2\ge 0\Leftrightarrow(R-2r)(15R-9r)\ge 0$$

Which is true from $R\ge 2r(Euler)\to(**)$ is true.

Therefore,

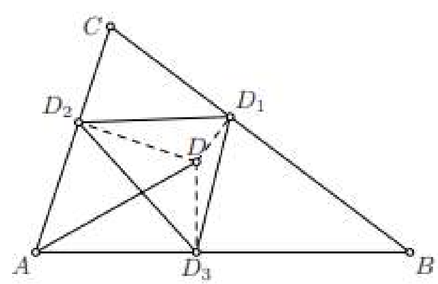
$$\frac{16}{9}\sum_{cyc}m_bm_c\le\frac{\sum a^4+3\sum b^2c^2}{\sum a^2}$$

2319. Let Δ be area of pedal triangle of first Brocard's point in ΔABC. Prove that:

$$2R\Delta\ge rF$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let O,Ω –be the circumcenter, the first Brocard's point of ΔABC.

We know that:

$$O\Omega = R\sqrt{1-4sin^2\omega} = R\sqrt{1-4\cdot\frac{4F^2}{(ab)^2+(bc)^2+(ca)^2}}$$

Using the known identity: $\Delta = \frac{R^2-O\Omega^2}{4R^2}\cdot F$

$$\to \Delta = \frac{1}{4}\left[1-\left(1-\frac{16F^2}{(ab)^2+(bc)^2+(ca)^2}\right)\right]\cdot F = \frac{4F^2}{(ab)^2+(bc)^2+(ca)^2}\cdot F$$

We have:

$$\sum_{cyc}(ab)^2 = \frac{1}{2}\sum_{cyc}[(ab)^2+(ca)^2] \overset{AM-GM}{\geq} \frac{1}{2}\sum_{cyc}2a^2bc = abc\sum_{cyc}a$$

$$\to \sum_{cyc}(ab)^2 \geq 8s^2Rr = \frac{8RF^2}{r} \to \Delta = \frac{4F^2}{(ab)^2+(bc)^2+(ca)^2}\cdot F \leq \frac{r}{2R}\cdot F$$

Therefore: $2R\Delta \leq rF$

2320. In $\Delta ABC, g_a$ –Gergonne cevian, the following relationship holds:

$$\frac{4R}{r} \geq 5 + \sum_{cyc}\frac{m_a^2w_a^2}{g_a^2h_ar_a}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$n_a^2 = s(s-a) + \frac{(b-c)^2}{a}\cdot s;\ \ g_a^2 = s(s-a) - \frac{(b-c)^2(s-a)}{a}$$

$$m_a^2w_a^2 = \frac{(b+c)^2+(b-c)^2-a^2}{4}\cdot\frac{4bcs(s-a)}{(b+c)^2} =$$

$$= s(s-a)\left[bc + \frac{bc[(b-c)^2-a^2]}{(b+c)^2}\right]$$

$$n_ag_a^2 \overset{?}{\geq} m_a^2w_a^2 \Leftrightarrow \left((s-a)+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right) \geq bc + \frac{bc((b-c)^2-a^2)}{(b+c)^2}$$

$$\Leftrightarrow s(s-a) + \frac{(b-c)^2}{a^2}(a^2-(b-c)^2) \geq bc + \frac{bc}{(b+c)^2}((b-c)^2-a^2)$$

$$\Leftrightarrow 4s(s-a)-4bc+4[a^2-(b-c)^2]\left[\frac{(b-c)^2}{a^2}+\frac{bc}{(b+c)^2}\right]\geq 0$$

$$\Leftrightarrow (b-c)^2-a^2+4(a^2-(b-c)^2)\left(\frac{(b-c)^2}{a^2}+\frac{bc}{(b+c)^2}\right)\geq 0$$

$$\Leftrightarrow (a-b+c)(a+b-c)\left(\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}-1\right)\geq 0$$

$$\Leftrightarrow (b-c)^2\left(\frac{4}{a^2}-\frac{1}{(b+c)^2}\right)\geq 0 \Leftrightarrow 2(b+c)\geq a - true \to n_a g_a \geq m_a w_a$$

$$\to 5+\sum_{cyc}\frac{m_a^2 w_a^2}{g_a^2 h_a r_a}\leq 5+\sum_{cyc}\frac{n_a^2}{h_a r_a}=5+\sum_{cyc}\frac{s^2-2r_a h_a}{h_a r_a}=$$

$$=s^2\sum_{cyc}\frac{1}{h_a r_a}-1=\frac{1}{2r^2}\sum_{cyc}a(s-a)-1=\frac{2r(4R+r)}{2r^2}-1=\frac{4R}{r}$$

Therefore,

$$\frac{4R}{r}\geq 5+\sum_{cyc}\frac{m_a^2 w_a^2}{g_a^2 h_a r_a}$$

2321. In ΔABC, n_a –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$\frac{R}{r}\geq\frac{5R-r+\sum\frac{n_a g_a}{h_a}}{h_a+h_b+h_c}\geq\frac{5R-r+\sum\frac{m_a w_a}{h_a}}{h_a+h_b+h_c}\geq\frac{9R}{h_a+h_b+h_c}\geq\frac{2(r_a+r_b+r_c)}{h_a+h_b+h_c}\geq 2$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}h_a=2sr\sum_{cyc}\frac{1}{a}\overset{Leuenberger}{\gtreqless}2sr\cdot\frac{\sqrt{3}}{2r}=\sqrt{3}s=\sqrt{3\sum_{cyc}r_a r_b}\leq\sum_{cyc}r_a$$

$$\to 2\frac{\sum r_a}{\sum h_a}\geq 2$$

$$\sum_{cyc}r_a=4R+r\overset{Euler}{\lesseqgtr}4R+\frac{R}{2}=\frac{9R}{2}$$

$$\because m_a w_a \ge s(s-a) \to \sum_{cyc} \frac{m_a w_a}{h_a} \ge \sum_{cyc} \frac{s(s-a)}{h_a} = \frac{1}{2r}\sum_{cyc} a(s-a) = \frac{2r(4R+r)}{2r}$$

$$\to \frac{5R - r + \sum \frac{m_a w_a}{h_a}}{h_a + h_b + h_c} \ge \frac{9R}{h_a + h_b + h_c}$$

$$n_a^2 = s(s-a) + \frac{(b-c)^2}{a} \cdot s; \;\; g_a^2 = s(s-a) - \frac{(b-c)^2(s-a)}{a}$$

$$m_a^2 w_a^2 = \frac{(b+c)^2 + (b-c)^2 - a^2}{4} \cdot \frac{4bcs(s-a)}{(b+c)^2} =$$

$$= s(s-a)\left[bc + \frac{bc[(b-c)^2 - a^2]}{(b+c)^2}\right]$$

$$n_a g_a^2 \overset{?}{\geq} m_a^2 w_a^2 \Leftrightarrow \left((s-a) + \frac{(b-c)^2}{a}\right)\left(s - \frac{(b-c)^2}{a}\right) \ge bc + \frac{bc((b-c)^2 - a^2)}{(b+c)^2}$$

$$\Leftrightarrow s(s-a) + \frac{(b-c)^2}{a^2}(a^2 - (b-c)^2) \ge bc + \frac{bc}{(b+c)^2}((b-c)^2 - a^2)$$

$$\Leftrightarrow 4s(s-a) - 4bc + 4[a^2 - (b-c)^2]\left[\frac{(b-c)^2}{a^2} + \frac{bc}{(b+c)^2}\right] \ge 0$$

$$\Leftrightarrow (b-c)^2 - a^2 + 4(a^2 - (b-c)^2)\left(\frac{(b-c)^2}{a^2} + \frac{bc}{(b+c)^2}\right) \ge 0$$

$$\Leftrightarrow (a-b+c)(a+b-c)\left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} - 1\right) \ge 0$$

$$\Leftrightarrow (b-c)^2\left(\frac{4}{a^2} - \frac{1}{(b+c)^2}\right) \ge 0 \Leftrightarrow 2(b+c) \ge a - true \to n_a g_a \ge m_a w_a$$

$$\to \frac{5R - r + \sum \frac{n_a g_a}{h_a}}{h_a + h_b + h_c} \ge \frac{5R - r + \sum \frac{m_a w_a}{h_a}}{h_a + h_b + h_c}$$

$$\because \; n_a^2 = s(s-a) + \frac{(b-c)^2}{a} \cdot s; \;\; g_a^2 = s(s-a) - \frac{(b-c)^2(s-a)}{a}$$

$$\to n_a^2 + g_a^2 = 2s(s-a) + (b-c)^2$$

$$\to \sum_{cyc} \frac{n_a g_a}{h_a} \overset{AM-GM}{\leq} \sum_{cyc} \frac{n_a^2 + g_a^2}{2h_a} = \sum_{cyc} \frac{2s(s-a) + (b-c)^2}{\frac{4sr}{a}} =$$

$$= \frac{1}{2r}\sum_{cyc} a(s-a) + \frac{1}{4sr}\left(\sum_{cyc} a^2b - 6abc\right) =$$

$$= \frac{2r(4R+r)}{2r} + \frac{(\sum a)(\sum ab) - 9abc}{4sr} = 4R + r + \frac{\sum ab}{2r} - 9R =$$

$$= r - 5R + \frac{\sum 2Rh_c}{2r} = r - 5R + \frac{R}{r}\sum_{cyc} h_a \rightarrow \frac{R}{r} \geq \frac{5R - r + \sum \frac{n_a g_a}{h_a}}{h_a + h_b + h_c}$$

Therefore,

$$\frac{R}{r} \geq \frac{5R - r + \sum \frac{n_a g_a}{h_a}}{h_a + h_b + h_c} \geq \frac{5R - r + \sum \frac{m_a w_a}{h_a}}{h_a + h_b + h_c} \geq \frac{9R}{h_a + h_b + h_c} \geq \frac{2(r_a + r_b + r_c)}{h_a + h_b + h_c} \geq 2$$

2322. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a}{b+c} h_b h_c \leq \frac{R}{2r} \cdot \sum_{cyc} \frac{a}{b+c} r_b r_c$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{a}{b+c} h_b h_c = \sum_{cyc} \frac{a}{b+c} \cdot \frac{4F^2}{bc} = \frac{4F^2}{abc} \cdot \sum_{cyc} \frac{a^2}{b+c} =$$

$$= \frac{4s^2r^2}{4Rrs} \cdot \sum_{cyc} \frac{a^2}{b+c} = \frac{rs}{R} \cdot \frac{2s(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr} = \frac{r}{R} \cdot \frac{2s^2(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr}; (1)$$

$$\sum_{cyc} \frac{a}{b+c} r_b r_c = \sum_{cyc} \frac{a}{b+c} \cdot \frac{F^2}{(s-b)(s-c)} = s\sum_{cyc} \frac{a(s-a)}{b+c} =$$

$$= s^2 \sum_{cyc} \frac{a}{b+c} - s\sum_{cyc} \frac{a^2}{b+c} = s^2 \cdot \frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} - s \cdot \frac{2s(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr} =$$

$$= \frac{2s^2r(3R + 2r)}{s^2 + r^2 + 2Rr}; (2)$$

From $(1), (2)$ we must to prove:

$$\frac{r}{R} \cdot \frac{2s^2(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr} \leq \frac{R}{2r} \cdot \frac{2s^2r(3R + 2r)}{s^2 + r^2 + 2Rr} \Leftrightarrow$$

$$2r(s^2 - 3r62 - 4Rr) \le R^2(3R + 2r); (3)$$

But $s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen); (4)$

From $(3), (4)$ we must to prove that:

$$2r \cdot 4R^2 \le R^2(3R + 2r) \Leftrightarrow 8r \le 3R + 2r \Leftrightarrow 6r \le 3R \Leftrightarrow 2r \le R(Euler).$$

2323. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \frac{m_a}{a}\right)\left(\sum_{cyc} \frac{m_a^2 + m_b^2}{c^2}\right) \le \frac{27\sqrt{3}}{32}\left(\frac{R}{r}\right)^3$$

Proposed by Kostas Geronikolas-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that:

$$\sum_{cyc} m_a^2 = \frac{3}{4}\sum_{cyc} a^2 \,;\ \sum_{cyc} \frac{1}{a^2} \le \frac{1}{4r^2} (Goldstone)$$

$$\to \sum_{cyc} \frac{m_a}{a} \overset{CBS}{\le} \sqrt{\sum_{cyc} m_a^2 \cdot \sum_{cyc} \frac{1}{a^2}} \overset{CBS}{\le} \sqrt{\frac{3}{4}\left(\sum_{cyc} a^2\right) \cdot \frac{1}{4r^2}} \overset{Leibniz}{\le} \frac{1}{4r}\sqrt{3 \cdot 9R^2} = \frac{3\sqrt{3}R}{4r}; (1)$$

$$a \ge b \ge c \to m_a \le m_b \le m_c \to m_a^2 + m_b^2 \le m_a^2 + m_c^2 \text{ and } \frac{1}{c^2} \ge \frac{1}{b^2} \ge \frac{1}{a^2}$$

$$\to \sum_{cyc} \frac{m_a^2 + m_b^2}{c^2} \overset{Chebyshev's}{\le} \frac{1}{3}\left(\sum_{cyc} (m_a^2 + m_b^2)\right)\left(\sum_{cyc} \frac{1}{c^2}\right) \overset{Goldstone}{\le}$$

$$\le \frac{2}{3}\left(\sum_{cyc} m_a^2\right) \cdot \frac{1}{4r^2} = \frac{1}{6r^2}\left(\frac{3}{4}\sum_{cyc} a^2\right) \overset{Leibniz}{\le} \frac{1}{8r^2} \cdot 9R^2 \to \sum_{cyc} \frac{m_a^2 + m_b^2}{c^2} \le \frac{9R^2}{8r^2}; (2)$$

From $(1), (2)$ it follows that:

$$\left(\sum_{cyc} \frac{m_a}{a}\right)\left(\sum_{cyc} \frac{m_a^2 + m_b^2}{c^2}\right) \le \frac{27\sqrt{3}}{32}\left(\frac{R}{r}\right)^3$$

2324. In ΔABC the following relationship holds:

$$\frac{\left(\sum sin^2\frac{A}{5}\right)\left(\sum sin^2\frac{A}{7}\right)\left(\sum sin^2\frac{A}{9}\right)}{\left(1-cos\frac{2\pi}{15}\right)\left(1-cos\frac{2\pi}{21}\right)\left(1-cos\frac{2\pi}{27}\right)}\ge\frac{27}{8}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $f(x)=sin^2x, x\in\left(0,\frac{\pi}{4}\right)$. We have: $f'(x)=sin2x, f''(x)=2cos2x\ge 0\to$

f –convex.

Since $\frac{A}{5},\frac{A}{7},\frac{A}{9}\in\left(0,\frac{\pi}{4}\right)$ (and analogs). Using Jensen inequality, we have:

$$\sum_{cyc} sin^2\frac{A}{5}\ge\sum_{cyc} f\left(\frac{A}{5}\right)=3f\left(\frac{1}{3}\sum_{cyc}\frac{A}{5}\right)=3f\left(\frac{\pi}{15}\right)=3sin^2\frac{\pi}{15}=\frac{3}{2}\left(1-cos\frac{2\pi}{15}\right)$$

Similarly:

$$\sum_{cyc} sin^2\frac{A}{7}\ge\frac{3}{2}\left(1-cos\frac{2\pi}{21}\right),\sum_{cyc} sin^2\frac{A}{9}\ge\frac{3}{2}\left(1-cos\frac{2\pi}{27}\right)$$

Hence,

$$\left(\sum_{cyc} sin^2\frac{A}{5}\right)\left(\sum_{cyc} sin^2\frac{A}{7}\right)\left(\sum_{cyc} sin^2\frac{A}{9}\right)\ge\left(\frac{3}{2}\right)^3\cdot\left(1-cos\frac{2\pi}{15}\right)\left(1-cos\frac{2\pi}{21}\right)\left(1-cos\frac{2\pi}{27}\right)$$

Therefore,

$$\frac{\left(\sum sin^2\frac{A}{5}\right)\left(\sum sin^2\frac{A}{7}\right)\left(\sum sin^2\frac{A}{9}\right)}{\left(1-cos\frac{2\pi}{15}\right)\left(1-cos\frac{2\pi}{21}\right)\left(1-cos\frac{2\pi}{27}\right)}\ge\frac{27}{8}$$

2325. In ΔABC the following relationship holds:

$$\frac{2r}{h_a}\left(\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)\le\left(\frac{R}{F}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

We have: $r = \frac{F}{s}, s = 2s = a + b + c, h_a = \frac{2F}{a}$, inequality can be write as: $\frac{\frac{2F}{s}}{\frac{2F}{a}} \cdot \frac{b^2+c^2}{4F^2} \le \frac{R^2}{F^2}$

$$\Leftrightarrow \frac{a}{s} \cdot \frac{b^2 + c^2}{4} \le R^2 \Leftrightarrow a(b^2 + c^2) \le 4sR^2; (1)$$

But in any ΔABC we have: $\frac{b}{c} + \frac{c}{b} \le \frac{R}{r}; (2) \Leftrightarrow b^2 + c^2 \le \frac{R}{r} bc \Leftrightarrow$

$a(b^2 + c^2) \le \frac{R}{r} \cdot abc; (3)$ and $abc = 4Rrs; (4)$

From (3),(4) we have:

$a(b^2 + c^2) \le 4sR^2 \Rightarrow (1)$ is true.

$$(2) \Leftrightarrow \frac{(x + y)(y + z)(z + x)}{4xyz} \ge \frac{x + z}{x + y} + \frac{x + y}{x + z} \Leftrightarrow$$

$$\frac{y + z}{4xyz} \ge \frac{1}{(x + y)^2} + \frac{1}{(y + z)^2}; (5)$$

But $\frac{1}{(x+y)^2} \le \frac{1}{4xy}; (6) \Leftrightarrow (x - y)^2 \ge 0$ and $\frac{1}{(x+z)^2} \le \frac{1}{4xz}; (7) \Leftrightarrow (x - z)^2 \ge 0$

From (6),(7) it follows that (5) is true.

2326. In ΔABC the following relationship holds:

$$\frac{(m_a^2 + m_b^2)^2 + (m_b^2 + m_c^2)^2 + (m_c^2 + m_a^2)^2}{m_a^2 + m_b^2 + m_c^2} \le 9R^2$$

Proposed by Marin Chirciu – Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{\sum(m_a^2 + m_b^2)^2}{\sum m_a^2} \le 9R^2$$

$$\sum m_a^2 \overset{Tereshin}{\ge} \sum \left(\frac{b^2 + c^2}{4R}\right)^2 = \frac{1}{16R^2}\left(2\sum a^4 + 2\sum (bc)^2\right)$$

$$= \frac{1}{8R^2}\left(\sum a^4 + \sum (bc)^2\right)$$

We have: $16 \sum m_a^4 = \sum(2b^2 + 2c^2 - a^2)^2 = 9 \sum a^4$ and

$$16 \sum (m_a m_b)^2 = \sum (2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2) = 9 \sum (bc)^2$$

$$\to \sum m_a^2 \ge \frac{1}{8R^2}\left(\frac{16}{9}\sum m_a^4 + \frac{16}{9}\sum (m_a m_b)^2\right) = \frac{1}{9R^2}\left(2\sum m_a^4 + 2\sum (m_a m_b)^2\right)$$

$$= \frac{1}{9R^2}\sum\left(m_a^2 + m_b^2\right)^2$$

Therefore, $\frac{\sum\left(m_a^2+m_b^2\right)^2}{\sum m_a^2} \leq 9R^2$

2327. In ΔABC the following relationship holds:

$$\frac{n_a g_a}{w_a s_a} + \frac{n_b g_b}{w_b s_b} + \frac{n_c g_c}{w_c s_c} \geq 3$$

Proposed by D.M.Bătineţu-Giurgiu, Gabriel Tică-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let us prove: $n_a g_a \geq m_a w_a$

$$n_a^2 = s(s-a) + \frac{(b-c)^2}{a}\cdot s;\ \ g_a^2 = s(s-a) - \frac{(b-c)^2(s-a)}{a}$$

$$m_a^2 w_a^2 = \frac{(b+c)^2 + (b-c)^2 - a^2}{4}\cdot\frac{4bcs(s-a)}{(b+c)^2} =$$

$$= s(s-a)\left[bc + \frac{bc[(b-c)^2 - a^2]}{(b+c)^2}\right]$$

$$4[n_a^2 g_a^2 - m_a^2 w_a^2] =$$

$$= s(s-a)\left(2(s-a) + \frac{2(b-c)^2}{a}\right)\left(2s - \frac{2(b-c)^2}{a}\right) - 4\cdot\frac{(b+c)^2 + (b-c)^2 - a^2}{4}$$

$$\cdot\frac{4bcs(s-a)}{(b+c)^2} =$$

$$= s(s-a)\left[(b+c)^2 - a^2 + 4(b-c)^2 - \frac{4(b-c)^4}{a^2} - 4bc + 4bc\cdot\frac{a^2-(b-c)^2}{(b+c)^2}\right] =$$

$$= s(s-a)\left[(b+c)^2 - 4bc - a^2 + \frac{4(b-c)^2(a^2-(b-c)^2)}{a^2} + 4bc\cdot\frac{a^2-(b-c)^2}{(b+c)^2}\right] =$$

$$= s(s-a)[a^2-(b-c)^2]\left(-1 + \frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2}\right) = 4F^2\left(\frac{4(b-c)^2}{a^2} - \frac{(b-c)^2}{(b+c)^2}\right)$$

$$= 4F^2(b-c)^2\cdot\frac{(2b+2c-a)(2b+2c+a)}{a^2(b+c)^2} \geq 0$$

$$\rightarrow n_a g_a \geq m_a w_a$$

We know that: $m_a \geq s_a (and\ analogs)$ and $n_a g_a \geq s_a w_a (and\ analogs)$

Therefore,

$$\frac{n_a g_a}{w_a s_a} + \frac{n_b g_b}{w_b s_b} + \frac{n_c g_c}{w_c s_c} \geq 3$$

2328. In $\Delta\, ABC$ the following relationship holds:

$$72sR^3r^2 \leq \sum a^6 tan\frac{A}{2} \leq 16sR^2(4R^3 - 23r^3)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Proof: Let\ s-a = x, s-b = y\ and\ s-c = z \therefore \sum(s-a)^2 = \left(\sum x\right)^2 - 2\sum xy$$

$$= s^2 - 2(4Rr + r^2) \Rightarrow \sum x^2 \stackrel{(1)}{=} s^2 - 8Rr - 2r^2$$

$$Now, \sum(s-a)^3 = \left(\sum(s-a)\right)^3 - 3\prod\left((s-a)+(s-b)\right) = s^3 - 12Rrs$$

$$\Rightarrow \sum x^3 \stackrel{(2)}{=} s^3 - 12Rrs\ and$$

$$\sum(s-a)^4 = \sum(s^4 + a^4 - 4a^3s - 4as^3 + 6s^2a^2)$$

$$= 3s^4 + 2(s^2 + 4Rr + r^2)^2 - 32Rrs^2 - 16s^2r^2 - 8s^2(s^2 - 6Rr - 3r^2) - 8s^4$$

$$+12s^2(s^2 - 4Rr - 4r^2) = s^4 - 16Rrs^2 + 2r^2(4R + r)^2$$

$$\Rightarrow \sum x^4 \stackrel{(3)}{=} s^4 - 16Rrs^2 + 2r^2(4R + r)^2$$

$$\sum(s-a)^2(s-b)^2 = \sum x^2y^2 = \left(\sum xy\right)^2 - 2xyz\left(\sum x\right)$$

$$= \left(\sum(s-a)(s-b)\right)^2 - 2r^2s^2 \Rightarrow \sum x^2y^2 \stackrel{(4)}{=} (4Rr + r^2)^2 - 2r^2s^2$$

$$\left(\sum x^2\right)\left(\sum x^3\right) = \sum x^5 + \sum\left(x^2y^2\left(\sum x - z\right)\right)$$

$$= \sum x^5 + \left(\sum x\right)\left(\sum x^2y^2\right) - xyz\sum xy$$

$$\Rightarrow \sum x^5 = \left(\sum x^2\right)\left(\sum x^3\right) - \left(\sum x\right)\left(\sum x^2y^2\right) + xyz\sum xy$$

$$\stackrel{via\ (1),(2),(4)}{=} (s^2 - 8Rr - 2r^2)(s^3 - 12Rrs) - s((4Rr + r^2)^2 - 2r^2s^2) + r^2s(4Rr + r^2)$$

$$\Rightarrow \sum x^5 \stackrel{(5)}{=} s[s^4 - 20Rrs^2 + 20Rr^2(4R + r)]$$

$$\sum a^6 tan\frac{A}{2} = r\sum\frac{(a-s+s)^6}{s-a}$$

$$= r\left[\sum(s-a)^5 + \sum\frac{s^6}{s-a} - 3s^5 - 6s\sum(s-a)^4 + 15s^2\sum(s-a)^3 - 20s^3\sum(s-a)^2\right]$$

$$\overset{via\ (1),(2),(3),(5)}{\cong} r\left[s[s^4 - 20Rrs^2 + 20Rr^2(4R+r)] + \frac{s^6(4Rr+r^2)}{r^2s} - 3s^5 - 6s[s^4 - 16Rrs^2 + 2r^2(4R+r)^2] + 15s^2(s^3 - 12Rrs) - 20s^3(s^2 - 8Rr - 2r^2)\right]$$

$$= \boxed{s[(4R-12r)s^4 + s^2r^2(56R+40r) - r^3(112R^2 + 76Rr + 12r^2)] \overset{(l)}{\cong} \sum a^6 tan\frac{A}{2}}$$

$$(l) \Rightarrow \sum a^6 tan\frac{A}{2} \le 16sR^2(4R^3 - 23r^3)$$

$$\Leftrightarrow (R-3r)s^4 + s^2r^2(14R+10r) - r^3(28R^2 + 19Rr + 3r^2) - 4R^2(4R^3 - 23r^3) \overset{(i)}{\le} 0$$

$Now, Rouche \Rightarrow s^2 - (m-n) \ge 0\ and\ s^2 - (m+n) \le 0, where\ m = 2R^2 + 10Rr - r^2\ and\ n = 2(R-2r)\sqrt{R^2 - 2Rr}$

$$\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \le 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0$$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \le 0$$

$$\Rightarrow (R-3r)s^4 - (R-3r)(4R^2 + 20Rr - 2r^2)s^2 + r(R-3r)(4R+r)^3 \le 0\ and$$

$\therefore$ *in order to prove* (a), *it suffices to prove*:

$$(R-3r)s^4 + s^2r^2(14R+10r) - r^3(28R^2 + 19Rr + 3r^2) - 4R^2(4R^3 - 23r^3) \le (R-3r)s^4 - (R-3r)(4R^2 + 20Rr - 2r^2)s^2 + r(R-3r)(4R+r)^3$$

$$\Leftrightarrow (4R^3 - 8R^2r + 5Rr^2 + r^3)s^2 \overset{(ii)}{\le} r^3(28R^2 + 19Rr + 3r^2) + 4R^2(4R^3 - 23r^3)$$

$Now, LHS\ of\ (ii) \overset{Gerretsen}{\le} (4R^3 - 8R^2r + 5Rr^2 + r^3)(4R^2 + 4Rr + 3r^2) \overset{?}{\le} r^3(28R^2 + 19Rr + 3r^2) + 4R^2(4R^3 - 23r^3)$

$$\Leftrightarrow 16rR^2(R^2-4r^2)\overset{?}{\geq}0 \to \text{true via Euler} \Rightarrow (ii) \Rightarrow (i) \text{ is true}$$

$$\therefore \boxed{\sum a^6\tan\frac{A}{2} \leq 16sR^2(4R^3-23r^3)}$$

$$\text{Again, } (l) \Rightarrow 72sR^3r^2 \leq \sum a^6\tan\frac{A}{2}$$

$$\Leftrightarrow (R-3r)s^4+s^2r^2(14R+10r)-r^3(28R^2+19Rr+3r^2)-18R^3r^2 \geq 0$$

$$\Leftrightarrow (R-2r)s^4-rs^4+s^2r^2(14R+10r)-r^3(28R^2+19Rr+3r^2)-18R^3r^2 \overset{(iii)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (iii) \overset{\text{Gerretsen}}{\geq} \big((R-2r)(16Rr-5r^2)-r(4R^2+4Rr+3r^2)$$
$$+r^2(14R+10r)\big)s^2-r^3(28R^2+19Rr+3r^2)-18R^3r^2$$
$$= r(12R^2-27Rr+17r^2)s^2-r^3(28R^2+19Rr+3r^2)-18R^3r^2$$
$$\overset{\text{Gerretsen}}{\geq} r(12R^2-27Rr+17r^2)(16Rr-5r^2)-r^3(28R^2+19Rr+3r^2)-18R^3r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 87t^3-260t^2+194t-44 \overset{?}{\geq} 0 \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(87t+88)+198\big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (iii) \text{ is true}$$

$$\therefore \boxed{72sR^3r^2 \leq \sum a^6\tan\frac{A}{2}} \ (QED)$$

2329. For an acute ΔABC and a positive integer n, prove that:

$$\left(\sum_{cyc}(\sin A\cdot\sin B\cdot\cos C)^{\frac{1}{n}}\right)^n \leq \frac{3^{n+1}}{8}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

From Holder inequality, we have:

$$\left(\sum_{cyc}(sinA\cdot sinB\cdot cosC)^{\frac{1}{n}}\right)^n\geq\frac{\left(\sum_{cyc}(sinA\cdot sinB\cdot cosC)^{\frac{1}{n}}\right)^n}{3^{n-1}}\Leftrightarrow$$

$$\left(\sum_{cyc}(sinA\cdot sinB\cdot cosC)^{\frac{1}{n}}\right)^n\leq 3^{n-1}\sum_{cyc}sinA\cdot sinB\cdot cosC$$

We must show that:

$$\sum_{cyc}ainA\cdot sinB\cdot cosC\leq\frac{9}{8};(1)$$

Now, $cos2A+cos2B-cos2C=2cos(A+B)cos(A-B)-2cos^2C+1=$

$$=-2cosC(cos(A-B)+cosC)+1=1-2cosC\cdot cos\left(\frac{A-B+C}{2}\right)cos\left(\frac{A-B-C}{2}\right)=$$

$$=1-4sinA\cdot sinB\cdot cosC\Rightarrow$$

$$sinA\cdot sinB\cdot cosC=\frac{1}{4}(1-cos2A-cos2B+cos2C);(2)$$

From (1),(2) we must to show:

$$\frac{3-(cos2A+cos2B+cos2C)}{4}\leq\frac{9}{8};(3)$$

But: $cos2A+cos2B+cos2C=-1-4cosA\cdot cosB\cdot cosC;(4)$

From (3),(4) we must to show:

$1+cosA\cdot cosB\cdot cosC\leq\frac{9}{8}\Leftrightarrow cosA\cdot cosB\cdot cosC\leq\frac{1}{8}$, which is clearly true.

2330. In ΔABC the following relationship holds:

$$9\sqrt{3}r^{\frac{3}{2}}\leq m_a\sqrt{w_a}+m_b\sqrt{w_b}+m_c\sqrt{w_c}\leq\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

For LHS, we have:

$$\sum_{cyc}m_a\sqrt{w_a}\geq 3\sqrt[3]{m_am_bm_c\sqrt{w_aw_bw_c}}$$

We must show that: $m_am_bm_c\sqrt{w_aw_bw_c}\geq 81\sqrt{3}\cdot\sqrt{r^3};(1)$

But $m_a \geq \sqrt{s(s-a)} \rightarrow m_a m_b m_c \geq sF = s^2 r; (2)$

From (1),(2) we must show that: $s^2\sqrt{w_a w_b w_c} \geq 81\sqrt{3}\cdot r^3\sqrt{r}; (3)$

$$w_a \geq h_a \rightarrow \sqrt{w_a w_b w_c} \geq \sqrt{h_a h_b h_c} = \sqrt{\frac{2s^2r^2}{R}} = sr\sqrt{\frac{2}{R}}; (4)$$

From $(3), (4)$ we must to prove that:

$s^2 \cdot s\sqrt{\frac{2}{R}} \geq 81\sqrt{3}\cdot r^2\sqrt{r}; (5)$ and $s \geq 3\sqrt{3}r(Mitrinovic); (6)$

From $(5), (6)$ we must to prove that: $s^2\sqrt{\frac{2}{R}} \geq 27r\sqrt{r}; (7)$

$$s^2 \geq \frac{27Rr}{2}(Cosnita - Turtoiu); (8)$$

From $(7), (8)$ we must to prove: $\sqrt{\frac{R}{2}} \geq \sqrt{r} \Leftrightarrow R \geq 2r(Euler)$.

For RHS, we have:

$$\left(m_a\sqrt{w_a} + m_b\sqrt{w_b} + m_c\sqrt{w_c}\right)^2 \leq \frac{243}{8}R^3; (9)$$

From Cauchy inequality:

$$\left(m_a\sqrt{w_a} + m_b\sqrt{w_b} + m_c\sqrt{w_c}\right)^2 \leq \left(m_a^2 + m_b^2 + m_c^2\right)(w_a + w_b + w_c); (10)$$

From $(9), (10)$ we must to show that:

$$\sum_{cyc} m_a^2 \cdot \sum_{cyc} w_a \leq \frac{243}{8}R^3; (11)$$

$$\sum_{cyc} m_a^2 = \frac{3}{4}\sum_{cyc} a^2 \leq \frac{3}{4}\cdot 9R^2 = \frac{27R^2}{4}; (12)$$

From $(11), (12)$ we must to show:

$\sum_{cyc} w_a \leq \frac{9}{2}R$, which is true, because $w_a \leq m_a$.

Remains to prove that:

$$\sum_{cyc} w_a \leq \sum_{cyc} m_a \leq \frac{9}{2}R \Leftrightarrow \left(\sum_{cyc} m_a\right) \leq \frac{81R^2}{4}$$

But $\left(\sum_{cyc} m_a\right)^2 \overset{CBS}{\leq} 3\sum_{cyc} m_a^2 = \frac{9}{4}\sum_{cyc} a^2 \leq \frac{81}{4}R^2$

2331. In ΔABC the following relationship holds:

$$\frac{r}{R}\left(4-\frac{7r}{4R}-\frac{r^2}{2R^2}\right) \leq \sum_{cyc}\frac{h_a^2}{r_br_c} \leq 2+\frac{5r}{R}+\frac{3r^2}{2R^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-DongThap-Vietnam

- $\sum(b^2c^2(p-b)(p-c)) = abcp(\sum ab) - (\sum a^2b^2)p^2 + \sum b^3c^3 = 4Rrp^2(p^2 +$
4Rr+r2−p4+2r2−8Rrp2+4Rr+r22p2+p6+3r2−12Rrp4+3r4p2+4Rr+r2 3=r2p4−22Rr3−r4p2+4Rr+r23=r2p4−22Rr−r2p2+r4R+r3;

$$\to \sum\frac{1}{(p-a)a^2} = \frac{\sum(b^2c^2(p-b)(p-c))}{(p-a)(p-b)(p-c)a^2b^2c^2}$$
$$= \frac{r^2(p^4-2(2Rr-r^2)p^2+r(4R+r)^3)}{pr^2.(4Rrp)^2}$$
$$= \frac{(p^4-2(2Rr-r^2)p^2+r(4R+r)^3)}{16R^2r^2p^3}$$

Let

$$\Omega = \sum\frac{h_a^2}{r_br_c} = \frac{4S^3}{r_ar_br_c}\sum\frac{1}{(p-a)a^2} = \frac{4p^3r^3}{p^2r}.\frac{(p^4-2(2Rr-r^2)p^2+r(4R+r)^3)}{16R^2r^2p^3} = \frac{p^4-2(2Rr-r^2)p^2+r(4R+r)^3}{4p^2R^2};$$

- $\Omega = \frac{p^4-2(2Rr-r^2)p^2+r(4R+r)^3}{4p^2R^2} \overset{(1)}{\leq} 2+\frac{5r}{R}+\frac{3r^2}{2R^2};$

$$\leftrightarrow \frac{p^4-2(2Rr-r^2)p^2+r(4R+r)^3}{4p^2R^2} \leq \frac{4R^2+10Rr+3r^2}{2R^2};$$
$$\leftrightarrow p^4-2(2Rr-r^2)p^2+r(4R+r)^3 \leq 2p^2(4R^2+10Rr+3r^2);$$
$$\leftrightarrow p^4-4(2R^2+6Rr+r^2)p^2+r(4R+r)^3 \leq 0;$$
$$\leftrightarrow p^4 \leq 4(2R^2+6Rr+r^2)p^2-r(4R+r)^3; \ (2)$$

But:

$$\alpha = 2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)} \leq p^2 \leq \beta$$
$$= 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)};$$
$$\to (p^2-\alpha)(p^2-\beta) \leq 0;$$
$$\to p^4-2(2R^2+10Rr-r^2)p^2+r(4R+r)^3 \leq 0;$$
$$\to p^4 \leq 2(2R^2+10Rr-r^2)p^2-r(4R+r)^3; \ (3)$$

From (2) & (3) we need to prove:

$$2(2R^2+10Rr-r^2)p^2-r(4R+r)^3 \leq 4(2R^2+6Rr+r^2)p^2-r(4R+r)^3;$$
$$\leftrightarrow 2(2R^2+2Rr+3r^2)p^2 \geq 0;$$

(true because: $R, r, p > 0$)
$\rightarrow (2) \rightarrow (1)$ is true.

- $\Omega = \frac{p^4-2(2Rr-r^2)p^2+r(4R+r)^3}{4p^2R^2} \overset{(4)}{\geq} \frac{r}{R}\left(7 - \frac{7r}{4R} - \frac{r^2}{2R^2}\right);$

$$\leftrightarrow \frac{p^4 - 2(2Rr - r^2)p^2 + r(4R + r)^3}{4p^2R^2} \geq \frac{(28rR^2 - 7Rr^2 - 2r^3)}{4R^3};$$

$\leftrightarrow Rp^4 + (-32rR^2 + 9Rr^2 + 2r^3)p^2 + Rr(4R + r)^3 \geq 0;$
$\leftrightarrow Rp^4 - (16rR^2 - 5Rr^2)p^2 - (16rR^2 - 4Rr^2 - 2r^2)p^2 + Rr(4R + r)^3 \geq 0;$
$\leftrightarrow \left(Rp^2 - (16rR^2 - 5Rr^2)\right)p^2 + Rr(4R + r)^3 - (16rR^2 - 4Rr^2 - 2r^2)p^2 \geq 0;$ (5)

Because:

- $p^2 \geq 16Rr - 5r^2 \rightarrow Rp^2 \geq 16rR^2 - 5Rr^2 \rightarrow Rp^2 - (16rR^2 - 5Rr^2) \geq 0;$ (6)
- $Rr(4R + r)^3 - (16rR^2 - 4Rr^2 - 2r^2)p^2 \geq 0;$ (7)

$\leftrightarrow Rr(4R + r)^3 \geq (16rR^2 - 4Rr^2 - 2r^2)p^2;$

But:

$$p^2 \leq R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)} \rightarrow (16rR^2 - 4Rr^2 - 2r^2)p^2 \leq (16rR^2 - 4Rr^2 - 2r^2)\Big(R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}\Big) \overset{(8)}{\leq} Rr(4R + r)^3;$$

(8) $\leftrightarrow (16t^2 - 4t - 2)\left(t^2 + 10t - 1 + 2(t - 2)\sqrt{t^2 - 2t}\right) \leq t(4t + 1)^3; \left(\because t = \frac{R}{r} \geq 2\right)$

$\leftrightarrow (16t^2 - 4t - 2)(t^2 + 10t - 1) + 2(16t^2 - 4t - 2)\left((t - 2)\sqrt{t^2 - 2t}\right) \leq t(4t + 1)^3;$

$\leftrightarrow 2(16t^2 - 4t - 2)(t - 2)\sqrt{t^2 - 2t} \leq t(4t + 1)^3 - (16t^2 - 4t - 2)(t^2 + 10t - 1);$

$\leftrightarrow 2(16t^2 - 4t - 2)(t - 2)\sqrt{t^2 - 2t} \leq (t - 2)(4t + 1)(8t^2 - 12t + 1);$

$\leftrightarrow (t - 2)\left((4t + 1)(8t^2 - 12t + 1) - 2(16t^2 - 4t - 2)\sqrt{t^2 - 2t}\right) \geq 0;$

Since: $t \geq 2 \rightarrow t - 2 \geq 0$. We need to prove that:

$(4t + 1)(8t^2 - 12t + 1) - 2(16t^2 - 4t - 2)\sqrt{t^2 - 2t} > 0;$

$\leftrightarrow (4t + 1)(8t^2 - 12t + 1) > 2(16t^2 - 4t - 2)\sqrt{t^2 - 2t};$

$\leftrightarrow \left((4t + 1)(8t^2 - 12t + 1)\right)^2 > 4(t^2 - 2t)(16t^2 - 4t - 2)^2;$

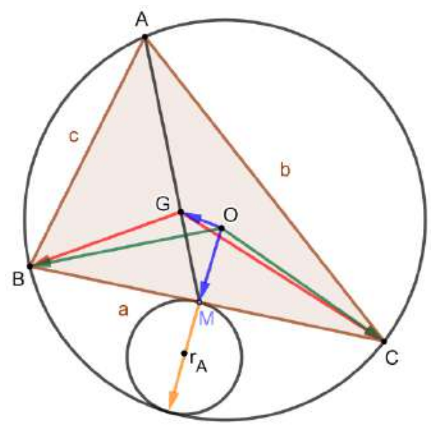
$\leftrightarrow 256t^4 + 256t^3 + 96t^2 + 16t + 1 > 0;$
$\leftrightarrow (4t + 1)^4 > 0;$ (true by $t \geq 2$)
$\rightarrow (7)$ is true $\overset{(7)\&(6)}{\Rightarrow} (5) \rightarrow (4)$ is true. Proved.

2332. In acute $\Delta ABC, r_A$ −radii of circle tangent simultaneous to BC in the middle of BC and circumcircle of ΔABC (internal tangent). If r_B, r_C −are similarly defined then:

$$4\left(\frac{2}{r}-\frac{1}{R}\right) \leq \frac{1}{r_A}+\frac{1}{r_B}+\frac{1}{r_C} \leq \frac{1}{r^2}(5R-4r)$$

Proposed by Mehmet Şahin-Turkiye

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let M be the midpoint of BC, we have: $\mu(OBM) = \frac{\pi}{2} - A \rightarrow$

$$tan(OBM) = tan\left(\frac{\pi}{2}-A\right) = \frac{2OM}{a}$$

$$\rightarrow OM = \frac{a}{2tanA} = \frac{2RsinA}{2tanA} = RcosA$$

We have: $R = 2r_A + OM = 2r_A + RcosA$

$$r_A = R \cdot \frac{1-cosA}{2} = Rsin^2\frac{A}{2}$$

$$\sum_{cyc}\frac{1}{r_A} = \frac{1}{R}\sum_{cyc}\frac{1}{sin^2\frac{A}{2}} = \frac{1}{R}\sum_{cyc}\frac{bc}{(s-b)(s-c)} = \frac{1}{Rr^2s}\sum_{cyc}bc(s-a) = \frac{s\sum bc - 3abc}{Rr^2s} =$$

$$= \frac{(s^2+r^2+4Rr)-12Rr}{Rr^2} = \frac{s^2+r^2-8Rr}{Rr^2}$$

By Gerretsen's inequality, we have:

$$\sum_{cyc}\frac{1}{r_A}=\frac{s^2+r^2-8Rr}{Rr^2}\ge\frac{(16Rr-5r^2)+r^2-8Rr}{Rr^2}=$$

$$=\frac{8Rr-4r^2}{Rr^2}=4\left(\frac{2}{r}-\frac{1}{R}\right)$$

$$\sum_{cyc}\frac{1}{r_A}=\frac{s^2+r^2-8Rr}{Rr^2}\le\frac{(4R^2+4Rr+3r^2)+r^2-8Rr}{Rr^2}=\frac{4R^2-4Rr+4r^2}{Rr^2}\le$$

$$\overset{Euler}{\le}\frac{4R^2-4Rr+R^2}{Rr^2}=\frac{1}{r^2}(5R-4r)$$

Therefore,

$$4\left(\frac{2}{r}-\frac{1}{R}\right)\le\frac{1}{r_A}+\frac{1}{r_B}+\frac{1}{r_C}\le\frac{1}{r^2}(5R-4r)$$

2333. In ΔABC the following relationship holds:

$$\frac{12r^2}{R^2}\le\frac{h_a^2}{r_br_c}+\frac{h_b^2}{r_cr_a}+\frac{h_c^2}{r_ar_b}\le 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

For RHS, using AM-GM:

$$\frac{h_a^2}{r_br_c}+\frac{h_b^2}{r_cr_a}+\frac{h_c^2}{r_ar_b}\ge 3\sqrt[3]{\frac{h_a^2h_b^2h_c^2}{r_a^2r_b^2r_c^2}}$$

We must to prove: $3\sqrt[3]{\frac{h_a^2h_b^2h_c^2}{r_a^2r_b^2r_c^2}}\ge 12\frac{r^2}{R^2}\Leftrightarrow\sqrt[3]{\frac{h_ah_bh_c}{r_ar_br_c}}\ge 2\frac{r}{R}\Leftrightarrow\frac{h_ah_bh_c}{r_ar_br_c}\ge 8\frac{r^3}{R^3};(1)$

But: $h_ah_bh_c=\frac{2s^2r^2}{R};(2)$ and $r_ar_br_c=s^2r;(3)$

From (1),(2),(3) we must show that: $2\frac{r}{R}\ge 8\frac{r^3}{R^3}\Leftrightarrow 1\ge 4\frac{r^2}{R^2}\Leftrightarrow$

$$R^2\ge 2r^2\Leftrightarrow R\ge 2r\ (Euler).$$

For LHS, we have:

$$\frac{h_a^2}{r_br_c}=\frac{\frac{4F^2}{a^2}}{\frac{F^2}{(s-b)(s-c)}}=\frac{4(s-b)(s-c)}{a^2};(4)$$

But: $\sqrt{(s-b)(s-c)} \le \frac{s-b+s-c}{2} = \frac{a}{2} \Rightarrow (s-b)(s-c) \le \frac{a^2}{4}$; (5)

From (4),(5) it follows that: $\frac{h_a^2}{r_b r_c} \le 1 \Rightarrow \sum_{cyc} \frac{h_a^2}{r_b r_c} \le 3$

2334. In ΔABC the following relationship holds:

$$\frac{2r(4R+r)^2}{3R^2 s} \le \frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \le \frac{(4R+r)^2}{3Rs}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{r_a}{a} = \sum_{cyc} \frac{F}{a(s-a)} = sr \sum_{cyc} \frac{1}{a(s-a)} = sr \cdot \frac{s^2 + (4R+r)}{4R+s^2} = \frac{s^2+(4R+r)^2}{4Rs}$$

For RHS, we have:

$$\frac{s^2+(4R+r)^2}{4Rs} \le \frac{(4R+r)^2}{3Rs} \Leftrightarrow 3s^2 + 3(4R+r)^2 \le 4(4R+r)^2$$

$$\Leftrightarrow 3s^2 \le (4R+r)^2 (Doucet)$$

For LHS, we have:

$$\frac{2r(4R+r)^2}{3R^2 s} \le \frac{s^2+(4R+r)^2}{4Rs} \Leftrightarrow 8r(4R+r)^2 \le 3Rs^2 + 3R(4R+r)^2; (1)$$

$$3s^2 \le (4R+r)^2 (Doucet); (2)$$

From (1), (2) we must to prove that:

$$3Rs^2 + 9Rs^2 \ge 8r(4R+r)^2 \Leftrightarrow 12Rs^2 \ge 8r(4R+r)^2 \Leftrightarrow 3Rs^2 \ge 2r(4R+r)^2; (3)$$

Again from $9s^2 \le 3(4R+r)^2 (Doucet); (4)$

From (3), (4) we must to show that: $9Rr(4R+r) \ge 2r(4R+r)^2$

$$\Leftrightarrow 9R \ge 8R + 2r \Leftrightarrow R \ge 2r (Euler).$$

2335. In ΔABC the following relationship holds:

$$\frac{a^3}{m_a} + \frac{b^3}{m_b} + \frac{c^3}{m_c} \ge 12\sqrt{3}Rr$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution 1 by Marian Ursărescu-Romania

$$\frac{a^3}{m_a}+\frac{b^3}{m_b}+\frac{c^3}{m_c}\overset{Holder}{\geq}\frac{(a+b+c)^3}{3(m_a+m_b+m_c)}=\frac{8s^2}{3(m_a+m_b+m_c)}$$

We must show: $\frac{s^2}{3(m_a+m_b+m_c)}\geq 3\sqrt{3}Rr;(1)$

But $m_a+m_b+m_c\leq s\sqrt{3}(Jun\ His\ Huang);(2)$

From $(1),(2)$ we must show that: $\frac{2s^3}{3\sqrt{3}s}\geq 3\sqrt{3}Rr\Leftrightarrow 2s^2\geq 27Rr(Cosnita-Turtoiu)$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}a^2=2m_a^2+\frac{3}{2}a^2\overset{AM-GM}{\geq}2\sqrt{(2m_a^2)\left(\frac{3}{2}a^2\right)}=2\sqrt{3}am_a$$

$$\to am_a\overset{(1)}{\leq}\frac{1}{2\sqrt{3}}\sum_{cyc}a^2\ (and\ analogs)$$

$$\to\sum_{cyc}\frac{a^3}{m_a}=\sum_{cyc}\frac{a^4}{am_a}\overset{CBS}{\geq}\frac{(\sum a^2)^2}{\sum am_a}\overset{(1)}{\geq}\frac{(\sum a^2)^2}{\frac{3}{2\sqrt{3}}\sum a^2}=\frac{2\sqrt{3}}{3}\sum_{cyc}a^2\overset{CBS}{\geq}$$

$$\overset{CBS}{\geq}\frac{2\sqrt{3}}{9}\left(\sum_{cyc}a^2\right)=\frac{8\sqrt{3}}{9}s^2\overset{s^2\geq\frac{27Rr}{2}}{\geq}\frac{8\sqrt{3}}{9}\cdot\frac{27Rr}{2}=12\sqrt{3}Rr$$

Therefore,

$$\frac{a^3}{m_a}+\frac{b^3}{m_b}+\frac{c^3}{m_c}\geq 12\sqrt{3}Rr$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$am_a+bm_b+cm_c\overset{CBS}{\leq}\sqrt{\sum_{cyc}a^2\cdot\sum_{cyc}m_a^2}=\sqrt{\sum_{cyc}a^2\cdot\frac{3}{4}\sum_{cyc}a^2}=\frac{\sqrt{3}}{2}\sum_{cyc}a^2;(1)$$

$$\frac{a^3}{m_a}+\frac{b^3}{m_b}+\frac{c^3}{m_c}=\frac{(a^2)^2}{m_a}+\frac{(b^2)^2}{m_b}+\frac{(c^2)^2}{m_c}\overset{CBS}{\geq}\frac{(a^2+b^2+c^2)^2}{am_a+bm_b+cm_c}\overset{by\ (1)}{\geq}$$

$$\geq\frac{(a^2+b^2+c^2)^2}{\frac{\sqrt{3}}{2}(a^2+b^2+c^2)}=\frac{2\sqrt{3}}{3}(a^2+b^2+c^2)=\frac{4\sqrt{3}}{3}(s^2-4Rr-r^2)\overset{(2)}{\geq}12\sqrt{3}Rr$$

$$(2) \Leftrightarrow s^2 - 4Rr - r^2 \geq 9Rr \Leftrightarrow s^2 \geq 13Rr + r^2$$

But: $s^2 \geq 16Rr - 5r^2 (Gerretsen)$. We need to prove that:

$16Rr - 5r^2 \geq 13Rr + r^2 \Leftrightarrow 3Rr \geq 6r^2 \Leftrightarrow R \geq 2r(Euler) \rightarrow (2)$ is true.

2336. In ΔABC the following relationship holds:

$$9r \leq \sqrt{m_b m_c} + \sqrt{m_c m_a} + \sqrt{m_a m_b} \leq \frac{9R}{2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc} \sqrt{m_b m_c} \overset{AM-GM}{\leq} \sum_{cyc} \frac{mb + m_c}{2} = \sum_{cyc} m_a \overset{CBS}{\leq} \sqrt{(1^2 + 1^2 + 1^2)\left(\sum_{cyc} m_a^2\right)} =$$

$$= \sqrt{3 \cdot \frac{3}{4} \sum_{cyc} a^2} \overset{Leibniz}{\leq} \sqrt{\frac{9}{4} \cdot 9R^2} = \frac{9R}{2}$$

$$\sum_{cyc} \sqrt{m_b m_c} \overset{AM-GM}{\geq} 3\sqrt[3]{m_a m_b m_c} = 3\sqrt[3]{\prod_{cyc} \sqrt{s(s-a)}} = 3\sqrt[3]{sF} = 3\sqrt[3]{s^2 r} \overset{Mitrinovic}{\geq}$$

$$\geq 3\sqrt[3]{\left(3\sqrt{3}r\right)^2 \cdot r} = 9r$$

Solution 2 by Samar Das-India

$$\Omega = \sum_{cyc} \sqrt{m_b m_c} \overset{AM-GM}{\geq} 3\sqrt[3]{m_a m_b m_c} = \frac{3}{4}\sqrt[3]{3\sqrt{3}abc} \overset{abc=4Rrs}{=} \frac{3}{4} \cdot \sqrt{3}\sqrt[3]{4Rrs} \geq$$

$$\geq \frac{3\sqrt{3}}{4} \cdot \sqrt[3]{4 \cdot Rr\sqrt{\frac{27Rr}{2}}} = \frac{3\sqrt{3} \cdot \sqrt[3]{4}}{4} \cdot \sqrt[3]{(Rr)^2 \sqrt{\frac{27}{2}}} = \frac{9r}{2}$$

$$\Omega = \sum_{cyc} \sqrt{m_b m_c} = \sum_{cyc} \sqrt{m_b} \cdot \sqrt{m_c} \leq \sqrt{\sum_{cyc} m_b} \cdot \sqrt{\sum_{cyc} m_c} = \sum_{cyc} m_a =$$

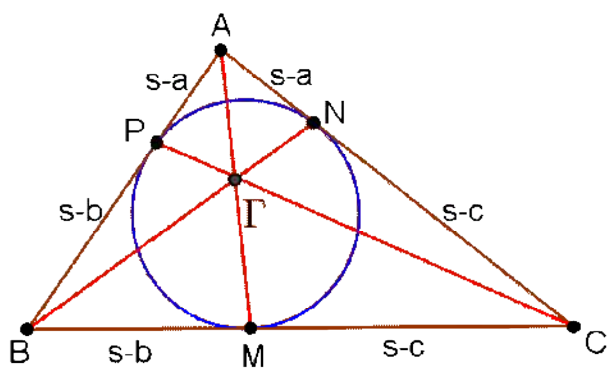
$$=\sum_{cyc}\sqrt{\frac{2b^2+2c^2-a^2}{4}}=\frac{\sqrt{3}}{2}(a+b+c)=s\sqrt{3}\overset{s\le\frac{3\sqrt{3}}{2}R}{\lesssim}\sqrt{3}\cdot\frac{3\sqrt{3}}{2}R=\frac{9R}{2}$$

2337. **ΔMNP – the intouch triangle of ΔABC, Γ – Gergonne's point. Prove that:**

$$\frac{3}{r^2s}\left(\frac{\Gamma M}{\Gamma A}+\frac{\Gamma N}{\Gamma B}+\frac{\Gamma P}{\Gamma C}\right)\le\sum_{cyc}\frac{1}{a}\cdot\sum_{cyc}\frac{1}{(s-a)^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



From Van Aubel's theorem, we have:

$$\frac{\Gamma A}{\Gamma M}=\frac{PA}{PB}+\frac{NA}{NC}=\frac{s-a}{s-b}+\frac{s-a}{s-c}=(s-a)\cdot\frac{(s-b)+(s-c)}{(s-b)(s-c)}=$$

$$=\frac{a(s-a)}{(s-b)(s-c)}=\frac{a(s-a)^2}{(s-a)(s-b)(s-c)}=\frac{a(s-a)^2}{r^2s}$$

$$\to\frac{\Gamma M}{\Gamma A}=\frac{r^2s}{a(s-a)^2}\,(and\ analogs)$$

$$\to\frac{3}{r^2s}\left(\frac{\Gamma M}{\Gamma A}+\frac{\Gamma N}{\Gamma B}+\frac{\Gamma P}{\Gamma C}\right)=\frac{3}{r^2s}\sum_{cyc}\frac{r^2s}{a(s-a)^2}=3\sum_{cyc}\frac{1}{a(s-a)^2}$$

$$a\ge b\ge c\to\frac{1}{a}\le\frac{1}{b}\le\frac{1}{c}\text{ and }\frac{1}{(s-a)^2}\ge\frac{1}{(s-b)^2}\ge\frac{1}{(s-c)^2}$$

$$\rightarrow \frac{3}{r^2 s}\left(\frac{\Gamma M}{\Gamma A}+\frac{\Gamma N}{\Gamma B}+\frac{\Gamma P}{\Gamma C}\right)=3\sum_{cyc}\frac{1}{s(s-a)^2}\overset{Chebyshev's}{\gtrless}\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{(s-a)^2}\right)$$

2338. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$n_a^8+n_b^8+n_c^8 \geq 3^9\cdot r^8$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution 1 by George Florin Şerban-Romania

$$n_a^2=s^2-2r_a h_a \rightarrow \sum_{cyc} n_a^8=\sum_{cyc}(n_a^2)^4 \overset{Holder}{\gtrless} \frac{1}{3^3}\cdot\left(\sum_{cyc} n_a^2\right)^4 \overset{?}{\gtrless} 3^9\cdot r^8$$

$$\left(\sum_{cyc} n_a^2\right)^4 \overset{?}{\gtrless} 3^{12}\cdot r^8 \Leftrightarrow \sum_{cyc} n_a^2 \overset{?}{\gtrless} 3^3\cdot r^2$$

$$\sum_{cyc} n_a^2=\sum_{cyc}(s^2-2r_a h_a)=\sum_{cyc}s^2-2\sum_{cyc}r_a h_a=3s^2-2\sum_{cyc}\frac{2F}{a}\cdot\frac{F}{s-a}=$$

$$=3s^2-4s^2r^2\sum_{cyc}\frac{1}{a(s-a)}=3s^2-4r^2s^2\cdot\frac{s^2+(4R+r)^2}{4Rrs^2}=$$

$$=3s^2-\frac{r}{R}\cdot(s^2+(4R+r)^2)\overset{Gerretsen}{\gtrless}$$

$$\geq 3(16Rr-5r^2)-\frac{r}{R}(4R^2+4R+3r^2+16R^2+8Rr+r^2)=$$

$$=\frac{48R^2r-15Rr^2-20R62\,r-12Rr^2-4r^3}{R}\overset{?}{\gtrless}27r^2$$

$$\Leftrightarrow 28R^2r-27Rr^2-4r^3\geq 27Rr^2 \Leftrightarrow 28R^2r-54Rr^2-4r^3\geq 0\left(\because t=\frac{R}{r}\geq 2\right)\rightarrow$$

$(t-2)(28t+2)\geq 0$, which is true from $t\geq 2$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\because \sum_{cyc}\frac{1}{h_a}=\frac{1}{r}\rightarrow \sum_{cyc}h_a \overset{CBS}{\gtrless} \frac{9}{\sum\frac{1}{h_a}}=9r;(1)$$

$$\sum_{cyc} n_a^8 \overset{n_a \ge h_a}{\ge} \sum_{cyc} h_a^8 \overset{Holder}{\ge} \frac{1}{3^7}\left(\sum_{cyc} h_a\right)^8 \overset{(1)}{\ge} \frac{1}{3^7}\cdot(9r)^8 = 3^9\cdot r^8$$

Therefore,

$$n_a^8 + n_b^8 + n_c^8 \ge 3^9\cdot r^8$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$n_a^8 + n_b^8 + n_c^8 \overset{AM-GM}{\ge} 3\cdot\sqrt[3]{(n_a n_b n_c)^8} \overset{n_a n_b n_c \ge m_a m_b m_c}{\ge} 3\cdot\sqrt[3]{(m_a m_b m_c)^8} \overset{m_a m_b m_c \ge s^2 r}{\ge}$$

$$\ge 3\cdot\sqrt[3]{(s^2 r)^8} \overset{(1)}{\ge} 3^9\cdot r^8$$

$$(1) \Leftrightarrow \sqrt[3]{(s^2 r)^8} \ge (3r)^8 \Leftrightarrow (s^2 r)^8 \ge (3r)^{24} \Leftrightarrow s^2 r \ge (3r)^3 \Leftrightarrow$$

$s^2 \ge 27r^2 \Leftrightarrow s \ge 3\sqrt{3}r(Mitrinovic) \to (1)$ is true.

2339. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \frac{a^3 b}{(a+b)^2}\right)\cdot\left(\sum_{cyc} \frac{a^2}{b^2+c^2}\right) \le \frac{9R^2}{8r}\cdot(2R-r)$$

Proposed by Kostas Geronikolas-Greece

Solution by Marian Ursărescu-Romania

$$(a+b)^2 \ge 4ab \to \frac{1}{(a+b)^2} \le \frac{1}{4ab} \to \sum_{cyc} \frac{a^3 b}{(a+b)^2} \le \frac{1}{4}\sum_{cyc} a^2 \le \frac{9R^2}{4}$$

We must show that:

$$\sum_{cyc} \frac{a^2}{b^2+c^2} \le \frac{2R-r}{2r}; (1)$$

$$b^2+c^2 \ge 2bc \to \frac{1}{b^2+c^2} \le \frac{1}{2bc} \to \sum_{cyc} \frac{a^2}{b^2+c^2} \le \frac{1}{2}\sum_{cyc} \frac{a^2}{bc}; (2)$$

From $(1), (2)$ we must show that:

$$\sum_{cyc} \frac{a^2}{bc} \le \frac{2R-r}{r} \Leftrightarrow \frac{a^3+b^3+c^3}{abc} \le \frac{2R-r}{r}; (3)$$

$a^3+b^3+c^3 = 2s(s^2-3r^2-6Rr)$ and $abc = 4Rrs; (4)$

From $(3), (4)$ we must show that:

$$\frac{2s(s^2-3r^2-6Rr)}{4Rrs} \le \frac{2R-r}{r} \Leftrightarrow s^2-3r^2-6Rr \le 4R^2-2Rr$$

$$\Leftrightarrow s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$$

2340. In ΔABC the following relationship holds:

$$\frac{2F}{r} < \sum_{cyc}\left(m_a + \frac{a^2}{4m_a}\right) \le 6R$$

Proposed by Rajeev Rastogi-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\left(m_a + \frac{a^2}{4m_a}\right) \overset{AM-GM}{\overset{(*)}{\ge}} \sum_{cyc} 2\cdot\sqrt{m_a \cdot \frac{a^2}{4m_a}} = \sum_{cyc} a = 2s = \frac{2F}{r}$$

$$\sum_{cyc}\left(m_a + \frac{a^2}{4m_a}\right) = \sum_{cyc}\frac{4m_a^2+a^2}{4m_a} = \sum_{cyc}\frac{(2b^2+2c^2-a^2)+a^2}{4m_a} =$$

$$= 2\sum_{cyc}\frac{b^2+c^2}{4m_a} \overset{Tereshin}{\overset{(*)}{\le}} 2\sum_{cyc} R = 6R$$

Therefore,

$$\frac{2F}{r} < \sum_{cyc}\left(m_a + \frac{a^2}{4m_a}\right) \le 6R$$

2341. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c} \ge 12$$

Proposed by D.M.Bătineţu-Giurgiu, Flaviu Cristian Verde-Romania

Solution by George Florin Şerban-Romania

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}=\sum_{cyc}\frac{\left(\frac{2F}{a}+\frac{2F}{b}\right)\left(\frac{2F}{a}+\frac{2F}{c}\right)}{\frac{2F}{b}\cdot\frac{2F}{c}}=$$

$$=\sum_{cyc}\frac{4F^2(a+b)(a+c)}{a^2bc}\cdot\frac{bc}{4F^2}=\sum_{cyc}\frac{(a+b)(a+c)}{a^2}=\sum_{cyc}\left(1+\frac{c}{a}+\frac{b}{a}+\frac{bc}{a^2}\right)=$$

$$=1+1+1+\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2}\overset{AM-GM}{\geq}$$

$$\geq 12\cdot\sqrt[12]{\frac{a^4b^4c^4}{a^4b^4c^4}}=12$$

Therefore,

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\geq 12$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\overset{AM-GM}{\geq}\sum_{cyc}\frac{2\sqrt{h_ah_b}\cdot 2\sqrt{h_ah_c}}{h_bh_c}=4\sum_{cyc}\frac{h_a\sqrt{h_bh_c}}{h_bh_c}=$$

$$=4\sum_{cyc}\frac{h_a}{\sqrt{h_bh_c}}\overset{AM-GM}{\geq}4\cdot 3\cdot\sqrt[3]{\frac{\prod h_a}{\prod\sqrt{h_bh_c}}}=12\cdot\sqrt[3]{\frac{\prod h_a}{\prod h_a}}=12$$

Therefore,

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}\geq 12$$

Solution 3 by Samar Das-India

$$\sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}=$$

$$=\frac{(h_a+h_b)(h_a+h_c)}{h_bh_c}+\frac{(h_b+h_a)(h_b+h_c)}{h_ch_a}+\frac{(h_c+h_a)(h_c+h_b)}{h_ah_b}\overset{AM-GM}{\geq}$$

$$\geq 3\cdot\sqrt[3]{\frac{\left((h_a+h_b)(h_b+h_c)(h_c+h_a)\right)^2}{(h_ah_bh_c)^2}}=$$

$$= 3 \cdot \sqrt[3]{\frac{4 \cdot 4 \cdot 4\left(\frac{h_a + h_b}{2}\right)^2\left(\frac{h_b + h_c}{2}\right)^2\left(\frac{h_c + h_a}{2}\right)^2}{(h_a h_b h_c)^2}} = 12$$

Therefore,

$$\sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \geq 12$$

2342. In acute ΔABC the following relationship holds:

$$\frac{\sin^5 A}{\sin^3 B} + \frac{\sin^5 B}{\sin^3 C} + \frac{\sin^5 C}{\sin^3 A} \geq \left(1 + \frac{r}{R}\right)^2$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{\sin^5 A}{\sin^3 B} + \frac{\sin^5 B}{\sin^3 C} + \frac{\sin^5 C}{\sin^3 A} = \frac{1}{(2R)^2}\sum \frac{a^5}{b^3} = \frac{1}{(2R)^2}\sum \frac{a^6}{(ab)\cdot b^2} \geq$$

$$\overset{Holder}{\geq} \frac{1}{(2R)^2} \cdot \frac{(a^2 + b^2 + c^2)^3}{(ab + bc + ca)(a^2 + b^2 + c^2)} \overset{\sum ab \leq \sum a^2}{\geq} \frac{1}{(2R)^2} \cdot \frac{(a^2 + b^2 + c^2)^2}{a^2 + b^2 + c^2}$$

$$= \frac{a^2 + b^2 + c^2}{(2R)^2} \overset{Walker}{\geq} \frac{4(R + r)^2}{(2R)^2} = \left(1 + \frac{r}{R}\right)^2$$

Therefore, $\frac{\sin^5 A}{\sin^3 B} + \frac{\sin^5 B}{\sin^3 C} + \frac{\sin^5 C}{\sin^3 A} \geq \left(1 + \frac{r}{R}\right)^2$

2343. Prove that for any triangle ABC the inequality:

$$\sqrt[6]{(1 + \cos A)(1 + \cos B)(1 + \cos C)} \leq \frac{\sqrt{6}}{486}\left(\frac{8(a + b + c)^3}{abc} + 27\right)$$

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by Avishek Mitra-West Bengal-India

$$\sqrt[6]{\prod (1 + \cos A)} = \sqrt[6]{\prod 2\cos^2 \frac{A}{2}} =$$

$$= 8^{\frac{1}{6}}\left(\prod \cos\frac{A}{2}\right)^{\frac{1}{3}} = \sqrt{2}\left(\prod\sqrt{\frac{s(s-a)}{bc}}\right)^{\frac{1}{3}} = \sqrt{2}\left(\frac{S\Delta}{abc}\right)^{\frac{1}{3}} = \sqrt{2}\left(\frac{S\Delta}{4R\Delta}\right)^{\frac{1}{3}}$$

2344. In ΔABC the following relationship holds:

$$\frac{\cos^5 A}{\cos^3 B} + \frac{\cos^5 B}{\cos^3 C} + \frac{\cos^5 C}{\cos^3 A} \ge 1 - \left(\frac{r}{R}\right)^2$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x = \cos A$, $y = \cos B$, $z = \cos C$

$$\frac{\cos^5 A}{\cos^3 B} + \frac{\cos^5 B}{\cos^3 C} + \frac{\cos^5 C}{\cos^3 A} = \sum\frac{x^5}{y^3} = \sum\frac{x^6}{(xy)\cdot y^2} \ge$$

$$\overset{Holder}{\ge} \frac{(x^2+y^2+z^2)^3}{(xy+yz+zx)(x^2+y^2+z^2)} \overset{\sum xy \le \sum x^2}{\ge} \frac{(x^2+y^2+z^2)^2}{x^2+y^2+z^2}$$

$$= x^2+y^2+z^2 = \sum\cos^2 A = 1 - 2\prod\cos A = 1 - 2\cdot\frac{s^2-(2R+r)^2}{4R^2} \ge$$

$$\overset{Gerretsen}{\ge} 1 - \frac{(4R^2+4Rr+3r^2)-(2R+r)^2}{2R^2} = 1 - \left(\frac{r}{R}\right)^2$$

Therefore, $\frac{\cos^5 A}{\cos^3 B} + \frac{\cos^5 B}{\cos^3 C} + \frac{\cos^5 C}{\cos^3 A} \ge 1 - \left(\frac{r}{R}\right)^2$

2345. In ΔABC the following inequality holds:

$$\sum\frac{h_a^2}{r_b r_c} \le \sum\frac{r_a^2}{h_b h_c}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum\frac{h_a^2}{r_b r_c} = \sum\frac{\frac{4S^2}{a^2}}{\frac{S^2}{(s-b)(s-c)}} = 4\sum\frac{(s-b)(s-c)}{a^2} \quad \textbf{(1)}$$

$$\sqrt{(s-b)(s-c)} \le \frac{s-b+s-c}{2} = \frac{a}{2} \Rightarrow (s-b)(s-c) \le \frac{a^2}{4} \quad (2)$$

From (1)+(2)$\Rightarrow \sum \frac{h_a^2}{r_b r_c} \le 3 \Rightarrow$ we must show: $\sum \frac{r_a^2}{h_b h_c} \ge 3$ (3)

$$\sum \frac{r_a^2}{h_b h_c} \ge 3\sqrt[3]{\frac{r_a^2 r_b^2 r_c^2}{h_a^2 h_b^2 h_c}} \quad (4)$$

From (3)+(4) we must show:

$$3\sqrt[3]{\frac{r_a^2 r_b r_c^2}{h_a^2 h_b^2 h_c}} \ge 3 \Leftrightarrow r_a r_b r_c \ge h_a h_b h_c \Leftrightarrow s^2 r \ge \frac{2s^2 r^2}{R} \Leftrightarrow R \ge 2r \quad \text{(True)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

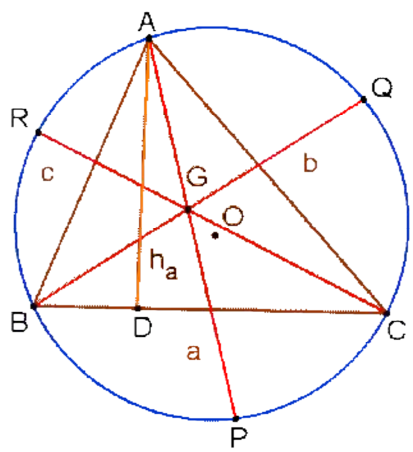
We know that: $\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c} \overset{AM-GM}{\ge} \frac{2}{\sqrt{r_b r_c}} \to \frac{h_a^2}{r_b r_c} \le 1$ (and analogs) (1)

$$(1) \to \prod \frac{h_a^2}{r_b r_c} \le 1 \to \prod h_a \le \prod r_a \quad (2)$$

$$(1) \to \sum \frac{h_a^2}{r_b r_c} \le \sum 1 = 3 \text{ and } \sum \frac{r_a^2}{h_b h_c} \overset{AM-GM}{\ge} 3\sqrt[3]{\prod \frac{r_a^2}{h_b h_c}} = 3\sqrt[3]{\frac{\prod r_a}{\prod h_a}}^{2} \overset{(2)}{\ge} 3$$

$$\text{Therefore, } \sum \frac{h_a^2}{r_b r_c} \le 3 \le \sum \frac{r_a^2}{h_b h_c}$$

2346.



Prove:

$$\frac{AG}{GP} + \frac{F\sqrt{3}}{h_a^2} \ge 2; \quad F = Area_{ABC}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Daniel Sitaru-Romania

$$\rho(G) = \frac{a}{2}\cdot\frac{a}{2} = m_a \cdot MP, \{M\} = AP \cap BC, MP = \frac{a^2}{4m_a}$$

$$GP = \frac{m_a}{3} + \frac{a^2}{4m_a} = \frac{4m_a^2 + 3a^2}{12m_a} = \frac{2a^2 + 2b^2 + 2c^2}{12m_a} = \frac{a^2 + b^2 + c^2}{6m_a}$$

$$\frac{AG}{GP} + \frac{F\sqrt{3}}{h_a^2} = \frac{\frac{2}{3}m_a \cdot 6m_a}{a^2 + b^2 + c^2} + \frac{F\sqrt{3}}{h_a^2} \overset{IONESCU-WEITZENBOCK}{\geq}$$

$$\geq \frac{4m_a^2}{a^2 + b^2 + c^2} + \frac{a^2 + b^2 + c^2}{4h_a^2} \geq \frac{4h_a^2}{a^2 + b^2 + c^2} + \frac{a^2 + b^2 + c^2}{4h_a^2} =$$

$$= \left(\frac{2h_a}{\sqrt{a^2 + b^2 + c^2}} - \frac{\sqrt{a^2 + b^2 + c^2}}{2h_a}\right)^2 + 2 \geq 2$$

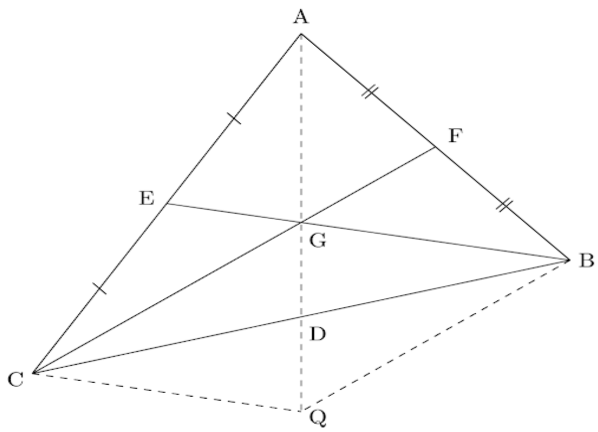
Equality holds for: $a = b = c$.

2347. In any $\Delta\, ABC$ the following relationship holds:

$$\frac{2bcm_bm_c}{\sqrt{8b^2c^2 + 2a^2b^2 + 2a^2c^2 - 2b^4 - 2c^4 + a^4}} \geq \sqrt{\frac{\prod_{cyc}(am_a + bm_b - cm_c)}{am_a + bm_b + cm_c}}$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



Proof :

Via Ptolemy's theorem on quadrilateral ABQC, $AB.CQ+AC.BQ\ge AQ.BC$

$$\Rightarrow c.\frac{2m_b}{3}+b.\frac{2m_c}{3}\ge\frac{4m_a}{3}.a\Rightarrow cm_b+bm_c\ge 2am_a$$

$$\overset{\text{upon squaring}}{\Rightarrow} c^2m_b^2+b^2m_c^2+2bcm_bm_c\ge 4a^2m_a^2$$

$$\Rightarrow 8bcm_bm_c\overset{(i)}{\ge}16a^2m_a^2-4c^2m_b^2-4b^2m_c^2$$

$$\text{Now, }(2bm_b+2cm_c)^2-4a^2m_a^2$$

$$=4b^2m_b^2+4c^2m_c^2+8bcm_bm_c-4a^2m_a^2\overset{\text{via (i)}}{\ge}4b^2m_b^2+4c^2m_c^2$$
$$+12a^2m_a^2-4c^2m_b^2-4b^2m_c^2$$
$$=b^2(4m_b^2-4m_c^2)-c^2(4m_b^2-4m_c^2)+12a^2m_a^2$$
$$=(b^2-c^2)(2c^2+2a^2-b^2-2a^2-2b^2+c^2)+12a^2m_a^2$$
$$=3(a^2.4m_a^2-(b^2-c^2)^2)=3(a^2(2b^2+2c^2-a^2)-b^4-c^4+2b^2c^2)$$
$$=3\left(2\sum a^2b^2-\sum a^4\right)=48F^2>0$$
$$\Rightarrow(2bm_b+2cm_c)^2>4a^2m_a^2$$

$$\Rightarrow \boxed{bm_b+cm_c-am_a>0\text{ and analogs}}\text{ and putting } bm_b+cm_c$$
$$-am_a=x,\ cm_c+am_a-bm_b=y$$

and $am_a+bm_b-cm_c=z$, we get : $\sum am_a=\sum x\Rightarrow am_a=\frac{y+z}{2}, bm_b$

$$=\frac{z+x}{2}, cm_c=\frac{x+y}{2}\text{ and of course }\boxed{x,y,z>0}$$

$$\text{Now, }8b^2m_b^2+8c^2m_c^2-4a^2m_a^2$$
$$=2b^2(2c^2+2a^2-b^2)+2c^2(2a^2+2b^2-c^2)-a^2(2b^2+2c^2-a^2)$$
$$=8b^2c^2+2a^2b^2+2a^2c^2-2b^4-2c^4+a^4$$

$$\Rightarrow 8b^2c^2+2a^2b^2+2a^2c^2-2b^4-2c^4+a^4\overset{(ii)}{=}2(z+x)^2+2(x+y)^2$$
$$-(y+z)^2$$

Via (ii) and substitutions mentioned above, proposed inequality

$$\Leftrightarrow\frac{(z+x)(x+y)}{2\sqrt{2(z+x)^2+2(x+y)^2-(y+z)^2}}\overset{(l)}{\ge}\sqrt{\frac{xyz}{x+y+z}}$$

$$\text{Let } y+z=a_1, z+x=b_1 \text{ and } x+y=c_1 \text{ and } \because a_1+b_1>c_1, b_1+c_1>a_1 \text{ and } c_1+a_1>b_1 \therefore a_1,b_1,c_1 \text{ are sides of a triangle}$$

$$\text{with circumradius, inradius and semiperimeter} = R_1, r_1, s_1 \text{ respectively (say) and}$$

$$2\sum x = 2s_1 \Leftrightarrow x = s_1 - a_1, y = s_1 - b_1$$

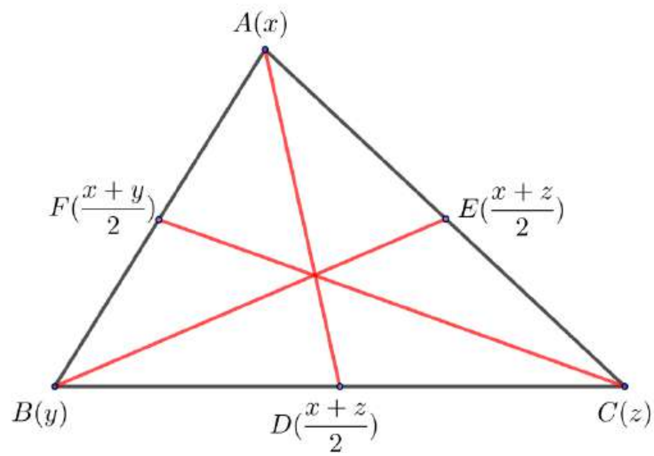
$$\text{and } z = s_1 - c_1 \therefore (l) \Leftrightarrow \frac{b_1c_1}{2\sqrt{2b_1^2+2c_1^2-a_1^2}} \ge \sqrt{\frac{r_1^2 s_1}{s_1}} \Leftrightarrow \frac{\frac{b_1c_1}{2R_1}}{4m_{a_1}} \ge \frac{r_1}{2R_1} \Leftrightarrow \frac{m_{a_1}}{h_{a_1}} \le \frac{R_1}{2r_1}$$

$$\to \text{true via Panaitopol} \Rightarrow (l) \text{ is true}$$

$$\therefore \frac{2bcm_bm_c}{\sqrt{8b^2c^2+2a^2b^2+2a^2c^2-2b^4-2c^4+a^4}} \ge \sqrt{\frac{\prod_{cyc}(am_a+bm_b-cm_c)}{am_a+bm_b+cm_c}} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{2bcm_bm_c}{\sqrt{8b^2c^2+2a^2b^2+2a^2c^2-2b^4-2c^4+a^4}} \ge \sqrt{\frac{\prod_{cyc}(am_a+bm_b-cm_c)}{am_a+bm_b+cm_c}}; (*)$$



Let x, y, z be the complexe coordinates of points A, B and C respectively.

We have: $am_a = |y-z|\left|\frac{y+z}{2}-x\right| \to am_a = \frac{1}{2}|y^2-z^2-2xy+2xz|$

Similarly, $bm_b = \frac{1}{2}|z^2-x^2-2zy+2yx|$ **and** $cm_c = \frac{1}{2}|y^2-x^2-2zy+2xz| \to$

$$am_a + bm_b = \frac{1}{2}|y^2-z^2-2xy+2xz| + \frac{1}{2}|z^2-x^2-2zy+2yx| \ge$$

$$\geq \frac{1}{2}\left|y^2 - x^2 - 2zy + 2xz\right| = cm_c \rightarrow$$

$$am_a + bm_b \geq cm_c(and\ analogs)$$

am_a, bm_b, cm_c – can be the sides of a triangle. Let $am_a = x, bm_b = y, cm_c = z$

$$\prod_{cyc}(am_a + bm_b - cm_c) = 8(s-x)(s-y)(s-z) = 8sr^2$$

$$2a^2b^2 + 2a^2c^2 + 8b^2c^2 + a^4 - 2b^4 - 2c^4 = 8b^28c^2m_c^2 - 4a^2m_a^2 =$$

$$= 4(2y^2 + 2z^2 - x^2) = 16m_x^2$$

$$(*) \Leftrightarrow \frac{2yz}{\sqrt{16m_x^2}} \geq \sqrt{\frac{8sr^2}{2s}} \Leftrightarrow \frac{yz}{2m_x} \geq 2r$$

$$\Leftrightarrow m_x \leq \frac{yz}{4r} = \frac{xyz}{4xr} = \frac{4srR}{4xr} = \frac{Rh_x}{2r}; (Panaitopol's)$$

Therefore,

$$\frac{2bcm_bm_c}{\sqrt{8b^2c^2 + 2a^2b^2 + 2a^2c^2 - 2b^4 - 2c^4 + a^4}} \geq \sqrt{\frac{\prod_{cyc}(am_a + bm_b - cm_c)}{am_a + bm_b + cm_c}}$$

2348. In any $\Delta\, ABC$, the following relationship holds:

$$\sum \frac{1}{a^3} tan\frac{A}{2} \leq \frac{1}{12Rr^2} \leq \frac{1}{24r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Proof : \sum \frac{1}{a^3} tan\frac{A}{2} = \sum\left(\frac{1}{a^2}\left(\frac{tan\frac{A}{2}}{4Rcos^2\frac{A}{2}tan\frac{A}{2}}\right)\right) = \frac{1}{4R}\sum\left(\frac{1}{a^2}\left(1 + tan^2\frac{A}{2}\right)\right)$$

$$= \frac{1}{4R}\left(\sum\frac{1}{a^2} + \sum\frac{tan^2\frac{A}{2}}{16R^2cos^4\frac{A}{2}tan^2\frac{A}{2}}\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2 + 4Rr + r^2)^2 - 16Rrs^2}{16R^2r^2s^2} + \frac{1}{16R^2}\sum\left(1 + tan^2\frac{A}{2}\right)^2\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2 + 4Rr + r^2)^2 - 16Rrs^2}{16R^2r^2s^2} + \frac{1}{16R^2}\sum\left(1 + tan^4\frac{A}{2} + 2tan^2\frac{A}{2}\right)\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}\right.$$

$$\left.+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(\left(\sum r_a^2\right)^2-2\sum r_a^2r_b^2\right)+\frac{2}{s^2}\left(\sum r_a^2\right)\right)\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}\right.$$

$$+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(((4R+r)^2-2s^2)^2-2\left(s^4-2rs^2(4R+r)\right)\right)\right.$$

$$\left.\left.+\frac{2}{s^2}((4R+r)^2-2s^2)\right)\right)$$

$$= \frac{(s^2+4Rr+r^2)^2-16Rrs^2}{64R^3r^2s^2}$$

$$+\frac{3s^4+2s^2(4R+r)^2-4s^4+(4R+r)^4-4s^2(4R+r)^2+4s^4-2s^4+4rs^2(4R+r)}{64R^3s^4}$$

$$= \frac{s^6-(8Rr-3r^2)s^4-r^2s^2(16R^2-8Rr-3r^2)+r^2(4R+r)^4}{64R^3r^2s^4} \le \frac{1}{12Rr^2}$$

$$\Leftrightarrow 3s^6-(16R^2+24Rr-9r^2)s^4-3r^2s^2(16R^2-8Rr-3r^2)+3r^2(4R+r)^4 \overset{(i)}{\le} 0$$

$$Now, Rouche \Rightarrow s^2-(m-n)\ge 0 \text{ and } s^2-(m+n)\le 0, \text{where } m$$

$$= 2R^2+10Rr-r^2 \text{ and } n = 2(R-2r)\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0 \Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0$$

$$\Leftrightarrow 3s^6-3s^4(4R^2+20Rr-2r^2)+3rs^2(4R+r)^3\le 0$$

$$\Rightarrow \text{in order to prove } (i), \text{it suffices to prove}:$$

$$3s^6-(16R^2+24Rr-9r^2)s^4-3r^2s^2(16R^2-8Rr-3r^2)+3r^2(4R+r)^4$$

$$\le 3s^6-3s^4(4R^2+20Rr-2r^2)+3rs^2(4R+r)^3$$

$$\Leftrightarrow (4R^2-36Rr-3r^2)s^4+3rs^2(64R^3+64R^2r+4Rr^2-2r^3)-3r^2(4R+r)^4\ge 0$$

$$\Leftrightarrow 4(R-2r)^2s^4-(20Rr+19r^2)s^4+3rs^2(64R^3+64R^2r+4Rr^2-2r^3)-3r^2(4R+r)^4 \overset{(ii)}{\ge} 0$$

$$Now, LHS\ of\ (ii) \overset{Gerretsen}{\gtreqless} \Big(4(16Rr - 5r^2)(R-2r)^2$$
$$-(20Rr + 19r^2)(4R^2 + 4Rr + 3r^2)$$
$$+3r(64R^3 + 64R^2r + 4Rr^2 - 2r^3)\Big)s^2$$
$$-3r^2(4R+r)^4 = r(176R^3 - 240R^2r + 212Rr^2 - 143r^3)s^2 - 3r^2(4R+r)^4$$
$$= r\Big((R-2r)(176R^2 + 112Rr + 436r^2) + 729r^3\Big)s^2$$
$$-3r^2(4R+r)^4 \overset{Gerretsen}{\gtreqless} r\Big((R-2r)(176R^2 + 112Rr + 436r^2) + 729r^3\Big)(16Rr - 5r^2) - 3r^2(4R+r)^4$$
$$\Leftrightarrow 512t^4 - 1372t^3 + 1076t^2 - 849t + 178 \overset{?}{\gtreqless} 0 \ \left(where\ t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big((t-2)(512t^2 + 676t + 1732) + 3375\Big) \overset{?}{\gtreqless} 0 \to true$$
$$\because t \overset{Euler}{\gtreqless} 2 \Rightarrow (ii) \Rightarrow (i) is\ true \therefore \sum \frac{1}{a^3}\tan\frac{A}{2} \le \frac{1}{12Rr^2} \overset{Euler}{\lesseqgtr} \frac{1}{24r^3}\ (QED)$$

2349. If ABC is an acute triangle with the circumradius R and the inradius r, then prove that :

$$\sum \frac{\cos A \cdot \cos^2 B}{\cos^2 C} \ge \frac{9R}{4(R+r)}.$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

ABC is an acute triangle→ $\cos A, \cos B, \cos C > 0$.

$$\sum \frac{\cos A \cdot \cos^2 B}{\cos^2 C} \cdot \sum \cos A \overset{CBS}{\gtreqless} \left(\sum \frac{\cos A \cdot \cos B}{\cos C}\right)^2 \to \sum \frac{\cos A \cdot \cos^2 B}{\cos^2 C}$$
$$\ge \frac{R}{R+r}\left(\sum \frac{\cos A \cdot \cos B}{\cos C}\right)^2$$

So, we need to prove : $\sum \frac{\cos A \cdot \cos B}{\cos C} \overset{(*)}{\gtreqless} \frac{3}{2}, \forall \Delta ABC$ acute.

Using the substitutions :

$$A = \frac{\pi - X}{2}, B = \frac{\pi - Y}{2}, C = \frac{\pi - Z}{2}, X, Y, Z \in (0, \pi), \sum X = \pi$$

$$\rightarrow (*) \leftrightarrow \sum \frac{\sin\frac{X}{2}.\sin\frac{Y}{2}}{\sin\frac{Z}{2}} \geq \frac{3}{2}, \forall \Delta XYZ \leftrightarrow \left(\prod \sin\frac{X}{2}\right)\sum \frac{1}{\sin^2\frac{Z}{2}} \geq \frac{3}{2}$$

We know that :

$$\prod \sin\frac{X}{2} = \frac{r}{4R} \text{ and } \sum \frac{1}{\sin^2\frac{X}{2}} = \frac{s^2+r^2-8Rr}{r^2}$$

$$\rightarrow (*) \leftrightarrow \frac{s^2+r^2-8Rr}{4Rr} \geq \frac{3}{2} \leftrightarrow s^2 \geq 14Rr - r^2$$

Which is true from Gerretsen, $s^2 \geq 16Rr - 5r^2 \overset{?}{\geq} 14Rr - r^2 \leftrightarrow R \geq 2r\ (Euler)$

$$\rightarrow (*) is\ true \rightarrow \sum \frac{\cos A.\cos^2 B}{\cos^2 C} \geq \frac{9R}{4(R+r)}.$$

Solution 2 by Mohammed Diai-Rabat-Morocco

$$\sum \frac{cos\,A.cos^2 B}{cos^2 C} \geq \frac{9R}{4(R+r)}; (*)($$

We know that: $\sum cosA = 1 + \frac{r}{R} = \frac{R+r}{R}$ then $(*) \Leftrightarrow \sum \frac{cos\,A.cos^2 B}{cos^2 C} \cdot \sum cos\,A \geq \frac{9}{4}$

By CBS inequality: $\sum \frac{cos\,A.cos^2 B}{cos^2 C} \cdot \sum cosA \geq \left(\sum \frac{cos\,A \cdot cos\,B}{cosC}\right)^2$

To prove $(*)$ we need to prove that: $\sum \frac{cos\,A\cdot cos\,B}{cosC} \geq \frac{3}{2}; (**)$

$$\sum \frac{cos\,A \cdot cos\,B}{cosC} = \sum \frac{cos\,A \cdot cos\,B}{-cos(A+B)} = \sum \frac{cos\,A \cdot cos\,B}{sin\,A \cdot sin\,B - cos\,A \cdot cos\,B} =$$

$$= \sum \frac{cot\,A \cdot cot\,B}{1 - cot\,A \cdot cot\,B} = \sum \frac{1}{1 - cot\,A \cdot cot\,B} - 3$$

Therefore, $(**) \Leftrightarrow \sum \frac{1}{1-cot\,A\cdot cot\,B} \geq \frac{9}{2}$

By CBS inequality: $\sum \frac{1}{1-cot\,A\cdot cot\,B} \geq \frac{9}{\sum(1-cot\,A\cdot cot\,B)} = \frac{9}{2}$

2350. In any $\Delta\ ABC$, the following relationship holds:

$$\frac{3}{4} \leq \sum \frac{a^2}{(b+c)^2} \leq \frac{R}{2r} - \frac{1}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof : } \sum \frac{a^2}{(b+c)^2} = \left(\sum \frac{a}{b+c}\right)^2 - 2\sum \frac{ab}{(b+c)(c+a)}$$

$$= \frac{1}{4s^2(s^2+2Rr+r^2)^2}\left(\sum a(c+a)(a+b)\right)^2 - \frac{2\sum ab(2s-c)}{2s(s^2+2Rr+r^2)}$$

$$= \frac{1}{4s^2(s^2+2Rr+r^2)^2}\left(\sum a\left(\sum ab + a^2\right)\right)^2 - \frac{2s(s^2+4Rr+r^2)-12Rrs}{s(s^2+2Rr+r^2)}$$

$$= \frac{\left(2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)\right)^2}{4s^2(s^2+2Rr+r^2)^2} - \frac{2(s^2-2Rr+r^2)}{s^2+2Rr+r^2}$$

$$\Rightarrow \sum \frac{a^2}{(b+c)^2} \overset{(i)}{=} \frac{2\left(s^4-2s^2(2Rr+3r^2)+r^2(6R^2+4Rr+r^2)\right)}{(s^2+2Rr+r^2)^2}$$

$$\sum \frac{a^2}{(b+c)^2} \le \frac{R}{2r} - \frac{1}{4} \overset{\text{via }(i)}{\Leftrightarrow} \frac{2\left(s^4-2s^2(2Rr+3r^2)+r^2(6R^2+4Rr+r^2)\right)}{(s^2+2Rr+r^2)^2} \le \frac{2R-r}{4r}$$

$$\Leftrightarrow (2R-9r)s^4 + rs^2(8R^2+32Rr+46r^2) + r^2(8R^3-44R^2r-34Rr^2-9r^3) \ge 0$$

$$\Leftrightarrow (2R-4r)s^4 + rs^2\left(8R^2+32Rr+46r^2\right) + r^2\left(8R^3-44R^2r-34Rr^2-9r^3\right) - 5rs^4 \overset{(a)}{\ge} 0$$

$$\text{Now, LHS of } (a) \overset{\text{Gerretsen}}{\ge} (2R-4r)\left(16Rr-5r^2\right)s^2 + rs^2\left(8R^2+32Rr+46r^2\right)$$
$$+ r^2\left(8R^3-44R^2r-34Rr^2-9r^3\right)$$
$$-5r\left(4R^2+4Rr+3r^2\right)s^2$$

$$= \left((R-2r)(20R-22r)+7r^2\right)s^2 + r\left(8R^3-44R^2r-34Rr^2-9r^3\right) \overset{\text{Gerretsen}}{\ge}$$

$$\left((R-2r)(20R-22r)+7r^2\right)(16Rr-5r^2) + r(8R^3-44R^2r-34Rr^2-9r^3) \overset{?}{\ge} 0$$

$$\Leftrightarrow 82t^3 - 284t^2 + 273t - 66 \overset{?}{\ge} 0 \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(82t+44)+121\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (a) \text{is true}$$

$$\Rightarrow \sum \frac{a^2}{(b+c)^2} \le \frac{R}{2r} - \frac{1}{4} \text{ and } \sum \frac{a^2}{(b+c)^2} \ge \frac{1}{3}\left(\sum \frac{a}{b+c}\right)^2$$

$$\overset{\text{Nesbitt}}{\ge} \frac{1}{3}\cdot\frac{9}{4} \Rightarrow \frac{3}{4} \le \sum \frac{a^2}{(b+c)^2} \text{ (QED)}$$

2351. **In ΔABC the following relationship holds:**

$$\sum_{cyc}\sqrt{\frac{m_a}{w_a h_b h_c}} \geq \frac{1}{r}$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\because m_a \geq \frac{b+c}{2}\cos\frac{A}{2} \rightarrow \frac{m_a}{w_a} \geq \frac{(b+c)^2}{4bc}$$

$$\rightarrow \frac{m_a}{w_a h_b h_c} \geq \frac{(b+c)^2}{4bc \cdot \frac{4s^2r^2}{bc}} = \left(\frac{b+c}{4sr}\right)^2 ; (and\ analogs)$$

$$\rightarrow \sum_{cyc}\sqrt{\frac{m_a}{w_a h_b h_c}} \geq \sum_{cyc}\frac{b+c}{4sr} = \frac{4s}{4sr} = \frac{1}{r}$$

Therefore,

$$\sum_{cyc}\sqrt{\frac{m_a}{w_a h_b h_c}} \geq \frac{1}{r}$$

Solution 2 by Ertan Yildirim-Izmir-Turkiye

$$\because \frac{m_a}{w_a} \geq \frac{(b+c)^2}{4bc}$$

$$\sum_{cyc}\sqrt{\frac{m_a}{w_a h_b h_c}} \geq \sum_{cyc}\sqrt{\frac{(b+c)^2}{4bc}\cdot\frac{1}{\frac{ac}{2R}\cdot\frac{ab}{2R}}} = \sum_{cyc}\frac{b+c}{abc}\cdot R$$

$$\rightarrow \sum_{cyc}\frac{R(b+c)}{abc} = \frac{R}{abc}\cdot\sum_{cyc}(b+c) = \frac{R}{4Rrs}\cdot 2(a+b+c) = \frac{R\cdot 4s}{Rr\cdot 4s} = \frac{1}{r}$$

Therefore,

$$\sum_{cyc}\sqrt{\frac{m_a}{w_a h_b h_c}} \geq \frac{1}{r}$$

2352. **In ΔABC the following relationship holds:**

$$\sum_{cyc} a^2\left(\frac{1}{m_a^2}+\frac{1}{h_bh_c}+\frac{1}{r_br_c}\right)\geq 12$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that :

$$h_bh_c \overset{h_a\leq m_a}{\leq} m_bm_c \overset{AM-GM}{\leq} \frac{{m_b}^2+{m_c}^2}{2} \text{ and } r_br_c = s(s-a)\leq {m_a}^2$$

$$\to {m_a}^2+h_bh_c+r_br_c\leq 2{m_a}^2+\frac{{m_b}^2+{m_c}^2}{2}=$$

$$=\frac{4(2b^2+2c^2-a^2)+(2c^2+2a^2-b^2)+(2a^2+2b^2-c^2)}{8}$$

$$\to {m_a}^2+h_bh_c+r_br_c\leq \frac{9}{8}(b^2+c^2)\quad (and\ analogs)\quad (1)$$

$$\to \sum a^2\left(\frac{1}{{m_a}^2}+\frac{1}{h_bh_c}+\frac{1}{r_br_c}\right)\overset{CBS}{\geq}$$

$$\sum a^2\cdot\frac{9}{{m_a}^2+h_bh_c+r_br_c}\overset{(1)}{\geq} 8\sum\frac{a^2}{b^2+c^2}\overset{Nesbitt}{\geq} 8\cdot\frac{3}{2}=12$$

Therefore,

$$\sum a^2\left(\frac{1}{{m_a}^2}+\frac{1}{h_bh_c}+\frac{1}{r_br_c}\right)\geq 12$$

Solution 2 by Marian Ursărescu-Romania

We must show:

$$\sum_{cyc}\frac{a^2}{m_a^2}+\sum_{cyc}\frac{a^2}{h_bh_c}+\sum_{cyc}\frac{a^2}{r_br_c}\geq 12;\ (1)$$

$$\sum_{cyc}\frac{a^2}{r_br_c}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{r_ar_b+r_br_c+r_cr_a}=\frac{4s^2}{s^2}=4;\ (2)$$

$$\sum_{cyc}\frac{a^2}{h_bh_c}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{h_ah_b+h_bh_c+h_ch_a}=\frac{4s^2}{\frac{2s^2r}{R}}=\frac{2R}{r}\geq 4;\ (3)$$

$$\because \sum_{cyc}\frac{a^2}{2b^2+2c^2-a^2}\ge 1 \to \sum_{cyc}\frac{a^2}{m_a^2}\ge 4;\ (4)$$

From $(2),(3),(4)$ it follows that (1) its true.

Therefore,

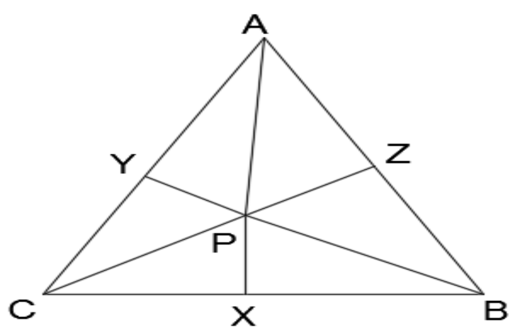
$$\sum_{cyc}a^2\left(\frac{1}{m_a^2}+\frac{1}{h_bh_c}+\frac{1}{r_br_c}\right)\ge 12$$

2353. Let ABC be an equilateral triangle of side length $\sqrt{3}$, let P be a point inside ΔABC and X,Y,Z be the feet of perpendiculars of P on sides BC,CA,AB respectively. Then prove:

$$\sum\frac{1+PX^2}{PY+PZ}\ge\frac{15}{4}$$

Proposed by Rajeev Rastogi-India

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let F be the area of ΔABC.

We have :

$$F=\frac{1}{2}.\sqrt{3}^2.\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{4}=[APB]+[BPC]+[CPA]=\frac{\sqrt{3}}{2}PZ+\frac{\sqrt{3}}{2}PX+\frac{\sqrt{3}}{2}PY$$

$$\to PX+PY+PZ=\frac{3}{2}\quad(1)$$

$$\sum\frac{1+PX^2}{PY+PZ}=\sum\frac{1}{PY+PZ}+\sum\frac{PX^2}{PY+PZ}\overset{CBS}{\ge}\frac{9}{2(PX+PY+PZ)}+\frac{(PX+PY+PZ)^2}{2(PX+PY+PZ)}$$

Therefore,

$$\sum \frac{1 + PX^2}{PY + PZ} \overset{(1)}{\geq} 3 + \frac{3}{4} = \frac{15}{4}.$$

Solution 2 by Abdul Aziz-Semarang-Indonesia

$$F = \frac{1}{2} \cdot \sqrt{3}(PX + PY + PZ) \Leftrightarrow PX + PY + PZ = \frac{3}{2}$$

$$\sum_{cyc} \frac{1 + PX^2}{PY + PZ} = \frac{3}{4}\sum_{cyc} \frac{1}{PY + PZ} + \sum_{cyc} \frac{\frac{1}{4} + PX^2}{PY + PZ} \overset{AM-GM}{\geq} \frac{3}{4}\sum_{cyc} \frac{1}{PY + PZ} + \sum_{cyc} \frac{PX}{PY + PZ} \geq$$

$$\overset{CBS}{\geq} \frac{3}{4} \cdot \frac{9}{2\sum PX} + \frac{(\sum PX)^2}{2(\sum PX \cdot PY)} \geq \frac{9}{4} + \frac{3}{2} \cdot \frac{\sum(PX \cdot PY)}{\sum(PX \cdot PY)} = \frac{15}{4}$$

Equality holds when $PX = PY = PZ = \frac{1}{2}$.

2354. In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, ω –Brocard's point, the following relationship holds:

$$4\left(1 + \frac{1}{2sin^2\omega}\right) \leq \sum_{cyc} \frac{n_a^2 + g_a^2 + 2m_a w_a}{h_a^2}$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let M be the midpoint of BC, E the foot of the angle bisector at A.

$$m_a w_a = |\overrightarrow{AM}||\overrightarrow{AD}| = \left|\frac{\overrightarrow{AB} + \overrightarrow{AC}}{2}\right|\left|\frac{b \cdot \overrightarrow{AB} + c \cdot \overrightarrow{AC}}{b + c}\right| \geq \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2} \cdot \frac{b \cdot \overrightarrow{AB} + c \cdot \overrightarrow{AC}}{b + c} =$$

$$= \frac{bc(b + c) + (b + c) \cdot \overrightarrow{AB} \cdot \overrightarrow{AC}}{2(b + c)} = \frac{bc(b + c) + \frac{b + c}{2}(b^2 + c^2 - a^2)}{2(b + c)}$$

$$= \frac{(b + c)^2 - a^2}{4} = s(s - a)$$

$$\rightarrow m_a w_a \geq s(s - a); (and\ analogs)$$

$$\sum_{cyc}\frac{n_a^2+g_a^2+2m_aw_a}{h_a^2}\ge\sum_{cyc}\frac{n_a^2+g_a^2+2s(s-a)}{h_a^2}$$

We know that: $n_a^2+g_a^2+2s(s-a)=4m_a^2$

$$\to\sum_{cyc}\frac{n_a^2+g_a^2+2m_aw_a}{h_a^2}\ge\sum_{cyc}\frac{4m_a^2}{h_a^2}=\frac{1}{4F^2}\sum_{cyc}(2b^2+2c^2-a^2)=$$

$$=\frac{1}{4F^2}\left(2\sum_{cyc}a^2b^2+\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)\right)=\frac{1}{4F^2}\left(2\sum_{cyc}a^2b^2+16F^2\right)=$$

$$=\frac{\sum a^2b^2}{2F^2}+4=4\left(1+\frac{1}{2sin^2\omega}\right)$$

Therefore,

$$4\left(1+\frac{1}{2sin^2\omega}\right)\le\sum_{cyc}\frac{n_a^2+g_a^2+2m_aw_a}{h_a^2}$$

Solution 2 by Alex Szoros-Romania

$$4\left(1+\frac{1}{2sin^2\omega}\right)=4+\frac{2}{sin^2\omega}=4+2(1+tan^2\omega)=6+2\left(\frac{a^2+b62+c^2}{4F}\right)^2=$$

$$=6+\frac{(\sum a^2)^2}{8F^2};(1)$$

From Stewart theorem, we have:

$$\begin{cases}an_a^2=b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\\ag_a^2=b^2(s-b)+c^2(s-c)-a(s-b)(s-c)\end{cases}\to$$

$$a(n_a^2+g_a^2)=ab^2+ac^2-2a(s-b)(s-c)$$

$$\to n_a^2+g_a^2=b^2+c^2-2(s-b)(s-c);(2)$$

On the other hand, $m_aw_a\ge s(s-a);(3)$

From $(2),(3)$ it follows tha:

$$n_a^2+g_a^2+2m_a\,w_a\ge b^2+c^2-2(s-b)(s-c)+2s(s-a)=4m_a^2$$

$$\to\frac{n_a^2+g_a^2+2m_aw_a}{h_a^2}\ge\frac{4m_a^2}{h_a^2}$$

$$\sum_{cyc}\frac{n_a^2+g_a^2+2m_aw_a}{h_a^2}\ge4\sum_{cyc}\frac{m_a^2}{h_a^2};(4)$$

$$4\sum_{cyc}\frac{m_a^2}{h_a^2}=4\sum_{cyc}\frac{a^2m_a^2}{4F^2}=\frac{1}{F^2}\sum_{cyc}a^2m_a^2=\frac{1}{4F^2}\sum_{cyc}a^2(2b^2+2c^2-a^2)=$$

$$=\frac{2\sum a^2b^2+2\sum a^2c^2-\sum a^4}{4F^2}=\frac{2\sum a^2b^2+16F^2}{4F^2}=\frac{\sum a^2b^2}{2F^2}+4;\ (5)$$

We want to prove that:

$$6+\frac{(\sum a^2)^2}{8F^2}=4+\frac{\sum a^2b^2}{2F^2};\ (6)\Leftrightarrow$$

$$2+\frac{(\sum a^2)^2}{8F^2}=\frac{\sum a^2b^2}{2F^2}\Leftrightarrow 2\sum a^2b^2-\sum a^4+\sum a^4+2\sum a^2b^2=4\sum a^2b^2$$

So,

$$4\left(1+\frac{1}{2sin^2\omega}\right)=6+\frac{(\sum a^2)^2}{8F^2}=4+\frac{\sum a^2b^2}{2F^2}=4\sum_{cyc}\frac{m_a^2}{h_a^2}\le\sum_{cyc}\frac{n_a^2+g_a^2+2m_aw_a}{h_a^2}$$

Therefore,

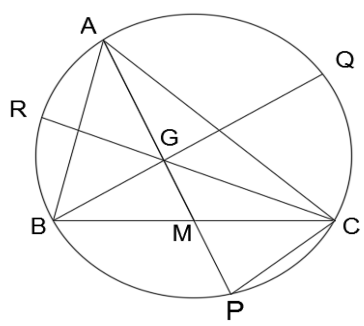
$$4\left(1+\frac{1}{2sin^2\omega}\right)\le\sum_{cyc}\frac{n_a^2+g_a^2+2m_aw_a}{h_a^2}$$

2355. Prove that :

$$\sum\frac{AG}{GP}+\frac{S\sqrt{3}}{4}\left(\frac{1}{r^2}+\sum\frac{1}{{r_a}^2}\right)\ge 6$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



Let M be the midpoint of BC, we know that : $AG=\frac{2}{3}m_a$ *and* $GM=\frac{1}{3}m_a$.

We have : $\mu(PCM)=\mu(BAM)$ *and* $\mu(MPC)=\mu(ABC)\to\Delta MPC\sim\Delta MBA$

$$\to MP=\frac{MB.MC}{MA}=\frac{a^2}{4m_a}$$

$$\rightarrow GP = GM + MP = \frac{1}{3}m_a + \frac{a^2}{4m_a} = \frac{4m_a{}^2 + 3a^2}{12m_a} = \frac{2b^2 + 2c^2 - a^2 + 3a^2}{12m_a}$$

$$= \frac{a^2 + b^2 + c^2}{6m_a}$$

$$\rightarrow \frac{AG}{GP} = \frac{2}{3}m_a \cdot \frac{6m_a}{a^2 + b^2 + c^2} = \frac{4m_a{}^2}{a^2 + b^2 + c^2} \ (and\ analogs)$$

$$\rightarrow \sum \frac{AG}{GP} = \sum \frac{4m_a{}^2}{a^2 + b^2 + c^2} = \frac{3(a^2 + b^2 + c^2)}{a^2 + b^2 + c^2} = 3 \quad (1)$$

$$We\ know\ that: \sum \frac{1}{r_a} = \frac{1}{r} \rightarrow \sum \frac{1}{r_a{}^2} \overset{CBS}{\geq} \frac{1}{3}\left(\sum \frac{1}{r_a}\right)^2 = \frac{1}{3r^2}$$

$$\rightarrow \frac{S\sqrt{3}}{4}\left(\frac{1}{r^2} + \sum \frac{1}{r_a{}^2}\right) \geq \frac{pr\sqrt{3}}{4}\left(\frac{1}{r^2} + \frac{1}{3r^2}\right) = \frac{p\sqrt{3}}{3r} \overset{Mitrinovic}{\geq} \frac{(3\sqrt{3}r).\sqrt{3}}{3r} = 3 \ (2)$$

$$(1)\ and\ (2) \rightarrow \sum \frac{AG}{GP} + \frac{S\sqrt{3}}{4}\left(\frac{1}{r^2} + \sum \frac{1}{r_a{}^2}\right) \geq 6$$

2356. In ΔABC the following relationship holds:

$$\frac{2r}{h_a}\left(\frac{1}{h_b^2} + \frac{1}{h_c^2}\right) \leq \left(\frac{R}{F}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

Using: $h_a = \frac{2F}{a}$, inequality becomes as:

$$\frac{\frac{2F}{s}}{\frac{2F}{a}} \cdot \frac{b^2 + c^2}{4F^2} \leq \frac{R^2}{F^2} \Leftrightarrow \frac{a}{s} \cdot \frac{b^2 + c^2}{4} \leq R^2 \Leftrightarrow a(b^2 + c^2) \leq 4sR^2; (1)$$

But in any ΔABC: $\frac{b}{c} + \frac{c}{b} \leq \frac{R}{r}; (2) \Leftrightarrow b^2 + c^2 \leq \frac{R}{r}bc \Leftrightarrow$

$a(b^2 + c^2) \leq \frac{R}{r} \cdot abc; (3)$ and $abc = 4Rrs; (4)$.

From (3),(4) it follows that: $a(b^2 + c^2) \leq 4R^2s \Rightarrow (1)$ is true.

Note:

$$(2) \Leftrightarrow \frac{(x + y)(y + z)(z + x)}{4xyz} \geq \frac{x + z}{x + y} + \frac{x + y}{x + z} \Leftrightarrow$$

$$\frac{y+z}{4xyz} \geq \frac{1}{(x+y)^2} + \frac{1}{(x+z)^2}; (5)$$

But: $\frac{1}{(x+y)^2} \leq \frac{1}{4xy}; (6) \Leftrightarrow (x-y)^2 \geq 0$ and $\frac{1}{(x+z)^2} \leq \frac{1}{4xz}; (7)$

From (6),(7) it follows that (5) is true.

2357. In ΔABC the following relationship holds:

$$2\sum_{cyc}\left(h_a \cdot \sqrt{\frac{m_c}{w_a h_a h_b}}\right) \geq 3 + \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\because m_c \geq \frac{a+b}{2}\cos\frac{C}{2} \to \frac{m_c}{w_c} \geq \frac{a+b}{2}\cos\frac{C}{2} \cdot \frac{a+b}{2ab \cdot \cos\frac{C}{2}} = \frac{(a+b)^2}{4ab}$$

$$\to h_a \cdot \sqrt{\frac{m_c}{w_a h_a h_b}} \geq \frac{2sr}{a} \cdot \sqrt{\frac{(a+b)^2}{4ab} \cdot \frac{ab}{(2sr)^2}} = \frac{a+b}{2a}$$

$$2\sum_{cyc}\left(h_a \cdot \sqrt{\frac{m_c}{w_a h_a h_b}}\right) \geq \sum_{cyc}\frac{a+b}{a} = 3 + \sum_{cyc}\frac{b}{a}$$

2358. In ΔABC the following relationship holds:

$$\sum \frac{a}{b} \cdot \frac{(b+c)^3}{b^3+c^3} \leq \frac{4p^2}{r(4R+r)}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ (b+c)^3 \overset{Hölder}{\leqq} 4(b^3+c^3) \to \frac{1}{4}\sum\frac{a}{b} \cdot \frac{(b+c)^3}{b^3+c^3} \leq \sum\frac{a}{b} \overset{(*)}{\leqq} \frac{p^2}{r(4R+r)}$$

From Oppenheim's inequality, we have:

$$\sum a^2 x \geq 4F(a,b,c).\sqrt{\sum xy}, \forall x,y,z > 0$$

Where $F(a,b,c)$ is the area of tiangle with sides a,b,c.

Appliying this inequality for a tiangle with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$

$$\to \sum \sqrt{a}^2 x \overset{(1)}{\gtreqless} 4F(\sqrt{a}, \sqrt{b}, \sqrt{c}).\sqrt{\sum xy}$$

$$We\ have: 16F(\sqrt{a}, \sqrt{b}, \sqrt{c})^2 = 2\sum(\sqrt{a}\sqrt{b})^2 - \sum \sqrt{a}^4 = 2\sum ab - \sum a^2$$
$$= 4r(4R + r)$$

$$\to F(\sqrt{a}, \sqrt{b}, \sqrt{c}) = \frac{1}{2}\sqrt{r(4R + r)} \to (1) \leftrightarrow \sum ax \ge 2\sqrt{r(4R + r)\sum xy}, \forall x, y, z > 0$$

$$Let\ x = \frac{b}{a}, y = \frac{c}{b}, z = \frac{a}{c} \to \sum a.\frac{b}{a} \ge 2\sqrt{r(4R + r)\sum \frac{b}{a}.\frac{c}{b}} \leftrightarrow \sum b$$
$$\ge 2\sqrt{r(4R + r)\sum \frac{c}{a}}$$

$$\leftrightarrow p \ge \sqrt{r(4R + r)\sum \frac{a}{b}} \leftrightarrow \sum \frac{a}{b} \le \frac{p^2}{r(4R + r)} \to (*)\ is\ true.$$

$$Therefore, \quad \sum \frac{a}{b}.\frac{(b + c)^3}{b^3 + c^3} \le \frac{4p^2}{r(4R + r)}$$

2359. Let a, b, c be the lengths of the sides of a triangle with inradius r, circumradius R. Let r_a, r_b, r_c be the exradii of triangle. Prove that:

$$1728r^5 \le \frac{a^6}{r_a} + \frac{b^6}{r_b} + \frac{c^6}{r_c} \le 108R^4(R - r)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$\frac{a^6}{r_a} + \frac{b^6}{r_b} + \frac{c^6}{r_c} \ge 3\sqrt[3]{\frac{(abc)^6}{r_a r_b r_c}} \Rightarrow 3\sqrt[3]{\frac{(abc)^6}{r_a r_b r_c}} \ge 2^6 \cdot 3^3 \cdot r^5$$

$$\frac{(abc)^6}{r_a r_b r_c} \ge 2^{18} \cdot 3^6 \cdot r^{15}; (1)$$

But: $abc = 4Rrs$ **and** $r_a r_b r_c = s^2 r; (2)$**. From (1),(2) we must to prove:**

$$\frac{2^{12}\cdot s^6R^6r^6}{s^2r}\ge 2^{18}\cdot 3^6\cdot r^{15}\Leftrightarrow R^6s^4\ge 6^6\cdot r^{10}\Leftrightarrow R^3s^2\ge 6^3\cdot r^5;(3)$$

But: $R\ge 2r\Rightarrow R^3\ge 8r^3$ and $s^2\ge 27r^2$, then $R^3s^2\ge 6^3\cdot r^5\Rightarrow(3)$ is true.

Now, use Schur's inequality:

$$a^5(a-b)(a-c)+b^5(b-c)(b-a)+c^5(c-a)(c-b)\ge 0$$

$$a^5(a^2-ab-ac+bc)+b^5(b^2-ab-bc+ac)+c^5(c^2-bc-ac+ab)\ge 0$$

$$a^6(a-b-c)+b^6(b-c-a)+c^6(c-a-b)+a^5bc+ab^5c+abc^5\ge 0\Leftrightarrow$$

$$a^6(b+c-a)+b^6(c+a-b)+c^6(a+b-c)\le abc(a+b+c)\Leftrightarrow$$

$$2a^6(s-a)+2b^6(s-b)+2c^6(s-c)\le abc(a+b+c)$$

But: $r_a=\frac{F}{s-a}\Rightarrow s-a=\frac{F}{r_a}\Rightarrow$

$$2F\left(\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_c}\right)\le abc(a^4+b^4+c^4);(4)$$

But: $F=\frac{abc}{4R};(5)$. From (4),(5) it follows that:

$$\frac{a^6}{r_a}+\frac{b^6}{r_b}+\frac{c^6}{r_c}\le 2R(a^4+b^4+c^4);(6)$$

We must show: $2R(a^4+b^4+c^4)\le 108R^4(R-r)\Leftrightarrow$

$$a^4+b^4+c^4\le 54R^3(R-r);(7)$$

But: $a^4+b^4+c^4=2(s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2);(8)$

From (7), (8) we must to prove that:

$$s^4-2s^2(4Rr+3r^2)+r^2(4R+r)^2-27R^3(R-r)\le 0;(9)$$

Now, let $f(p^2)=(p^2-x_1)(p^2-x_2);(9)\Leftrightarrow$

$$\left[s^2-r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)}\right]$$
$$\cdot\left[s^2-r(4R+3r)+\sqrt{8r^3(2R+r)+27R^3(R-r)}\right]\le 0;(10)$$

(10) is true if $x_1\le s\le x_2;x_1,x_2$ –square, then we must show that:

$$s^2-r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3(R-r)}\le s^2;(11)\text{ and}$$

$$s^2-r(4R+3r)+\sqrt{8r^3(2R+r)+27R^3(R-r)}\ge s^2;(12)$$

For (11) using Gerretsen inequality $s^2\ge 16Rr-5r^2$ it follows that:

$$4Rr+3r^2-\sqrt{8r^3(2R+r)+27R^3\,(R-r)}\le 16Rr-5r^2 \Leftrightarrow$$

$$8r^2-12Rr\le\sqrt{8r^3(2R+r)+27R^3\,(R-r)} \Leftrightarrow$$

$$(8r^2-12Rr)^2\le 8r^3(2R+r)+27R^3(R-r)\Leftrightarrow R\ge 2r(Euler).$$

For (12) using Gerretsen inequality $s^2\le 4R^2+4Rr+3r^2$ it follows that:

$$4R^2+4Rr+3r^2\le r(4R+3r)-\sqrt{8r^3(2R+r)+27R^3\,(R-r)} \Leftrightarrow$$

$$8r^4+16Rr^3-27R^3r+11R^4\ge 0$$

Let $x=\frac{R}{2r}\ge 1(Euler)$, we must show that: $22x^4-27x^3+4x+1\ge 0$ and with Horner and Rolle sequence, it follows that: $(x-1)\left(11x^3+(x-1)(11x^2+6x+1)\right)\ge 0$

2360. If $m\in[0,\infty)$, $x,y,z,t\in(0,\infty)$, then in any triangle ABC, with usual notations holds

$$\sum_{cyclic}\frac{(xa^2+ymb^2)^{m+1}}{(zw_c^2+tw_a^2)^m}\ge\frac{(4x+3y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S$$

Proposed by D.M Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Tran Hong-DongThap-Vietnam

$$\sum\frac{\left(xa^2+ym_b^2\right)^{m+1}}{(zw_c^2+tw_a^2)^m}\overset{Holder}{\ge}\frac{\left(xa^2+ym_b^2+xb^2+ym_c^2+xc^2+ym_a^2\right)^{m+1}}{\left(zw_c^2+tw_a^2+zw_a^2+tw_b^2+zw_b^2+tw_c^2\right)^m}$$

$$=\frac{\left(x(a^2+b^2+c^2)+y\left(m_a^2+m_b^2+m_c^2\right)\right)^{m+1}}{(z+t)^m\left(w_a^2+w_b^2+w_c^2\right)^m}=$$

$$=\frac{\left(\frac{4x}{3}\left(m_a^2+m_b^2+m_c^2\right)+y\left(m_a^2+m_b^2+m_c^2\right)\right)^{m+1}}{(z+t)^m\left(w_a^2+w_b^2+w_c^2\right)^m}$$

$$=\frac{(4x+3y)^{m+1}\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{3^{m+1}(z+t)^m\left(w_a^2+w_b^2+w_c^2\right)^m}\overset{m_a\ge w_a(analogs)}{\ge}\frac{(x+y)^{m+1}}{(z+t)^m}\cdot\left(m_a^2+m_b^2+m_c^2\right)=$$

$$=\frac{(x+y)^{m+1}}{3^{m+1}(z+t)^m}\cdot\frac{3}{4}\cdot\left(a^2+b^2+c^2\right)$$

$$\overset{Finsler-Hadwinger}{\geq} \frac{(x+y)^{m+1}}{3^{m+1}(z+t)^m}\cdot\frac{3}{4}\cdot 4\sqrt{3}\cdot S = \frac{(x+y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}\cdot S$$

2361. In ΔABC, I –incenter, R_a, R_b, R_c –circumradii of $\Delta IBC, \Delta ICA, \Delta IAB$. Prove that:

$$\frac{a^2\cdot R_b^3R_c^3}{R_a}+\frac{b^2\cdot R_c^3R_a^3}{R_b}+\frac{c^2\cdot R_a^3R_b^3}{R_c}\geq\frac{16R^3F^2}{3}$$

Proposed by Florică Anastase-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have that: $\mu(BIC) = \pi-\frac{B+C}{2} = \frac{\pi}{2}+\frac{A}{2}\to sin\left(\frac{\pi}{2}+\frac{A}{2}\right) = sin(BIC) = \frac{a}{2R_a}$

$$R_a = \frac{a}{2cos\frac{A}{2}} = 2R\cdot sin\frac{A}{2}; (and\ analogs)$$

$$\to\sum_{cyc}\frac{a^2\cdot R_b^3R_c^3}{R_a} = (2R)^5\sum_{cyc}\frac{a^2\cdot sin^3\frac{B}{2}\cdot sin^3\frac{C}{2}}{sin\frac{A}{2}} =$$

$$= (2R)^5\sum_{cyc}\frac{\left(4R\cdot cos\frac{A}{2}\cdot sin\frac{A}{2}\right)^2\cdot sin^3\frac{B}{2}sin^3\frac{C}{2}}{sin\frac{A}{2}} =$$

$$= 8R^4\sum_{cyc}\left(4R\prod_{cyc}sin\frac{A}{2}\right)\left(4R\cdot cos\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\right)^2 = 8R^4\sum_{cyc}r(s-a)^2\overset{CBS}{\geq}$$

$$\geq 8R^4r\cdot\frac{1}{3}\left(\sum_{cyc}(s-a)\right)^2 = \frac{8R^4rs^2}{3}$$

$$\to\sum_{cyc}\frac{a^2\cdot R_b^3R_c^3}{R_a}\geq\frac{8R^4rs^2}{3}\overset{EULER}{\geq}\frac{16R^3(sr)^2}{3} = \frac{16R^3F^2}{3}$$

Therefore,

$$\frac{a^2\cdot R_b^3R_c^3}{R_a}+\frac{b^2\cdot R_c^3R_a^3}{R_b}+\frac{c^2\cdot R_a^3R_b^3}{R_c}\geq\frac{16R^3F^2}{3}$$

Solution 2 by proposer

$$\because R_aR_bR_c = 2R^2r;\ (1)$$

$$\mu(\angle BIC) = \pi - \frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$$

$$2R_a = \frac{a}{\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)} = \frac{2R\cdot\sin A}{\cos\frac{A}{2}} = 4R\cdot\sin\frac{A}{2} \Rightarrow R_a = 2R\cdot\sin\frac{A}{2}$$

$$R_aR_bR_c = 8R^3\cdot\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2R^2r$$

$$\because \sum_{cyc} a\cdot R_b^2R_c^2 = R^2\cdot abc;\ (2)$$

$$\sum_{cyc} a\cdot R_b^2R_c^2 = 32R^5\cdot\sum_{cyc}\sin A\cdot\sin^2\frac{B}{2}\cdot\sin^2\frac{C}{2} =$$

$$= 64R^5\cdot\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\left(\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\right)$$

But,

$$\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} = \sin\frac{A}{2}\left(\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{B}{2}\right) + \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} =$$

$$= \sin\frac{A}{2}\sin\frac{B+C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} = \cos\frac{A}{2}\left(\cos\frac{B+C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right) =$$

$$= \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Hence,

$$\sum_{cyc} a\cdot R_b^2R_c^2 = 8R^5\cdot\sin A\sin B\sin C = 4R^3F = \frac{4R^3\cdot abc}{4R} = R^2\cdot abc$$

$$\frac{a^2\cdot R_b^3R_c^3}{R_a} + \frac{b^2\cdot R_c^3R_a^3}{R_b} + \frac{c^2\cdot R_a^3R_b^3}{R_c} = \frac{a^2\cdot R_b^4R_c^4}{R_aR_bR_c} + \frac{b^2\cdot R_c^4R_a^2}{R_aR_bR_c} + \frac{c^2\cdot R_a^4R_b^4}{R_aR_bR_c} =$$

$$= \frac{\left(a\cdot R_b^2R_c^2\right)^2}{R_aR_bR_c} + \frac{(b\cdot R_c^2R_a^2)^2}{R_aR_bR_c} + \frac{\left(c\cdot R_a^2R_b^2\right)^2}{R_aR_bR_c} \overset{Bergstrom}{\geq}$$

$$\geq \frac{\left(a\cdot R_b^2R_c^2 + b\cdot R_c^2R_a^2 + c\cdot R_a^2R_b^2\right)^2}{3R_aR_bR_c} \overset{(1),(2)}{=} \frac{(R^2\cdot abc)^2}{3\cdot 2R^2r} = \frac{R^2\cdot(abc)^2}{6r} =$$

$$= \frac{8R^4F^2}{3r} \overset{Euler}{\geq} \frac{16R^3F^2}{3}$$

2362. In ΔABC the following relationship holds:

$$\sum \frac{a^4}{h_b h_c} \leq \sum \frac{a^4}{r_b r_c}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c} \overset{AM-GM}{\geq} \frac{2}{\sqrt{r_b r_c}} \to \frac{2}{h_a} \geq \frac{2}{\sqrt{r_b r_c}} \to \frac{1}{r_b r_c} \leq \frac{1}{h_a^2}; (and\ analogs)$$

$$\sum_{cyc} \frac{a^4}{r_b r_c} \leq \sum_{cyc} \frac{a^4}{h_a^2} = \frac{1}{4F^2} \sum_{cyc} a^6 = \frac{a^6 + b^6 + c^6}{4F^2}; (1)$$

$$\sum_{cyc} \frac{a^4}{h_b h_c} = \sum_{cyc} \frac{a^4}{\frac{2F}{b} \cdot \frac{2F}{c}} = \frac{1}{4F^2} \sum_{cyc} bca^4 = \frac{abc(a^3 + b^3 + c^3)}{4F^4}; (2)$$

From $(1), (2)$ we need to prove that:

$$\frac{a^6 + b^6 + c^6}{4F^2} \geq \frac{abc(a^3 + b^3 + c^3)}{4F^4} \Leftrightarrow (a^3)^2 + (b^3)^2 + (c^3)^2 \geq abc(a^3 + b^3 + c^3); (2)$$

Because:

$$abc \overset{AM-GM}{\leq} \frac{a^3 + b^3 + c^3}{3} \to abc(a^3 + b^3 + c^3) \leq$$

$$\leq \frac{(a^3 + b^3 + c^3)^2}{3} \overset{(3)}{\leq} (a^3)^2 + (b^3)^2 + (c^3)^2$$

$$(3) \Leftrightarrow 3(x^2 + y^2 + z^2) \geq (x + y + z)^2; (x = a^3, y = b^3, z = c^3) \Leftrightarrow$$

$x^2 + y^2 + z^2 \geq xy + yz + zx; (true) \to (3) \to (2)$ is true. Proved.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{a^4}{h_b h_c} \leq \sum \frac{a^4}{r_b r_c} (*)$$

$$(*) \leftrightarrow \frac{1}{4} \sum a^4 bc \leq \sum a^4 (s - b)(s - c) \leftrightarrow \sum a^4 [(a - b + c)(a + b - c) - bc] \geq 0$$

$$\leftrightarrow \sum a^4 [(a - b)(a - c) + ab - b^2 + ac - c^2] \geq 0$$

$$\leftrightarrow \sum a^4(a-b)(a-c) + \sum a^4 b(a-b) - \sum a^4 c(c-a) \geq 0$$

$$From\ Schur's\ inequality, we\ have: \sum a^4(a-b)(a-c) \geq 0$$

$$So, it\ is\ suffices\ to\ prove: \sum a^4 b(a-b) - \sum a^4 c(c-a) \geq 0$$

$$\leftrightarrow \sum a^4 b(a-b) - \sum b^4 a(a-b) \geq 0 \leftrightarrow \sum ab(a-b)(a^3-b^3) \geq 0$$

$$\leftrightarrow \sum ab(a-b)^2(a^2+ab+b^2) \geq 0, which\ is\ true.$$

$$Therefore, \quad \sum \frac{a^4}{h_b h_c} \leq \sum \frac{a^4}{r_b r_c}$$

2363. In ΔABC the following relationship holds:

$$\frac{12r^2}{R} \leq \frac{a^2}{m_b+m_c} + \frac{b^2}{m_c+m_a} + \frac{c^2}{m_a+m_b} < 4R$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

From Bergstrom inequality we have:

$$\frac{a^2}{m_b+m_c} + \frac{b^2}{m_c+m_a} + \frac{c^2}{m_a+m_b} \geq \frac{(a+b+c)^2}{2(m_a+m_b+m_c)} = \frac{2s^2}{m_a+m_b+m_c}; (1)$$

From (1) we must to prove: $\frac{2s^2}{m_a+m_b+m_c} \geq \frac{12r^2}{R} \Leftrightarrow \frac{s^2}{m_a+m_b+m_c} \geq \frac{6r^2}{R}; (2)$

But: $m_a + m_b + m_c \leq \frac{9R}{2}; (3)$. From (2),(3) we must to show:

$\frac{2s^2}{9R} \geq \frac{6r^2}{R} \Leftrightarrow s^2 \geq 27r^2$, which is true from Mitrinovic.

In ΔABC: $m_a \geq \frac{b^2+c^2}{4R} \Rightarrow m_b + m_c \geq \frac{2a^2+b^2+c^2}{4R} \Rightarrow$

$$\frac{a^2}{m_b+m_c} + \frac{b^2}{m_c+m_a} + \frac{c^2}{m_a+m_b} \geq 4R\left(\frac{a^2}{2a^2+b^2+c^2} + \frac{b^2}{a^2+2b^2+c^2} + \frac{c^2}{a^2+b^2+2c^2}\right)$$

$$\Leftrightarrow \sum_{cyc} \frac{a^2}{2a^2+b^2+c^2} < 1; (4)$$

But: $\frac{a^2}{2a^2+b^2+c^2} < \frac{a^2}{a^2+b^2+c^2} \Rightarrow \sum_{cyc}\frac{a^2}{2a^2+b^2+c^2} < \sum_{cyc}\frac{a^2}{a^2+b^2+c^2} = 1$.

2364. In ΔABC the following relationship holds:

$$4\sqrt{3}\frac{r}{R} \le \frac{a}{\sqrt{m_b m_c}} + \frac{b}{\sqrt{m_c m_a}} + \frac{c}{\sqrt{m_a m_b}} \le \sqrt{3}\frac{R}{r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

For LHS, using AM-GM: $\frac{a}{\sqrt{m_b m_c}} + \frac{b}{\sqrt{m_c m_a}} + \frac{c}{\sqrt{m_a m_b}} \ge 3\sqrt[3]{\frac{abc}{m_a m_b m_c}}$.

We must to prove that: $3\sqrt[3]{\frac{abc}{m_a m_b m_c}} \ge 4\sqrt{3}\frac{r}{R}; (1)$

But: $abc \ge 24\sqrt{3}r^3; (2)$. From (1),(2) it is suffices to prove that:

$$\frac{3\cdot 2\sqrt{3}r}{\sqrt[3]{m_a m_b m_c}} \ge 4\sqrt{3}\frac{r}{R} \Leftrightarrow \frac{1}{\sqrt[3]{m_a m_b m_c}} \ge \frac{2}{3R} \Leftrightarrow \sqrt[3]{m_a m_b m_c} \le \frac{3}{2}R; (3)$$

But: $\sqrt[3]{m_a m_b m_c} \le \frac{m_a+m_b+m_c}{3} (AM-GM); (4)$

From (3),(4) it follows that: $m_a + m_b + m_c \le \frac{9}{2}R$, which is clearly true.

For RHS, we have: $m_a \ge \frac{b^2+c^2}{4R} \ge \frac{bc}{2R} \Rightarrow m_a m_b \ge \frac{abc^2}{4R^2} \Rightarrow \frac{1}{\sqrt{m_a m_b}} \le \frac{2R}{\sqrt{abc^2}} \Rightarrow \frac{c}{\sqrt{m_a m_b}} \le \frac{2R}{\sqrt{ab}}$

$$\frac{a}{\sqrt{m_b m_c}} + \frac{b}{\sqrt{m_c m_a}} + \frac{c}{\sqrt{m_a m_b}} \le 2R\left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}\right)$$

It remains to prove that: $2R\left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}\right) \le \sqrt{3}\frac{R}{r} \Leftrightarrow \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \le \frac{\sqrt{3}}{2r}; (5)$

From BCS inequality: $\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \le \sqrt{3\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)}; (6)$

But: $\frac{1}{ab} + \frac{1}{bc} + 1ca \le \frac{1}{4r^2}; (7)$. From (6),(7) it follows that:

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \le \frac{\sqrt{3}}{2r} \Rightarrow (5) \text{ is true.}$$

2365. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(r_b+r_c)^2}{b^2+c^2}\le\frac{9R}{4r}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$Now,\ (b+c)^2\ge 32Rr\cos^2\frac{A}{2}\overset{by\ (i)}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\ge 0\Leftrightarrow (b+c-2a)^2\ge 0\to true\ \therefore b^2+c^2\ge\frac{(b+c)^2}{2}$$

$$\ge 16Rr\cos^2\frac{A}{2}\Rightarrow\frac{(r_b+r_c)^2}{b^2+c^2}\le\frac{(r_b+r_c)^2}{16Rr\cos^2\frac{A}{2}}$$

$$\overset{via\ (i)}{=}\frac{16R^2\cos^4\frac{A}{2}}{16Rr\cos^2\frac{A}{2}}=\frac{R}{2r}(1+\cos A)\Rightarrow\frac{(r_b+r_c)^2}{b^2+c^2}$$

$$\le\frac{R}{2r}(1+\cos A)\text{ and analogs}\overset{summing\ up}{\Rightarrow}\sum\frac{(r_b+r_c)^2}{b^2+c^2}$$

$$\le\frac{R}{2r}\left(3+1+\frac{r}{R}\right)=\frac{8R+2r}{4r}$$

$$\overset{Euler}{\le}\frac{8R+R}{4r}\ \therefore\sum\frac{(r_b+r_c)^2}{b^2+c^2}\le\frac{9R}{4r}\ (QED)$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc}\frac{(r_b+r_c)^2}{b^2+c^2}=\sum_{cyc}\frac{\left(\frac{F}{s-b}+\frac{F}{s-c}\right)^2}{b^2+c^2}=F^2\sum_{cyc}\frac{a^2}{((s-b)(s-c))^2(b^2+c^2)}\overset{\frac{1}{b^2+c^2}\le\frac{1}{2bc}}{\le}$$

$$\leq \frac{F^2}{2}\sum_{cyc}\frac{a^2}{bc\left((s-b)(s-c)\right)^2} = \frac{F^2}{2}\cdot\frac{\sum a^3(s-a)^2}{abc\left((s-a)(s-b)(s-c)\right)^2} = \Omega$$

$$\sum a^3(s-a)^2 = \sum a^3(s^2-2as+a^2) = \left(\sum a^3\right)s^2 - 2\left(\sum a^4\right)s + \sum a^5 =$$

$$= (2s^3-(12Rr+6r^2)s)s^2 - 2(2s^4-(16Rr+12r^2)s^2+2(4Rr+r^2)^2)s +$$

$$+2s^5-(20Rr+20r^2)s^3+(80R^2r^2+60Rr^3+10r^4)s =$$

$$= s(-(12Rr+6r^2)s^2+2(16Rr+12r^2)s^2-4(4Rr+r^2)^2 -$$

$$-(20Rr+20r^2)s^2+80R^2r^2+60Rr^3+10r^4 =$$

$$= s(-2r^2s^2+16R^2r^2+28Rr^3+6r^4) = 2sr^2(-s^2+8R^6+14Rr+3r^2)$$

$$\sum_{cyc}\frac{(r_b+r_c)^2}{b^2+c^2} \leq \Omega = \frac{s^2r^2}{2}\cdot\frac{2sr^2(-s^2+8R^2+14Rr+3r^2)}{4Rrs(sr^2)^2} =$$

$$= \frac{-s^2+8R62+14Rr+3r^2}{4Rr} \overset{(1)}{\leq} \frac{9R}{4r}$$

$$(1) \Leftrightarrow -s^2+8R^2+14Rr+3r^2 \leq 9R^2 \Leftrightarrow s^2+R^2 \geq 14Rr+3r^2$$

But $s^2 \geq 16Rr-5r^2(Gerretsen) \to s^2+R^2 \geq 16Rr-5r^2+R^2 \overset{(2)}{\geq} 14Rr+3r^2$

$$(2) \Leftrightarrow R^2+2Rr-8r^2 \geq 0 \Leftrightarrow (R-2r)(R+4r) \geq 0\ \left(true\ by R \geq 2r(Euler)\right)$$

$\to (2) \to (1)$ is true. Proved.

2366. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\prod_{cyc}\left(\sqrt{\frac{2r_a(n_a+h_a)}{w_a^2}}-1\right) \leq \frac{R}{2r}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tager-Morocco

Let's to prove: $\frac{n_a}{h_a} \leq \frac{R}{r} - 1$

$$\frac{n_a^2}{h_a^2} = \frac{s^2-2r_ah_a}{h_a^2} = \frac{a^2}{4r^2} - \frac{a}{s-a} = \frac{a^2}{4r^2} - \frac{s}{s-a} + 1 =$$

$$= \frac{a^2(s-a)-4sr^2}{4r^2(s-a)} + 1 = \frac{a^2(s-a)-4(s-a)(s-b)(s-c)}{4r^2(s-a)} + 1 =$$

$$=\frac{a^2-(a-b+c)(a+b-c)}{4r^2}+1\le\frac{(b-c)^2}{4r^2}+1$$

$$\to\frac{n_a^2}{h_a^2}=\frac{R^2(sinB-sinC)^2}{r^2}+1\overset{?}{\le}\left(\frac{R}{r}-1\right)^2=\frac{R^2}{r^2}\left(1-\frac{2r}{R}\right)+1$$

$$\left(2sin\frac{B-C}{2}cos\frac{B+C}{2}\right)^2\le1-8\prod_{cyc}sin\frac{A}{2}=1-4sin\frac{A}{2}\left(2sin\frac{B}{2}sin\frac{C}{2}\right)$$

$$4sin^2\frac{A}{2}\left(1-cos^2\frac{B-C}{2}\right)\le1-4sin\frac{A}{2}\left(cos\frac{B-C}{2}-cos\frac{B+C}{2}\right)$$

$$4sin^2\frac{A}{2}-4sin^2\frac{A}{2}cos^2\frac{B-C}{2}\le1-4sin\frac{A}{2}cos\frac{B-C}{2}+4sin^2\frac{A}{2}$$

$$\left(2sin\frac{A}{2}cos\frac{B-C}{2}-1\right)^2\ge0-true.$$

$$\to\frac{n_a}{h_a}\overset{(1)}{\le}\frac{R}{r}-1;(and\ analogs)$$

$$\frac{2r_a(n_a+h_a)}{w_a^2}=\frac{2sr}{s-a}\cdot\frac{(b+c)^2(n_a+h_a)}{4bc\cdot s(s-a)}=$$

$$=\frac{r(b+c)^2(n_a+h_a)}{2(s-a)^2\cdot2Rh_a}=\frac{r(b+c)^2}{4R(s-a)^2}\cdot\left(\frac{n_a}{h_a}+1\right)\overset{(1)}{\le}\frac{r(b+c)^2}{4R(s-a)^2}\cdot\frac{R}{r}=\frac{(b+c)^2}{4(s-a)^2}$$

$$\to\prod_{cyc}\left(\sqrt{\frac{2r_a(n_a+h_a)}{w_a^2}}-1\right)\le\prod_{cyc}\left(\frac{b+c}{2(s-a)}-1\right)=\prod_{cyc}\frac{a}{2(s-a)}=$$

$$=\frac{4Rrs}{8sr^2}=\frac{R}{2r}$$

Therefore,

$$\prod_{cyc}\left(\sqrt{\frac{2r_a(n_a+h_a)}{w_a^2}}-1\right)\le\frac{R}{2r}$$

2367. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\prod_{cyc}\left(\sqrt{\frac{2r_a(n_a+h_a)}{w_a^2}}-1\right)\le\frac{R}{2r}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tager-Morocco

Let's to prove: $\frac{n_a}{h_a} \le \frac{R}{r} - 1$

$$\frac{n_a^2}{h_a^2} = \frac{s^2 - 2r_a h_a}{h_a^2} = \frac{a^2}{4r^2} - \frac{a}{s-a} = \frac{a^2}{4r^2} - \frac{s}{s-a} + 1 =$$

$$= \frac{a^2(s-a) - 4sr^2}{4r^2(s-a)} + 1 = \frac{a^2(s-a) - 4(s-a)(s-b)(s-c)}{4r^2(s-a)} + 1 =$$

$$= \frac{a^2 - (a-b+c)(a+b-c)}{4r^2} + 1 \le \frac{(b-c)^2}{4r^2} + 1$$

$$\to \frac{n_a^2}{h_a^2} = \frac{R^2(\sin B - \sin C)^2}{r^2} + 1 \overset{?}{\le} \left(\frac{R}{r} - 1\right)^2 = \frac{R^2}{r^2}\left(1 - \frac{2r}{R}\right) + 1$$

$$\left(2\sin\frac{B-C}{2}\cos\frac{B+C}{2}\right)^2 \le 1 - 8\prod_{cyc}\sin\frac{A}{2} = 1 - 4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$4\sin^2\frac{A}{2}\left(1 - \cos^2\frac{B-C}{2}\right) \le 1 - 4\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right)$$

$$4\sin^2\frac{A}{2} - 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2} \le 1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}$$

$$\left(2\sin\frac{A}{2}\cos\frac{B-C}{2} - 1\right)^2 \ge 0 - true.$$

$$\to \frac{n_a}{h_a} \overset{(1)}{\le} \frac{R}{r} - 1; (and\ analogs)$$

$$\frac{2r_a(n_a + h_a)}{w_a^2} = \frac{2sr}{s-a} \cdot \frac{(b+c)^2(n_a + h_a)}{4bc \cdot s(s-a)} =$$

$$= \frac{r(b+c)^2(n_a + h_a)}{2(s-a)^2 \cdot 2Rh_a} = \frac{r(b+c)^2}{4R(s-a)^2} \cdot \left(\frac{n_a}{h_a} + 1\right) \overset{(1)}{\le} \frac{r(b+c)^2}{4R(s-a)^2} \cdot \frac{R}{r} = \frac{(b+c)^2}{4(s-a)^2}$$

$$\to \prod_{cyc}\left(\sqrt{\frac{2r_a(n_a + h_a)}{w_a^2}} - 1\right) \le \prod_{cyc}\left(\frac{b+c}{2(s-a)} - 1\right) = \prod_{cyc}\frac{a}{2(s-a)} = \frac{4Rrs}{8sr^2} = \frac{R}{2r}$$

Therefore,

$$\prod_{cyc}\left(\sqrt{\frac{2r_a(n_a + h_a)}{w_a^2}} - 1\right) \le \frac{R}{2r}$$

2368. In any scalene ΔABC the following relationship holds:

$$S=\frac{bc(1+a^2)}{a(b-a)(c-a)}+\frac{ca(1+b^2)}{b(a-b)(c-b)}+\frac{ab(1+c^2)}{c(c-a)(c-b)}>\frac{\sqrt{3}}{R}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\frac{1}{(b-a)(c-a)}=\frac{-1}{c-b}\left(\frac{1}{a-b}+\frac{1}{c-a}\right)$$

$$\to S=\left(-\frac{bc(1+a^2)}{a(a-b)(c-b)}+\frac{ca(1+b^2)}{b(a-b)(c-b)}\right)+\left(-\frac{bc(1+a^2)}{a(c-a)(c-b)}+\frac{ab(1+c^2)}{c(c-a)(c-b)}\right)$$

$$=c.\frac{-b^2(1+a^2)+a^2(1+b^2)}{ab(a-b)(c-b)}+b.\frac{-c^2(1+a^2)+a^2(1+c^2)}{ac(c-a)(c-b)}$$

$$=\frac{c(a-b)(a+b)}{ab(a-b)(c-b)}+\frac{b(a-c)(a+c)}{ac(c-a)(c-b)}$$

$$=\frac{c(a+b)}{ab(c-b)}-\frac{b(a+c)}{ac(c-b)}=\frac{c^2(a+b)-b^2(a+c)}{abc(c-b)}=\frac{a(c-b)(c+b)+bc(c-b)}{abc(c-b)}$$

$$=\frac{ab+bc+ca}{abc}$$

$$\to S=\sum\frac{1}{a}\overset{CBS}{\geq}\frac{9}{\sum a}=\frac{9}{2s}\overset{Mitrinovic}{\geq}\frac{9}{3\sqrt{3}R}=\frac{\sqrt{3}}{R}\quad (\Delta ABC \text{ is scalene})$$

Therefore,

$$\frac{bc(1+a^2)}{a(b-a)(c-a)}+\frac{ca(1+b^2)}{b(a-b)(c-b)}+\frac{ab(1+c^2)}{c(c-a)(c-b)}>\frac{\sqrt{3}}{R}$$

2369. In ΔABC the following relationship holds:

$$\frac{6R}{r}\leq\sum\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\leq\frac{3}{2}\left(\frac{R}{r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{\prod(r_a+r_b)}{r_ar_br_c}\cdot\sum\frac{r_a}{r_b+r_c}=\frac{4Rs^2}{rs^2}\sum\frac{r_a}{r_b+r_c}\overset{Nesbitt}{\geq}\frac{4R}{r}\cdot\frac{3}{2}=\frac{6R}{r}$$

Now,

$$\sum\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{4R}{r}\sum\frac{r_a}{r_b+r_c}=\frac{4R}{r}\sum\frac{r_a(r_b+r_c)}{(r_b+r_c)^2}\overset{AM-GM}{\leq}\frac{4R}{r}\sum\frac{r_a(r_b+r_c)}{4r_br_c}$$

$$=\frac{R}{rr_ar_br_c}\sum r_a{}^2(r_b+r_c)=\frac{R}{r^2s^2}\left[\left(\sum r_a\right)\left(\sum r_ar_b\right)-3r_ar_br_c\right]$$

$$=\frac{R}{r^2s^2}[(4R+r)s^2-3rs^2]$$

$$\to\sum\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\leq\frac{2R(2R-r)}{r^2}\overset{?}{\leq}\frac{3}{2}\left(\frac{R}{r}\right)^3$$

$$\leftrightarrow 3R^2-8Rr+4r^2\geq 0$$

$\leftrightarrow (R-2r)\big(2R+(R-2r)\big)\geq 0$, which is true from Euler $(R\geq 2r)$.

Therefore,

$$\frac{6R}{r}\leq\sum\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\leq\frac{3}{2}\left(\frac{R}{r}\right)^3$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc}r_a^3=(4R+r)^2-12s^2R;\ \sum_{cyc}r_br_c(r_b+r_c)=(4R-2r)s^2;\ \ r_ar_br_c=s^2r$$

$$Let\ \Omega=\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{\sum r_a(r_a+r_b)(r_a+r_c)}{r_ar_br_c}=$$

$$=\frac{r\sum_a^3+\sum r_br_c(r_b+r_c)+3r_ar_br_c}{r_ar_br_c}=\frac{(4R+r)^3-12s^2R+(4R-2r)s^2+3s^2r}{s^2r}=$$

$$=\frac{(4R+r)^3+(r-8R)s^2}{s^2r}$$

$$\Omega\overset{(1)}{\leq}\frac{3}{2}\left(\frac{R}{r}\right)^3\Leftrightarrow\frac{(4R+r)^3+(r-8R)s^2}{s^2r}\leq\frac{3}{2}\left(\frac{R}{r}\right)^3\Leftrightarrow$$

$$2r^2[(4R+r)^3+(r-8R)s^2]\leq 3R^3s^2\Leftrightarrow 2r^2(4R+r)^2+(2r^3-16Rr^2)s^2\leq 3R^3s^2$$

$$(3R^3+16Rr^2-2r^3)s^2 \ge 2r^2(4R+r)^3$$

But $s^2 \ge 16Rr-5r^2(Gerretsen)$; $3R^3+16Rr^2-2r^3 > 3R^3-2r^3 \overset{Euler}{\gtrsim} 22r^3 > 0$

$$\to (3R^3+16Rr^2-2r^3)s^2 \ge (3R^3+16Rr^2-2r^3)(16Rr-5r^2) \overset{(2)}{\gtrsim} 2r^2(4R+r)^3$$

$$(2) \overset{t=\frac{R}{r}\ge 2}{\Longleftrightarrow} (3t^3+16t-2)(16t-2) \ge 2(4t+1)^3$$

$$\Leftrightarrow 48t^4-143t^3+160t^2-136t+8 \ge 0 \Leftrightarrow (t-2)(t^2(48t-47)+66t-4) \ge 0$$

True because $t \ge 2 \to t-2 \ge 0 \to t^2(48t-47)+66t-4 \ge 324 > 0$

$\to (2) \to (1)$ is true.

$$\Omega \overset{(3)}{\gtrsim} \frac{6R}{r};\ (3) \Leftrightarrow \frac{(4R+r)^3+(r-8R)s^2}{s^2r} \ge \frac{6R}{r} \Leftrightarrow (4R+r)^3+(r-8R)s^2 \ge 6Rs^2$$

$$\Leftrightarrow (4R+r)^3 \ge (14R-r)s^2$$

But $s^2 \le 4R^2+4Rr+3r^2(Gerretsen) \to$

$$(14Rr-r)s^{22} \le (14R-r)(4R^2+4Rr+3r^2) \overset{(4)}{\lesssim} (4R+r)^3$$

$$(4) \overset{t=\frac{R}{r}\ge 2}{\Longleftrightarrow} (14t-1)(4t^2+4t+3) \le (4t+1)^3 \Leftrightarrow 8t^3-4t^2-26t+4 \ge 0$$

$\Leftrightarrow 2(t-2)(4t^2+6t-1) \ge 0$, which is true because $t \ge 2 \to t-2 \ge 0$,

$4t^2+6t-1 > 6t-1 > 11 > 0 \to (4) \to (3)$ is true. Proved.

2370. In ΔABC the following relationship holds:

$$\frac{rr_a}{r_br_c}+\frac{rr_b}{r_cr_a}+\frac{rr_c}{r_ar_b} \le \frac{R^2}{2r^2}-1$$

Proposed by Kostas Geronikolas-Greece

Solution by Marian Ursărescu-Romania

$$r\sum_{cyc}\frac{r_a}{r_b+r_c} = r\cdot\frac{\sum r_a^2}{r_ar_br_c};(1)$$

$$r_a^2+r_b^2+r_c^2 = (4R+r)^2-2s^2;(2);\ \ r_ar_br_c = s^2r;(3)$$

From $(1),(2),(3)$ we must to prove that:

$$\frac{r[(4R+r)^2-2s^2]}{s^2r} \le \frac{R^2-2r^2}{2r^2} \Leftrightarrow 2r^2(4R+r)^2-4s^2r^2 \le s^2R^2-2s^2r^2$$

$$\Leftrightarrow 2r^2(4R+r)^2 \le s^2(R^2+2r^2); (4)$$

From $3r(4R+r) \le s^2(Doucet); (5)$. From $(4), (5)$ we must to show that:

$$3(R^2+2r^2) \ge 2r(4R+r) \Leftrightarrow 3R^2+6r^2 \ge 8Rr+2r^2 \Leftrightarrow$$

$3R^2-8Rr+4r^2 \ge 0 \Leftrightarrow (R-2r)(3R-2r) \ge 0$, which is true from $R \ge 2r(Euler)$.

2371. In any $\Delta\, ABC$ the following relationship holds:

$$\frac{h_a+h_b+h_c}{2r} \ge \sum \frac{n_a\sqrt{r_br_c-r^2}}{m_a^2}$$

Proposed by Bogdan Fuştei-Romania

Solution by Soumava-Chakraborty-Kolkata-India

$$Stewart's\ theorem \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$

$$= an_a^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2-as^2 \Rightarrow an_a^2 = as^2+s(2bccosA-2bc)$$

$$= as^2-4sbcsin^2\frac{A}{2} = as^2-4s(s-b)(s-c)$$

$$\Rightarrow a^2n_a^2 = a^2s^2-sa[a^2-(b-c)^2] \therefore a^2n_a^2 \overset{(i)}{=} sa[a(s-a)+(b-c)^2]$$

$$Again, r_br_c-r^2 = s(s-a)-\frac{s(s-b)(s-c)(s-a)}{s^2} = s(s-a)\left[1-\frac{a^2-(b-c)^2}{4s^2}\right]$$

$$\therefore r_br_c-r^2 \overset{(ii)}{=} s(s-a)\left[\frac{4s^2-a^2+(b-c)^2}{4s^2}\right]$$

$$Now, \frac{h_a}{2r} \ge \frac{n_a\sqrt{r_br_c-r^2}}{m_a^2} \Leftrightarrow \frac{s^2}{a^2} \ge \frac{n_a^2(r_br_c-r^2)}{m_a^4}$$

$$\overset{via\ (i)\ and\ (ii)}{\Leftrightarrow} \frac{s^2[4s(s-a)+(b-c)^2]^2}{16}$$

$$\ge sa[a(s-a)+(b-c)^2].s(s-a)\left[\frac{4s^2-a^2+(b-c)^2}{4s^2}\right]$$

$$\Leftrightarrow s^2[(b-c)^4+16s^2(s-a)^2+8s(s-a)(b-c)^2]$$
$$\geq [4a^2(s-a)^2+4a(s-a)(b-c)^2][4s^2-a^2+(b-c)^2]$$
$$\Leftrightarrow s^2[(b-c)^4+16s^2(s-a)^2+8s(s-a)(b-c)^2]$$
$$\geq 16s^2a^2(s-a)^2+16s^2a(s-a)(b-c)^2-4a^4(s-a)^2-4a^3(s-a)(b-c)^2$$
$$+4a^2(s-a)^2(b-c)^2+4a(s-a)(b-c)^4$$
$$\Leftrightarrow (b-c)^4(s^2-4sa+4a^2)+4(s-a)^2(4s^4-4s^2a^2+a^4)$$
$$+4(s-a)(b-c)^2[2s^3-4s^2a+a^3-a^2(s-a)]\geq 0$$
$$\Leftrightarrow (b-c)^4(s-2a)^2+4(s-a)^2\left(2s^2-a^2\right)^2+4(s-a)(s-2a)\left(2s^2-a^2\right)(b-c)^2\geq 0$$
$$\Leftrightarrow [(s-2a)(b-c)^2+2(s-a)(2s^2-a^2)]^2\geq 0\rightarrow true\ \therefore \frac{h_a}{2r}$$
$$\geq \frac{n_a\sqrt{r_br_c-r^2}}{m_a^2}\overset{summing\ up}{\Rightarrow}\frac{h_a+h_b+h_c}{2r}\geq \sum\frac{n_a\sqrt{r_br_c-r^2}}{m_a^2}\ (QED)$$

2372. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\geq \sum_{cyc}\sqrt{\frac{m_am_b}{r_ar_b}}$$

Proposed by Bogdan Fuştei-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$Since: \frac{2}{h_a}=\frac{1}{r_b}+\frac{1}{r_c}\rightarrow \frac{1}{h_a}=\frac{1}{2}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\rightarrow \frac{m_a}{h_a}=\frac{1}{2}\left(\frac{m_a}{r_b}+\frac{m_a}{r_c}\right); (and\ analogs)$$
$$\rightarrow \sum_{cyc}\frac{m_a}{h_a}=\frac{1}{2}\left(\frac{m_a}{r_b}+\frac{m_a}{r_c}+\frac{m_b}{r_c}+\frac{m_b}{r_a}+\frac{m_c}{r_a}+\frac{m_c}{r_b}\right)=$$
$$=\frac{1}{2}\left(\frac{m_a}{r_b}+\frac{m_b}{r_a}\right)+\frac{1}{2}\left(\frac{m_a}{r_c}+\frac{m_c}{r_a}\right)+\frac{1}{2}\left(\frac{m_b}{r_c}+\frac{m_c}{r_b}\right)\overset{AM-GM}{\geq}\frac{1}{2}\cdot 2\sum_{cyc}\sqrt{\frac{m_am_b}{r_ar_b}}=\sum_{cyc}\sqrt{\frac{m_am_b}{r_ar_b}}$$

2373. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\geq \frac{1}{2}\sum_{cyc}\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$$

Proposed by Bogdan Fuştei-Romania

Solution by Alex Szoros-Romania

$$h_a=\frac{2F}{a}=\frac{bc\cdot\sin A}{a}=\frac{bc}{2R}$$

$$From: m_a\ge\frac{b^2+c^2}{4R}\to m_a\ge\frac{bc}{2R}\cdot\frac{b^2+c^2}{2bc}\to m_a\ge h_a\cdot\frac{b^2+c^2}{2bc}\to$$

$$\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right);(1)$$

On the other hand, in any ΔABC we have:

$$\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right);(Adil\ Abdullayev);(2)$$

From $(1),(2)$ it follows that:

$$\left(\frac{m_a}{h_a}\right)^2\ge\frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)\to\frac{m_a}{h_a}\ge\frac{1}{2}\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$$

Therefore,

$$\sum_{cyc}\frac{m_a}{h_a}\ge\frac{1}{2}\sum_{cyc}\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$$

2374. In ΔABC the following relationship holds:

$$\frac{24r^2}{R}\le\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\le\frac{4R^2-2Rr}{r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\overset{Bergstrom}{\ge}\frac{(a+b+c)^2}{m_a+m_b+m_c}=\frac{4s^2}{m_a+m_b+m_c};(1)$$

But: $m_a+m_b+m_c\le\frac{9R^2}{2};(2)$. From (1),(2) it follows that:

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\ge\frac{8s^2}{9R};(3)$$

Now, we must to prove that: $\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\le\frac{4R^2-2Rr}{r}$

We know that: $m_a \ge \frac{b^2+c^2}{4R} \ge \frac{bc}{2R} \Rightarrow \frac{1}{m_a} \le \frac{2R}{bc} \Rightarrow$

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le 2R\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right); (4)$$

$$2R\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right) \le \frac{4R^2 - 2Rr}{r} \Leftrightarrow \frac{a^3 + b^3 + c^3}{abc} \le \frac{2R - r}{r}; (5)$$

But: $a^3 + b^3 + c^3 = 2(s^2 - 3r^2 - 6Rr)$ and $abc = 4Rrs; (6)$

From (5),(6) we must show that:

$$\frac{2s(s^2 - 3r^2 - 6Rr)}{4Rrs} \le \frac{2R - r}{r} \Leftrightarrow s^2 - 3r^2 - 6Rr \le 4R^2 - 2Rr \Leftrightarrow$$

$$s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen).$$

2375. If $m, n \in \mathbb{R}_+^*$, then prove in any triangle ABC is true the following relationship holds:

$$\frac{ma^2 + nb^2}{(a + b - c) \cdot c} + \frac{mb^2 + nc^2}{(b + c - a) \cdot a} + \frac{mc^2 + na^2}{(c + a - b) \cdot b} \ge 3(m + n)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$\frac{ma^2 + nb^2}{(a + b - c) \cdot c} + \frac{mb^2 + nc^2}{(b + c - a) \cdot a} + \frac{(mc^2 + na^2)}{(c + a - b) \cdot b} =$$

$$= m\left(\frac{a^2}{(a + b - c) \cdot c} + \frac{b^2}{(b + c - a) \cdot a} + \frac{c^2}{(c + a - b) \cdot b}\right) +$$

$$+n\left(\frac{a^2}{(a + b - c) \cdot c} + \frac{b^2}{(b + c - a) \cdot a} + \frac{c^2}{(c + a - b) \cdot b}\right) \overset{Bergstrom}{\ge}$$

$$\ge \frac{m \cdot (\sum a)^2}{\sum(a + b - c) \cdot c} + \frac{n \cdot (\sum a)^2}{\sum(a + b - c) \cdot c} = (m + n) \cdot \frac{(\sum a)^2}{\sum(a + b - c) \cdot c} =$$

$$= (m + n) \cdot \frac{(\sum a)^2}{\sum ab} \ge 3(m + n)$$

$$\because (a + b + c)^2 \ge 3(ab + bc + ca), \forall a, b, c \in R$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\frac{ma^2+nb^2}{(a+b-c)\cdot c}=m\sum_{cyc}\frac{a^2}{(a+b-c)\cdot c}+m\sum_{cyc}\frac{b^2}{(a+b-c)\cdot c}\overset{Bergstrom}{\geq}$$

$$\geq m\cdot\frac{(\sum a)^2}{\sum(ac+bc-c^2)}+n\cdot\frac{(\sum b)^2}{\sum(ac+bc-c^2)}\geq$$

$$\geq(m+n)\cdot\frac{(2s)^2}{2\sum ab-\sum a^2}=(m+n)\cdot\frac{4s^2}{(2s^2+2r^2+8Rr-2s^2+8Rr+2r^2)}=$$

$$=(m+n)\cdot\frac{4s^2}{16Rr+4r^2}$$

$$Need\ to\ show:(m+n)\cdot\frac{4s^2}{16Rr+4r^2}\geq 3(m+n)\to 4s^2\geq 48Rr+12r^2$$

$$But: s^2\geq 16Rr-5r^2\ (Gerretsen), then\ need\ to\ show:$$

$$64Rr-20r^2\geq 48Rr+12r^2\to$$

$$16Rr\geq 32r^2\to R\geq 2r\ (Euler) true.$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{ma^2+nb^2}{(a+b-c)\cdot c}+\frac{mb^2+nc^2}{(b+c-a)\cdot a}+\frac{mc^2+na^2}{(c+a-b)\cdot b}\geq 3(m+n)$$

$$m\left(\frac{a^2}{(a+b-c)\cdot c}+\frac{b^2}{(b+c-a)\cdot a}+\frac{c^2}{(c+a-b)\cdot b}\right)+$$

$$+n\left(\frac{a^2}{(a+b-c)\cdot c}+\frac{b^2}{(b+c-a)\cdot a}+\frac{c^2}{(c+a-b)\cdot b}\right)\geq 3(m+n)$$

$$Let\ x=a+b-c, y=b+c-a, z=c+a-b, hence$$

$$m\left(\frac{a^2}{(a+b-c)\cdot c}+\frac{b^2}{(b+c-a)\cdot a}+\frac{c^2}{(c+a-b)\cdot b}\right)=$$

$$=m\left(\frac{(x+z)^2}{2x(y+z)}+\frac{(x+y)^2}{y(z+x)}+\frac{(y+z)^2}{z(x+y)}\right)\geq 3m$$

$$Iff\ \frac{(x+y+y+z+z+x)^2}{x(y+z)+y(z+x)+z(x+y)}=\frac{4\left(x^2+y^2+z^2+2(xy+yz+zx)\right)}{2(xy+yz+zx)}\geq 6$$

$$2\left(x^2+y^2+z^2+2(xy+yz+zx)\right)\geq 6(xy+yz+zx)\ which\ is\ true.$$

$$Similarly, n\left(\frac{(x+z)^2}{2x(y+z)}+\frac{(x+y)^2}{y(z+x)}+\frac{(y+z)^2}{z(x+y)}\right)\ge 3n$$

2376. In any $\Delta\, ABC$ the following relationship holds:

$$16R^2rs^3\le\sum b^3c^3\cot\frac{A}{2}\le\frac{R^6s^3}{r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof}: 16R^2rs^3\le\sum b^3c^3\cot\frac{A}{2}\le\frac{R^6s^3}{r^3}\Leftrightarrow\boxed{\frac{1}{4Rr^2}\le\sum\frac{1}{a^3}\cot\frac{A}{2}\le\frac{R^3}{64r^6}}$$

$$\text{Now}, \sum\frac{1}{a^3}\cot\frac{A}{2}=s\sum\left(\frac{1}{a^3}\left(\frac{s-a}{rs}\right)\right)$$

$$=\frac{1}{r}\left(\frac{s}{64R^3r^3s^3}\left(\left(\sum ab\right)^3-3.4Rrs.2s(s^2+2Rr+r^2)\right)\right.$$

$$\left.-\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}\right)$$

$$=\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)-4Rr((s^2+4Rr+r^2)^2-16Rrs^2)}{64R^3r^4s^2}$$

$$=\frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{64R^3r^4s^2}\overset{(i)}{=}\sum\frac{1}{a^3}\cot\frac{A}{2}\ \therefore\ (i)$$

$$\Rightarrow\frac{1}{4Rr^2}\le\sum\frac{1}{a^3}\cot\frac{A}{2}$$

$$\Leftrightarrow\frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{64R^3r^4s^2}\ge\frac{1}{4Rr^2}$$

$$\Leftrightarrow s^6-(16Rr-3r^2)s^4+r^2s^2(16R^2-8Rr+3r^2)+r^4(4R+r)^2\overset{(a)}{\gtrapprox}0$$

$$\text{Now, LHS of }(a)\overset{\text{Gerretsen}}{\gtrapprox}-2r^2s^4+r^2s^2(16R^2-8Rr+3r^2)+r^4(4R+r)^2$$

$$\overset{\text{Gerretsen}}{\gtrapprox}r^2s^2\left(-2(4R^2+4Rr+3r^2)+(16R^2-8Rr+3r^2)\right)+r^4(4R+r)^2$$

$$=r^2s^2(8R^2-16Rr-3r^2)+r^4(4R+r)^2\overset{?}{\gtrapprox}0$$

$$\Leftrightarrow s^2(8R^2-16Rr-3r^2)+r^2(4R+r)^2 \underset{(b)}{\overset{?}{\geq}} 0$$

Case 1 $8R^2-16Rr-3r^2 \geq 0$ and then, LHS of (b) $>0 \Rightarrow$ (b) $\Rightarrow$ (a) is true $\therefore \frac{1}{4Rr^2} < \sum \frac{1}{a^3}\cot\frac{A}{2}$

Case 2 $8R^2-16Rr-3r^2 < 0$ and then, LHS of (b)

$$= -s^2\left(-(8R^2-16Rr-3r^2)\right)+r^2(4R+r)^2$$

$$\overset{\text{Gerretsen}}{\geq} -(4R^2+4Rr+3r^2)\left(-(8R^2-16Rr-3r^2)\right)+r^2(4R+r)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 8t^4-8t^3-9t^2-13t-2 \overset{?}{\geq} 0 \left(\text{where } t=\frac{R}{r}\right)$$

$\Leftrightarrow (t-2)(8t^3+8t^2+7t+1) \overset{?}{\geq} 0 \to$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ (b) $\Rightarrow$ (a) is true $\therefore \frac{1}{4Rr^2} \leq \sum \frac{1}{a^3}\cot\frac{A}{2}$ and $\therefore$ combining cases 1, 2,

$$\boxed{\text{in any } \Delta ABC, \frac{1}{4Rr^2} \leq \sum \frac{1}{a^3}\cot\frac{A}{2} \Rightarrow 16R^2rs^3 \leq \sum b^3c^3\cot\frac{A}{2}}$$ and again,

via (i), $\sum \frac{1}{a^3}\cot\frac{A}{2} \leq \frac{R^3}{64r^6}$

$$\Leftrightarrow \frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{64R^3r^4s^2} \leq \frac{R^3}{64r^6}$$

$$\Leftrightarrow r^2s^6-r^2(16Rr-3r^2)s^4+r^4s^2(32R^2-8Rr+3r^2)+r^6(4R+r)^2-R^6s^2 \overset{(1)}{\leq} 0$$

Now, Rouche $\Rightarrow s^2-(m-n) \geq 0$ and $s^2-(m+n) \leq 0$, where $m = 2R^2+10Rr-r^2$ and $n = 2(R-2r)\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right) \leq 0 \Rightarrow s^4-s^2(2m)+m^2-n^2 \leq 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(l)}{\leq} 0$$

$$\Rightarrow r^2s^6-r^2s^4(4R^2+20Rr-2r^2)+r^3s^2(4R+r)^3 \overset{(m)}{\leq} 0 \therefore (m)$$

$\Rightarrow$ inorder to prove (1), it suffices to prove :

$$r^2s^6 - r^2(16Rr - 3r^2)s^4 + r^4s^2(32R^2 - 8Rr + 3r^2) + r^6(4R + r)^2 - R^6s^2$$
$$\leq r^2s^6 - r^2s^4(4R^2 + 20Rr - 2r^2) + r^3s^2(4R + r)^3$$
$$\Leftrightarrow r^2s^4(4R^2 + 4Rr + r^2) + s^2(-R^6 + r^4(32R^2 - 8Rr + 3r^2) - r^3(4R + r)^3)$$
$$+ r^6(4R + r)^2 \overset{(2)}{\leqq} 0$$

$Again, (l) \Rightarrow r^2(4R^2 + 4Rr + r^2)s^4 - s^2r^2(4R^2 + 4Rr + r^2)(4R^2 + 20Rr - 2r^2)$
$$+ r^3(4R + r)^3(4R^2 + 4Rr + r^2) \overset{(n)}{\leqq} 0 \therefore (n) \Rightarrow$$

inorder to prove (2), it suffices to prove

$$: r^2s^4(4R^2 + 4Rr + r^2)$$
$$+ s^2(-R^6 + r^4(32R^2 - 8Rr + 3r^2) - r^3(4R + r)^3) + r^6(4R + r)^2$$
$$\leq r^2(4R^2 + 4Rr + r^2)s^4 - s^2r^2(4R^2 + 4Rr + r^2)(4R^2 + 20Rr - 2r^2)$$
$$+ r^3(4R + r)^3(4R^2 + 4Rr + r^2)$$
$$\Leftrightarrow (R^5 - 16R^3r^2 - 32R^2r^3 - 60Rr^4 + 8r^5)s^2$$
$$+ r^3(16R^2 + 20Rr + 8r^2)(4R + r)^2 \overset{(3)}{\geqq} 0$$

Case i $R^5 - 16R^3r^2 - 32R^2r^3 - 60Rr^4 + 8r^5 \geq 0$ **and then, LHS of (3) > 0 ⇒ (3)**

$$\Rightarrow (2) \Rightarrow (1) \text{ is true } \therefore \sum \frac{1}{a^3}\cot\frac{A}{2} < \frac{R^3}{64r^6}$$

Case ii $R^5 - 16R^3r^2 - 32R^2r^3 - 60Rr^4 + 8r^5 < 0$ **and then, LHS of (3)**

$$= -s^2\left(-(R^5 - 16R^3r^2 - 32R^2r^3 - 60Rr^4 + 8r^5)\right)$$
$$+r^3(16R^2 + 20Rr + 8r^2)(4R + r)^2 \overset{\text{Gerretsen}}{\geqq}$$
$$-(4R^2 + 4Rr + 3r^2)\left(-(R^5 - 16R^3r^2 - 32R^2r^3 - 60Rr^4 + 8r^5)\right)$$
$$+r^3(16R^2 + 20Rr + 8r^2)(4R + r)^2 \overset{?}{\geqq} 0$$
$$\Leftrightarrow 4t^7 + 4t^6 - 61t^5 + 64t^4 + 32t^3 - 64t + 32 \overset{?}{\geqq} 0 \left(\text{where } t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(4t^5 + 20t^4 + t^3 + 2t^2(t-2) + 4t + 32) + 48\right) \overset{?}{\geqq} 0 \rightarrow \text{true}$$
$$\because t \overset{\text{Euler}}{\geqq} 2 \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \text{ is true}$$

$$\therefore \sum \frac{1}{a^3}\cot\frac{A}{2} \le \frac{R^3}{64r^6} \text{ and}$$

$$\therefore \text{ combining cases i, ii, } \boxed{\text{in any } \Delta ABC, \sum \frac{1}{a^3}\cot\frac{A}{2} \le \frac{R^3}{64r^6} \Rightarrow \sum b^3c^3\cot\frac{A}{2} \le \frac{R^6s^3}{r^3}} \text{ (QED)}$$

2377. In ΔABC the following relationship holds:

$$\frac{1}{2R^3} \le \frac{r_a}{a^4} + \frac{r_b}{b^4} + \frac{r_c}{c^4} \le \frac{1}{16r^3}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$\frac{r_a}{a^4} + \frac{r_b}{b^4} + \frac{r_c}{c^4} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{r_ar_br_c}{(abc)^4}}$$

$$\sqrt[3]{\frac{r_ar_br_c}{(abc)^4}} \ge \frac{1}{2R^3} \Leftrightarrow \frac{r_ar_br_c}{(abc)^4} \ge \frac{1}{8 \cdot 27 \cdot R^9}; (1)$$

But: $abc = 4Rrs$ and $r_ar_br_c = s^2r$; (2)

From (1),(2) we must show that: $\frac{s^2r}{256R^4r^4s^4} \ge \frac{1}{8 \cdot 27R^9} \Leftrightarrow R^5 \ge \frac{32}{27}s^2r^3$; (3)

From Euler: $R \ge 2r \Rightarrow R^3 \ge 8r^3$ and from $R^2 \ge \frac{4}{27}s^2 \Rightarrow^5 \ge \frac{32}{27}s^2r^3$.

$$\frac{r_a}{a^4} + \frac{r_b}{b^4} + \frac{r_c}{c^4} \le \frac{1}{16r^3} \Leftrightarrow \frac{F}{a^4(s-a)} + \frac{F}{b^4(s-b)} + \frac{F}{c^4(s-c)} \le \frac{1}{16\frac{F^3}{s^3}} \Leftrightarrow$$

$$\frac{F}{a^4(s-a)} + \frac{F}{b^4(s-b)} + \frac{F}{c^4(s-c)} \le \frac{s^3}{16F^4}; (4)$$

Let: $x = s - a; y = s - b; z = s - c \Rightarrow x + y + z = 2s$ and

$$a = y + z; b = x + z; c = x + y;\ F = \sqrt{(x+y+z)xyz}; (5)$$

From (4),(5) we must show that: $\frac{1}{x(y+z)^4} + \frac{1}{y(x+z)^4} + \frac{1}{z(x+y)^4} \le \frac{x+y+z}{16x^2y^2z^2}$; (6).

$$y + z \ge 2\sqrt{yz} \Rightarrow (y+z)^4 \ge 16y^2z^2 \Rightarrow \frac{1}{(y+z)^2} \le \frac{1}{16y^2z^2} \Rightarrow$$

$$\frac{1}{x(y+z)^4} + \frac{1}{y(x+z)^4} + \frac{1}{z(x+y)^4} \le \frac{1}{16}\left(\frac{1}{xy^2z^2} + \frac{1}{x^2yz^2} + \frac{1}{x^2y^2z}\right); (7)$$

From (6),(7) we must show that:

$$\frac{1}{16xyz}\left(\frac{1}{yz}+\frac{1}{xz}+\frac{1}{xy}\right)\le\frac{x+y+z}{16x^2y^2z^2}\Leftrightarrow\frac{x+y+z}{xyz}\le\frac{x+y+z}{xyz}$$, which is true.

2378. In any $\Delta\, ABC$, the following relationship holds:

$$\sum\frac{a}{\sqrt{h_a-2r}}\ge\sqrt{R}\sum\frac{n_a}{h_a}+\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)}$$

Proposed by Bogdan Fuştei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Proof: b+c-a=4Rcos\frac{A}{2}cos\frac{B-C}{2}-4Rsin\frac{A}{2}cos\frac{A}{2}$$

$$=4Rcos\frac{A}{2}\left(cos\frac{B-C}{2}-cos\frac{B+C}{2}\right)=8Rcos\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

$$\Rightarrow s-a\overset{(1)}{=}4Rcos\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

$$\frac{a}{\sqrt{h_a-2r}}=\frac{a}{\sqrt{\frac{2rs}{a}-2r}}=\frac{a}{\sqrt{2r}}\sqrt{\frac{a}{s-a}}\overset{via\ (1)}{=}\frac{a}{\sqrt{2r}}\sqrt{\frac{4Rcos\frac{A}{2}sin^2\frac{A}{2}}{4Rcos\frac{A}{2}sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}}}$$

$$=\frac{asin\frac{A}{2}}{\sqrt{2r}}\sqrt{\frac{4R}{r}}=\frac{\sqrt{2R}}{r}asin\frac{A}{2}\ and\ analogs\overset{summing\ up}{\Rightarrow}$$

$$\sum\frac{a}{\sqrt{h_a-2r}}=\frac{\sqrt{2R}}{r}\sum asin\frac{A}{2}\ge\frac{s\sqrt{2R}}{r}\Leftrightarrow\sum cos\frac{A}{2}sin^2\frac{A}{2}$$

$$\ge cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}\ applying\ which\ on\ a\ triangle\ with\ angles\ \pi-2A,$$

$$\pi-2B,\pi-2C, we\ get:\sum cos\frac{\pi-2A}{2}sin^2\frac{\pi-2A}{2}$$

$$\ge cos\frac{\pi-2A}{2}cos\frac{\pi-2B}{2}cos\frac{\pi-2B}{2}\Leftrightarrow\sum sinAcos^2A$$

$$\ge sinAsinBsinC$$

$$\Leftrightarrow \sum sinA(1-sin^2A) \geq \frac{4Rrs}{8R^3} \Leftrightarrow \sum \frac{a}{2R} - \sum \frac{a^3}{8R^3} \geq \frac{4Rrs}{8R^3}$$

$$\Leftrightarrow \frac{s}{R} - \frac{2s(s^2-6Rr-3r^2)}{8R^3} - \frac{4Rrs}{8R^3} \geq 0$$

$$\Leftrightarrow \frac{8R^2-2(s^2-6Rr-3r^2)-4Rr}{8R^3} \geq 0 \Leftrightarrow s^2 \leq 4R^2+8Rr+3r^2$$

$$\to true\ (Gerretsen) \therefore \sum \frac{a}{\sqrt{h_a-2r}} \overset{(a)}{\geq} \frac{s\sqrt{2R}}{r}$$

$$r_b + r_c = s\left(\frac{sin\frac{B}{2}}{cos\frac{B}{2}} + \frac{sin\frac{C}{2}}{cos\frac{C}{2}}\right) = \frac{ssin\left(\frac{B+C}{2}\right)cos\frac{A}{2}}{cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}} = \frac{scos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4Rcos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4Rcos^2\frac{A}{2}$$

$$Now,\ (b+c)^2 \geq 32Rrcos^2\frac{A}{2} \overset{by\ (i)}{=} 8r(r_b+r_c) = 8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \to true \therefore b+c$$

$$\geq 4\sqrt{2Rr}cos\frac{A}{2} and\ analogs \Rightarrow \sum m_a \overset{Ioscu}{\geq} \sum \frac{b+c}{2}cos\frac{A}{2}$$

$$\geq \sqrt{2Rr}\sum\left(2cos^2\frac{A}{2}\right) = \sqrt{2Rr}\sum(1+cosA) = \sqrt{2Rr}\left(\frac{4R+r}{R}\right) = \sqrt{\frac{2r}{R}}(4R+r)$$

$$\Rightarrow \frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)}$$

$$\leq \sqrt{\frac{R}{2r}}\sum\sqrt{2\left(\frac{rs}{s-a}-\frac{rs}{s}\right)} = \sqrt{R}\sum\sqrt{\frac{a}{s-a}} = \sqrt{R}\sum\sqrt{\frac{4Rcos\frac{A}{2}sin^2\frac{A}{2}}{4Rcos\frac{A}{2}sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}}} = \sqrt{\frac{4R^2}{r}}\sum sin\frac{A}{2}$$

$$\Rightarrow \frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)} \overset{(b)}{\leq} \frac{2R}{\sqrt{r}}\sum sin\frac{A}{2}$$

$$Now, Stewart's\ theorem \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+s(2bccosA-2bc)$$

$$=as^2-4sbcsin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a$$

$$\Rightarrow n_a^2\overset{(ii)}{=}s^2-2h_ar_a\text{and analogs}$$

$$\therefore\sqrt{R}\sum\frac{n_a}{h_a}+\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)}\overset{via\ (b)}{\leq}\sqrt{R}\sum\frac{n_a}{h_a}+\frac{2R}{\sqrt{r}}\sum sin\frac{A}{2}$$

$$=\sqrt{R}\sum\left(\frac{n_a}{h_a}+2\sqrt{\frac{R}{r}}sin\frac{A}{2}\right)$$

$$\overset{via\ CBS\ and\ (ii)and\ analogs}{\leq}\sqrt{2R}\sum\sqrt{\frac{s^2-2h_ar_a}{h_a^2}+\frac{4R}{r}sin^2\frac{A}{2}}$$

$$=\sqrt{2R}\sqrt{\frac{s^2}{h_a^2}-\frac{8Rcos\frac{A}{2}sin\frac{A}{2}stan\frac{A}{2}}{2rs}+\frac{4R}{r}sin^2\frac{A}{2}}$$

$$=\sqrt{2R}\sum\sqrt{\frac{s^2}{h_a^2}-\frac{4R}{r}sin^2\frac{A}{2}+\frac{4R}{r}sin^2\frac{A}{2}}=s\sqrt{2R}\sum\frac{1}{h_a}$$

$$=\frac{s\sqrt{2R}}{r}\overset{via\ (a)}{\leq}\sum\frac{a}{\sqrt{h_a-2r}}(QED)$$

2379. In ΔABC the following relationship holds:

$$16pr(2R-r)^2\leq\sum b^2c^2\cot\frac{A}{2}\leq\frac{16p}{r}(R^4-7r^4)$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$r=(p-a)\tan\frac{A}{2}\Rightarrow\cot\frac{A}{2}=\frac{p-a}{r}\text{ (analogs)}$$

$$\Rightarrow\Omega=\sum b^2c^2\cot\frac{A}{2}=\frac{1}{r}\sum b^2c^2(p-a)=\frac{1}{r}\left(p\sum b^2c^2-abc\sum bc\right)$$

$$=\frac{1}{r}\left(p(p^4+(2r^2-8Rr)p^2+(4Rr+r^2)^2)-4Rrp(p^2+4Rr+r^2)\right)$$

$$=\frac{p^5+(2r^2-8Rr)p^3+(4Rr+r^2)^2p-4Rrp^3-(16R^2r^2+4Rr^3)p}{r}$$

$$=\frac{p^5+(2r^2-12Rr)p^3+(4Rr^3+r^4)p}{r}=\frac{p\left(p^4+(2r^2-12Rr)p^2+(4Rr^3+r^4)\right)}{r}$$

$$\Omega\overset{(1)}{\geq}16pr(2R-r)^2$$

$$\Leftrightarrow p^4+(2r^2-12Rr)p^2+(4Rr^3+r^4)\geq 16r^2(2R-r)^2$$

$$\Leftrightarrow p^4+(2r^2-12Rr)p^2+4Rr^3+r^4-64R^2r^2+64Rr^3-16r^4\geq 0$$

$$\Leftrightarrow p^4+(2r^2-12Rr)p^2+68Rr^3-64R^2r^2-15r^4\geq 0$$

Let: $\varphi(u)=u^2+(2r^2-12Rr)u+68Rr^3-64R^2r^2-15r^4$

(where: $16Rr-5r^2\leq u=p^2\leq 4R^2+4Rr+3r^2$)

$$\Rightarrow \varphi'(u)=2u-12Rr+2r^2\overset{u\geq 16Rr-5r^2}{\geq}20Rr-8r^2\overset{R\geq 2r}{\geq}32r^2>0$$

$$\Rightarrow \varphi(u)\uparrow[16Rr-5r^2,4R^2+4Rr+3r^2]$$

$$\Rightarrow \varphi(u)\geq\varphi(16Rr-5r^2)\overset{(2)}{=}0$$

(2) $\Leftrightarrow (16Rr-5r^2)^2+(2r^2-12Rr)(16Rr-5r^2)+68Rr^3-64R^2r^2-15r^4=0$

$$\Leftrightarrow 64R^2r^2-68Rr^3+15r^4+68Rr^3-64R^2r^2-15r^4=0$$

$\Leftrightarrow 0=0$ **(true)**$\Rightarrow$ **(2)** $\Rightarrow$ **(1) is true.**

$$\Omega\overset{(3)}{\leq}\frac{16p}{r}(R^4-7r^4)$$

(3) $\Leftrightarrow p^4+(2r^2-12Rr)p^2+4Rr^3+r^4\leq 16R^4-112r^4$

$$\Leftrightarrow p^4+(2r^2-12Rr)p^2+4Rr^3-16R^4+113r^4\leq 0$$

Let $\varphi(t)=t^2+(2r^2-12Rr)t+4Rr^3-16R^4+113r^4$

(Where: $16Rr-5r^2\leq t=p^2\leq 4R^2+4Rr+3r^2$)

$$\Rightarrow \varphi'(t)=2t-12Rr+2r^2\overset{t\geq 16Rr-5r^2}{\geq}20Rr-8r^2\overset{R\geq 2r}{\geq}32r^2>0$$

$$\Rightarrow \varphi(t)\uparrow[16Rr-5r^2,4R^2+4Rr+3r^2]\Rightarrow\varphi(t)\leq\varphi(4R^2+4Rr+3r^2)\overset{(4)}{\leq}0$$

(4) $\Leftrightarrow (4R^2+4Rr+3r^2)^2+(2r^2-12Rr)(4R^2+4Rr+3r^2)+4Rr^3-16R^4+113r^4\leq 0$

$$\Leftrightarrow 16R^4 - 16R^3r - 4Rr^3 + 15r^4 + 4Rr^3 - 16R^4 + 113r^4 \leq 0$$

$$\Leftrightarrow 16R^3r \geq 128r^4R^3 \geq 8r^3 \Leftrightarrow R \geq 2r$$ **(Euler)⇒ (4) ⇒ (3) is true. Proved.**

2380. In ΔABC the following relationship holds:

$$\frac{2\sqrt{2}}{3}\sum_{cyc} m_a \leq \sum_{cyc} \frac{n_a^2 + g_a^2 + 2rr_a}{\sqrt{4m_bm_c + 3bc}}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$n_a^2 = s(s-a) + \frac{(b-c)^2}{a} \cdot s \text{ and } g_a^2 = s(s-a) - \frac{(b-c)^2}{a} \cdot (s-a)$$

$$\to n_a^2 + g_a^2 = 2s(s-a) + (b-c)^2 = b^2 + c^2 - (2bc - 2r_br_c)$$

$$2bc - 2r_br_c = 2rr_a \to n_a^2 + g_a^2 + 2rr_a = b^2 + c^2; (1)$$

$$4m_bm_c + 3bc \overset{AM-GM}{\leq} 2(m_b^2 + m_c^2) + \frac{3(b^2 + c^2)}{2} =$$

$$= \frac{2(c^2 + a^2) - b^2}{4} + \frac{2(a^2 + b^2) - c^2}{4} + \frac{3(b^2 + c^2)}{2}$$

$$\to 4m_bm_c + 3bc \leq 2\sum_{cyc} a^2 ; (2)$$

$$\to \sum_{cyc} \frac{n_a^2 + g_a^2 + 2rr_a}{\sqrt{4m_bm_c + 3bc}} \overset{(1),(2)}{\geq} \sum_{cyc} \frac{b^2 + c^2}{\sqrt{2\sum a^2}} = \sqrt{2} \cdot \sqrt{\sum a^2} = \sqrt{2} \cdot \sqrt{\frac{4}{3}\sum_{cyc} m_a^2} \overset{CBS}{\geq} \frac{2\sqrt{2}}{3}\sum_{cyc} m_a$$

Therefore,

$$\frac{2\sqrt{2}}{3}\sum_{cyc} m_a \leq \sum_{cyc} \frac{n_a^2 + g_a^2 + 2rr_a}{\sqrt{4m_bm_c + 3bc}}$$

2381. In ΔABC the following relationship holds:

$$\left(\frac{ab}{a+b}\right)^2 + \left(\frac{bc}{b+c}\right)^2 + \left(\frac{ca}{c+a}\right)^2 \geq 9r^2$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\left(\frac{ab}{a+b}\right)^2 \overset{CBS}{\geq} \frac{1}{3}\left(\sum\frac{ab}{a+b}\right)^2 \overset{?}{\geq} 9r^2 \leftrightarrow \sum\frac{ab}{a+b} \overset{(*)}{\geq} 3\sqrt{3}r.$$

$$\sum\frac{ab}{a+b} = \sum\frac{1}{\frac{1}{a}+\frac{1}{b}} \overset{CBS}{\geq} \frac{9}{\sum\left(\frac{1}{a}+\frac{1}{b}\right)} = \frac{9abc}{2\sum ab} \overset{3\sum ab\leq(\sum a)^2}{\geq} \frac{27abc}{2(\sum a)^2} =$$

$$= \frac{27.4Rrs}{2.(2s)^2} = \frac{27R}{2s}.r \overset{Mitrinovic}{\geq} \frac{27.2}{2.3\sqrt{3}}.r = 3\sqrt{3}r$$

$$\rightarrow (*)\ is\ true \rightarrow \sum\left(\frac{ab}{a+b}\right)^2 \geq 9r^2$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}\left(\frac{ab}{a+b}\right)^2 \overset{Bergstrom}{\geq} \frac{(\sum ab)^2}{\sum(a+b)^2} \overset{Power\ Mean}{\geq} \frac{(\sum ab)^2}{2\sum(a^2+b^2)} = \frac{(\sum ab)^2}{4\sum a^2} \overset{Leibniz}{\geq} \frac{(\sum ab)^2}{4\cdot 9R}$$

$$= \left(\frac{\sum ab}{6R}\right)^2$$

$$Need\ to\ show: \left(\frac{\sum ab}{6R}\right)^2 \geq 9r^2 \rightarrow s^2 \geq 14Rr - r^2$$

$$But\ s^2 \geq 16Rr - 5r^2 (Gerretsen) \rightarrow 16Rr - 5r^2 \geq 14Rr - r^2 \rightarrow$$

$$2Rr \geq 4r^2 \rightarrow R \geq 2r (Euler)$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$\left(\frac{ab}{a+b}\right)^2 + \left(\frac{bc}{b+c}\right)^2 + \left(\frac{ca}{c+a}\right)^2 \overset{AM-GM}{\geq} 3\cdot\sqrt[3]{\frac{(abc)^4}{\left((a+b)(b+c)(c+a)\right)^2}} =$$

$$= \frac{3\sqrt[3]{(abc)^4}}{\left(\sqrt[3]{(a+b)(b+c)(c+a)}\right)^2} \overset{AM-GM}{\geq} \frac{3\sqrt[3]{(abc)^4}}{\left(\frac{a+b+b+c+c+a}{3}\right)^2} =$$

$$= \frac{27\sqrt[3]{(4Rrs)^4}}{16s^2} \overset{(1)}{\geq} 9r^2$$

$$(1) \Leftrightarrow 3\sqrt[2]{(4Rrs)^4} \geq 4^2r^2s^2 \Leftrightarrow 27(4Rrs)^4 \geq 4^6r^6s^6 \Leftrightarrow 27R^4 \geq 4^2r^2s^2$$

$$3\sqrt{3}R^2 \geq 4rs\ true\ by\ s \leq \frac{3\sqrt{3}}{2}R\ (Mitrinovic), 2r \leq R\ (Euler) \rightarrow$$

$$2rs \le \frac{3\sqrt{3}}{2}R \cdot R \Leftrightarrow 4rs \le 3\sqrt{3}R^2 \to (1) is\ true. Proved.$$

2382. In ΔABC the following relationship holds:

$$\frac{3r^2}{2R^2} \le \sum \frac{rr_a}{b^2+c^2} \le \frac{2R-r}{8r}$$

Proposed by Kostas Geronikolas-Greece

Solurion 1 by Marian Ursărescu-Romania

For LHS: $rr_a \le \frac{a^2}{2}$. *We must show:*

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2} \le \frac{2R-r}{2r}; (1)$$

$$a^2+b^2 \ge 2ab;\ b^2+c^2 \ge 2bc, c^2+a^2 \ge 2ca; (2)$$

From $(1),(2)$ *we must to show that:*

$$\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} \le \frac{2R-r}{r} \Leftrightarrow \frac{a^3+b^3+c^3}{abc} \le \frac{2R-r}{r}$$

$$\Leftrightarrow \frac{2s(s^2-3r^2-6Rr)}{4Rrs} \le \frac{2R-r}{r} \Leftrightarrow s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$$

$$For\ RHS: \sum_{cyc} \frac{r_a}{b^2+c^2} \ge 3\sqrt[3]{\frac{r_ar_br_c}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}}. We\ must\ show\ that:$$

$$\sqrt[3]{\frac{r_ar_br_c}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}} \ge \frac{r}{2R^2} \Leftrightarrow \sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$\le \frac{2R^2}{r} \cdot \sqrt[3]{r_ar_br_c}; (3)$$

$$\sqrt[3]{r_ar_br_c} = \sqrt[3]{s^2r} \overset{Mitrinovic}{\gtrapprox} \sqrt[3]{27r^3} = 3r; (4)$$

From $(3),(4)$ *we must to prove that:*

$$\sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le 2 \cdot 3R^2; (5)$$

From $(4),(5)$ *we must show that:*

$$\sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le \frac{2(a^2+b^2+c^2)}{3} \Leftrightarrow a^2+b^2+c^2 \le 9R^2\ true.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a\ge b\ge c\to r_a\ge r_b\ge r_c \text{ and } \frac{1}{b^2+c^2}\ge\frac{1}{a^2+c^2}\ge\frac{1}{b^2+a^2}\overset{\text{Using Chebyshev}}{\Rightarrow}$$

$$\sum\frac{rr_a}{b^2+c^2}\ge\frac{r}{3}\left(\sum r_a\right)\left(\sum\frac{1}{b^2+c^2}\right)\overset{CBS}{\ge}\frac{r}{3}(4R+r).\frac{9}{2\sum a^2}\overset{\substack{Euler\\Leibniz}}{\ge}\frac{r}{3}(4.2r+r).\frac{9}{2.9R^2}=\frac{3r^2}{2R^2}$$

$$Now,\quad \sum\frac{rr_a}{b^2+c^2}\overset{AM-GM}{\le}\sum\frac{rr_a}{2bc}=\frac{r}{2abc}\sum a\,r_a=\frac{r.sr}{2.4sRr}\sum\frac{a}{s-a}$$

$$=\frac{r}{8R}\sum\left(\frac{s}{s-a}-1\right)=$$

$$=\frac{1}{8R}\sum r_a-\frac{3r}{8R}=\frac{(4R+r)-3r}{8R}=\frac{2R-r}{4R}\overset{Euler}{\le}\frac{2R-r}{8r}$$

$$Therefore,\quad \frac{3r^2}{2R^2}\le\sum\frac{rr_a}{b^2+c^2}\le\frac{2R-r}{8r}$$

2383. In acute ΔABC the following relationship holds:

$$\sum\frac{\cos^{2n+1}A}{\cos^{2n-1}B}\ge 1-\left(\frac{r}{R}\right)^2, n\in\mathbb{N}^*$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x=\cos A, y=\cos B, z=\cos C.\ \ \Delta ABC \text{ is acute }\to x,y,z>0$$

$$\sum\frac{x^{2n+1}}{y^{2n-1}}=\sum\frac{(x^2)^{n+1}}{(xy)(y^2)^{n-1}}\overset{H\ddot{o}lder}{\ge}\frac{(\sum x^2)^{n+1}}{(\sum xy)(\sum y^2)^{n-1}}=\frac{(\sum x^2)^2}{\sum xy}\overset{\sum xy\le\sum x^2}{\ge}\sum x^2$$

$$\to\sum\frac{\cos^{2n+1}A}{\cos^{2n-1}B}\ge\sum\cos^2A=1-2\prod\cos A=1-2.\frac{s^2-(2R+r)^2}{4R^2}\ge$$

$$\overset{Gerretsen}{\ge}1-\frac{(4R^2+4Rr+3r^2)-(2R+r)^2}{2R^2}=1-\left(\frac{r}{R}\right)^2$$

Therefore,

$$\sum\frac{\cos^{2n+1}A}{\cos^{2n-1}B}\ge 1-\left(\frac{r}{R}\right)^2, \forall n\in\mathbb{N}^*$$

2384. In ΔABC the following relationship holds:

$$3\sum_{cyc}(bc)^3\tan\frac{A}{2}\le\sum_{cyc}(bc)^3\cot\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$3\sum_{cyc}(bc)^3\tan\frac{A}{2}\le\sum_{cyc}(bc)^3\cot\frac{A}{2};(*)\Leftrightarrow\sum_{cyc}(bc)^3\left(\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}-\frac{3\sin\frac{A}{2}}{\cos\frac{A}{2}}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{(bc)^3}{a}\left(\cos^2\frac{A}{2}-\sin^2\frac{A}{2}\right)\ge 0\Leftrightarrow\frac{1}{abc}\sum_{cyc}(bc)^4\left(\frac{1+\cos A}{2}-3\cdot\frac{1-\cos A}{2}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}(bc)^4\left(\frac{b^2+c^2-a^2}{bc}-1\right)\ge 0\Leftrightarrow\sum_{cyc}(bc)^4(2\cos A-1)\ge 0$$

$$\sum_{cyc}b^5c^3+\sum_{cyc}b^3c^5\ge\sum_{cyc}a^2(bc)^3+\sum_{cyc}(bc)^4;(**)$$

$$We\ have:\sum_{cyc}b^5c^3+\sum_{cyc}b^3c^5=\sum_{cyc}(bc)^3(b^2+c^2)\overset{(AM-GM)}{\gtreqqless}2\sum_{cyc}(bc)^4;(1)$$

$$and\sum_{cyc}b^5a^3+\sum_{cyc}a^3c^5+\sum_{cyc}(bc)^4=\sum_{cyc}\left(b^5a^3+a^3c^5+(bc)^4\right)\overset{AM-GM}{\gtreqqless}$$

$$\ge 3\sum_{cyc}a^2(bc)^3;(2)$$

From $(1),(2)\to(**)$

Therefore,

$$3\sum_{cyc}(bc)^3\tan\frac{A}{2}\le\sum_{cyc}(bc)^3\cot\frac{A}{2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$3\sum b^3c^3\tan\frac{A}{2}\le\sum b^3c^3\cot\frac{A}{2}\Leftrightarrow\sum\frac{1}{a^3}\tan\frac{A}{2}\le\sum\frac{1}{a^3}\cot\frac{A}{2}$$

$$Now, \sum \frac{1}{a^3}\cot\frac{A}{2} = s\sum\left(\frac{1}{a^3}\left(\frac{s-a}{rs}\right)\right)$$

$$= \frac{1}{r}\left(\frac{s}{64R^3r^3s^3}\left(\left(\sum ab\right)^3 - 3.4Rrs.2s(s^2+2Rr+r^2)\right)\right.$$

$$\left. - \frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{16R^2r^2s^2}\right)$$

$$= \frac{(s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2) - 4Rr((s^2+4Rr+r^2)^2 - 16Rrs^2)}{64R^3r^4s^2}$$

$$= \frac{s^6 - (16Rr-3r^2)s^4 + r^2s^2(32R^2-8Rr+3r^2) + r^4(4R+r)^2}{64R^3r^4s^2} \stackrel{(i)}{=} \sum\frac{1}{a^3}\cot\frac{A}{2}$$

$$\sum\frac{1}{a^3}\tan\frac{A}{2} = \sum\left(\frac{1}{a^2}\left(\frac{\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\right)\right) = \frac{1}{4R}\sum\left(\frac{1}{a^2}\left(1+\tan^2\frac{A}{2}\right)\right)$$

$$= \frac{1}{4R}\left(\sum\frac{1}{a^2} + \sum\frac{\tan^2\frac{A}{2}}{16R^2\cos^4\frac{A}{2}\tan^2\frac{A}{2}}\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{16R^2r^2s^2} + \frac{1}{16R^2}\sum\left(1+\tan^2\frac{A}{2}\right)^2\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{16R^2r^2s^2} + \frac{1}{16R^2}\sum\left(1+\tan^4\frac{A}{2} + 2\tan^2\frac{A}{2}\right)\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{16R^2r^2s^2}\right.$$

$$\left. + \frac{1}{16R^2}\left(3 + \frac{1}{s^4}\left(\left(\sum r_a^2\right)^2 - 2\sum r_a^2r_b^2\right) + \frac{2}{s^2}\left(\sum r_a^2\right)\right)\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{16R^2r^2s^2}\right.$$

$$+ \frac{1}{16R^2}\left(3 + \frac{1}{s^4}\left(((4R+r)^2 - 2s^2)^2 - 2\left(s^4 - 2rs^2(4R+r)\right)\right)\right.$$

$$\left.\left. + \frac{2}{s^2}((4R+r)^2 - 2s^2)\right)\right)$$

$$= \frac{(s^2+4Rr+r^2)^2-16Rrs^2}{64R^3r^2s^2}$$

$$+\frac{3s^4+2s^2(4R+r)^2-4s^4+(4R+r)^4-4s^2(4R+r)^2+4s^4-2s^4+4rs^2(4R+r)}{64R^3s^4}$$

$$= \frac{s^6-(8Rr-3r^2)s^4-r^2s^2(16R^2-8Rr-3r^2)+r^2(4R+r)^4}{64R^3r^2s^4} \overset{(ii)}{=} \sum \frac{1}{a^3}\tan\frac{A}{2}$$

$$\therefore (i),(ii) \Rightarrow 3\sum \frac{1}{a^3}\tan\frac{A}{2} \le \sum \frac{1}{a^3}\cot\frac{A}{2}$$

$$\Leftrightarrow \frac{3s^6-3(8Rr-3r^2)s^4-3r^2s^2(16R^2-8Rr-3r^2)+3r^2(4R+r)^4}{64R^3r^2s^4}$$

$$\le \frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{64R^3r^4s^2}$$

$$\Leftrightarrow s^8-16Rrs^6+(32R^2+16Rr-6r^2)r^2s^4+r^4(64R^2-16Rr-8r^2)s^2$$

$$-3r^4(4R+r)^4 \overset{(1)}{\gtreqless} 0$$

$$Now, LHS\ of\ (1) \overset{Gerretsen}{\ge} -5r^2s^6+(32R^2+16Rr-6r^2)r^2s^4$$

$$+r^4(64R^2-16Rr-8r^2)s^2-3r^4(4R+r)^4$$

$$\overset{Gerretsen}{\ge} r^2s^4\left(32R^2+16Rr-6r^2-5(4R^2+4Rr+3r^2)\right)$$

$$+r^4(64R^2-16Rr-8r^2)s^2-3r^4(4R+r)^4$$

$$\overset{Gerretsen}{\ge} r^2s^2\left((12R^2-4Rr-21r^2)(16Rr-5r^2)+r^2(64R^2-16Rr-8r^2)\right)$$

$$-3r^4(4R+r)^4$$

$$= r^3s^2(192R^3-60R^2r-332Rr^2+97r^3)$$

$$-3r^4(4R+r)^4 \overset{Gerretsen}{\ge} r^4\left((192R^3-60R^2r-332Rr^2+97r^3)(16R\right.$$

$$\left.-5r)-3(4R+r)^4\right)$$

$$\overset{?}{\ge} 0 \Leftrightarrow 576t^4-672t^3-1325t^2+791t-122 \overset{?}{\ge} 0 \left(where\ t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(576t^3+480t^2-365t+61) \overset{?}{\ge} 0 \rightarrow true \because t \overset{Euler}{\ge} 2$$

$$\Rightarrow (1) is\ true \therefore 3\sum\frac{1}{a^3}tan\frac{A}{2}\le\sum\frac{1}{a^3}cot\frac{A}{2}\Rightarrow 3\sum b^3c^3tan\frac{A}{2}$$

$$\le\sum b^3c^3cot\frac{A}{2}\ (QED)$$

2385. **In any ΔABC holds:**

$$3\sum a^5\tan\frac{A}{2}\ge\sum a^5\cot\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum(s-a)^3=\left(\sum(s-a)\right)^3-3\prod\left((s-a)+(s-b)\right)=s^3-12Rrs$$

$$\Rightarrow\sum(s-a)^3\overset{(i)}{=}s^3-12Rrs\ and$$

$$\sum(s-a)^4=\sum(s^4+a^4-4a^3s-4as^3+6s^2a^2)$$

$$=3s^4+2(s^2+4Rr+r^2)^2-32Rrs^2-16s^2r^2-8s^2(s^2-6Rr-3r^2)$$

$$-8s^4$$

$$+12s^2(s^2-4Rr-4r^2)=s^4-16Rrs^2+2r^2(4R+r)^2$$

$$\Rightarrow\sum(s-a)^4\overset{(ii)}{=}s^4-16Rrs^2+2r^2(4R+r)^2$$

$$\sum a^5\tan\frac{A}{2}=r\sum\frac{(a-s+s)^5}{s-a}$$

$$=r\left[-\sum(s-a)^4+\sum\frac{s^5}{s-a}-15s^4+5s\sum(s-a)^3\right.$$

$$\left.+10s^2\sum a(s-a)\right]$$

$$\overset{via\ (i)\ and\ (ii)}{=}r\left[-\{s^4-16Rrs^2+2r^2(4R+r)^2\}+\frac{s^5r(4R+r)}{r^2s}-15s^4\right.$$

$$\left.+5s(s^3-12Rrs)+10s^2\{s(2s)-2s^2(s^2-4Rr-4r^2)\}\right]$$

$$= (4R-10r)s^4 + s^2r^2(36R+20r) - 2r^3(4R+r)^2 \overset{(1)}{=} \sum a^5 \tan\frac{A}{2}$$

$$\sum a^5 \cot\frac{A}{2} = s\sum\frac{a^5}{r_a} = \sum\frac{a^5(s-a)}{r} = \frac{1}{2r}\left(2s\sum a^5 - 2\sum a^6\right) \overset{(iii)}{=} \sum a^5\cot\frac{A}{2}$$

$$\text{Now, } \left(\sum a\right)\left(\sum a^5\right) = \sum a^6 + \sum ab(a^4+b^4) \Rightarrow 2s\sum a^5 - \sum a^6$$

$$= \sum\left\{ab\left(\sum a^4 - c^4\right)\right\} \overset{(m)}{=} \left(\sum a^4\right)\left(\sum ab\right) - abc\sum a^3$$

$$\text{Now, } \left(\sum a^2\right)\left(\sum a^4\right) = \sum a^6 + \sum\left\{a^2b^2\left(\sum a^2 - c^2\right)\right\}$$

$$= \left(\sum a^2\right)\left(\sum a^2b^2\right) + \sum a^6 - 3a^2b^2c^2$$

$$\Rightarrow -\sum a^6 \overset{(n)}{=} \left(\sum a^2\right)\left(\sum a^2b^2\right) - \left(\sum a^2\right)\left(\sum a^4\right) - 3a^2b^2c^2$$

$$(m)+(n) \Rightarrow 2s\sum a^5 - 2\sum a^6$$

$$= \left(\sum a^4\right)\left(\sum ab\right) + \left(\sum a^2\right)\left(\sum a^2b^2\right) - \left(\sum a^2\right)\left(\sum a^4\right) - abc\sum a^3 - 3a^2b^2c^2$$

$$= \left(\sum a^2\right)\left(\sum a^2b^2 - 2\sum a^2b^2 + 16r^2s^2\right) + \left(2\sum a^2b^2 - 16r^2s^2\right)\left(\sum ab\right) - abc(2s(s^2-6Rr-3r^2)+12Rrs)$$

$$= 16r^2s^2\left(\sum a^2 - \sum ab\right) + \left(\sum a^2b^2\right)\left(2\sum ab - \sum a^2\right) - 8Rrs^2(s^2-3r^2)$$

$$= 16r^2s^2(s^2-12Rr-3r^2) + ((s^2+4Rr+r^2)^2 - 16Rrs^2)(16Rr+4r^2) - 8Rrs^2(s^2-3r^2)$$

$$\Rightarrow \frac{1}{2r}\left(2s\sum a^5 - 2\sum a^6\right)$$

$$= \frac{16r^2s^2(s^2-12Rr-3r^2) + ((s^2+4Rr+r^2)^2-16Rrs^2)(16Rr+4r^2) - 8Rrs^2(s^2-3r^2)}{2r}$$

$$= (4R+10r)s^4 - 2rs^2(32R^2+42Rr+10r^2) + 2r^2(4R+r)^3$$

$$\overset{via\ (iii)}{\Rightarrow}\sum a^5\cot\frac{A}{2}\overset{(2)}{=}(4R+10r)s^4-2rs^2(32R^2+42Rr+10r^2)+2r^2(4R+r)^3$$

$$\therefore (1),(2)\Rightarrow 3\sum a^5\tan\frac{A}{2}\geq\sum a^5\cot\frac{A}{2}$$

$$\Leftrightarrow 3(4R-10r)s^4+s^2r^2(36R+20r)-2r^3(4R+r)^2$$

$$\geq(4R+10r)s^4-2rs^2(32R^2+42Rr+10r^2)+2r^2(4R+r)^3$$

$$\Leftrightarrow(8R-16r)s^4-24rs^4+3s^2r^2(36R+20r)$$

$$+2rs^2(32R^2+42Rr+10r^2)\overset{(l)}{\geq}6r^3(4R+r)^2++2r^2(4R+r)^3$$

$$\text{Now, LHS of}(l)\overset{Gerretsen}{\geq}(8R-16r)(16Rr-5r^2)s^2-24r(4R^2+4Rr+3r^2)s^2$$

$$+3s^2r^2(36R+20r)+2rs^2(32R^2+42Rr+10r^2)$$

$$\overset{?}{\geq}6r^3(4R+r)^2+2r^2(4R+r)^3$$

$$\Leftrightarrow\left((48R-4r)(R-2r)+36r^2\right)s^2\overset{?}{\geq}r(4R+r)^3+3r^2(4R+r)^2\ and$$

$$\because\left((48R-4r)(R-2r)+36r^2\right)s^2\overset{Gerretsen}{\geq}\left((48R-4r)(R-2r)+36r^2\right)(16Rr$$

$$-5r^2)\ \therefore\text{ it suffices to prove :}$$

$$\left((48R-4r)(R-2r)+36r^2\right)(16Rr-5r^2)\overset{?}{\geq}r(4R+r)^3+3r^2(4R+r)^2$$

$$\Leftrightarrow 44t^3-121t^2+73t-14\overset{?}{\geq}0\ \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(44t^2-33t+7)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{Euler}{\geq}2\Rightarrow(l)\text{is true}\therefore 3\sum a^5\tan\frac{A}{2}$$

$$\geq\sum a^5\cot\frac{A}{2}\ (QED)$$

2386. In ΔABC the following relationship holds:

$$\sqrt[4]{\frac{r_a}{h_a}}+\sqrt[4]{\frac{r_b}{h_b}}+\sqrt[4]{\frac{r_c}{h_c}}\leq\sqrt[4]{\frac{27(R^2-r^2)}{r^2}}$$

Proposed by Kostas Geronikolas-Greece

Solution by Adrian Popa-Romania

$$\left(\sqrt[4]{\frac{r_a}{h_a}}+\sqrt[4]{\frac{r_b}{h_b}}+\sqrt[4]{\frac{r_c}{h_c}}\right)^4=\left(1\cdot1\cdot1\cdot\sqrt[4]{\frac{r_a}{h_a}}+1\cdot1\cdot1\cdot\sqrt[4]{\frac{r_b}{h_b}}+1\cdot1\cdot1\cdot\sqrt[4]{\frac{r_c}{h_c}}\right)^4\overset{Holder}{\leq}$$

$$\leq(1+1+1)(1+1+1)(1+1+1)\left(\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\right)=27\left(\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\right)$$

$$\because \frac{r_a}{h_a}=\frac{\frac{F}{s-a}}{\frac{2F}{a}}=\frac{a}{2(s-a)}\to\sum_{cyc}\frac{r_a}{h_a}=\frac{1}{2}\sum_{cyc}\frac{a}{s-a}=\frac{1}{2}\cdot\frac{2(2R-r)}{r}=\frac{2R-r}{r}$$

Now, we must to prove that:

$$27\cdot\frac{2R-r}{r}\leq\frac{27(R^2-r^2)}{r^2}\Leftrightarrow 2Rr-r^2\leq R^2-r^2\Leftrightarrow R\geq 2r\ (Euler)$$

Therefore,

$$\sqrt[4]{\frac{r_a}{h_a}}+\sqrt[4]{\frac{r_b}{h_b}}+\sqrt[4]{\frac{r_c}{h_c}}\leq\sqrt[4]{\frac{27(R^2-r^2)}{r^2}}$$

2387. Prove that in any acute ΔABC holds:

$$\sum\left[\left(\frac{\sin A\sin B}{\sin C}\right)^2+\left(\frac{\cos A\cos B}{\cos C}\right)^2\right]\geq 3$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$ABC\text{ is an acute triangle }\to\cos A,\cos B,\cos C>0$$

$$We\ have:\ \sum\left(\frac{\sin A\sin B}{\sin C}\right)^2\overset{\sum x^2\geq\sum xy}{\geq}\sum\frac{\sin A\sin B}{\sin C}\cdot\frac{\sin A\sin C}{\sin B}=\sum\sin^2 A\quad(1)$$

$$and:\ \sum\left(\frac{\cos A\cos B}{\cos C}\right)^2\overset{\sum x^2\geq\sum xy}{\geq}\sum\frac{\cos A\cos B}{\cos C}\cdot\frac{\cos A\cos C}{\cos B}=\sum\cos^2 A\quad(2)$$

$$(1),(2)\to\sum\left[\left(\frac{\sin A\sin B}{\sin C}\right)^2+\left(\frac{\cos A\cos B}{\cos C}\right)^2\right]\geq\sum(\sin^2 A+\cos^2 A)=\sum 1=3$$

$$Therefore,\quad\sum\left[\left(\frac{\sin A\sin B}{\sin C}\right)^2+\left(\frac{\cos A\cos B}{\cos C}\right)^2\right]\geq 3$$

2388. In any $\Delta\, ABC$ the following relationship holds:

$$3 \geq \sum \sqrt{\frac{2r}{r_b + r_c}} + \sum \frac{m_a + \sqrt{g_a}\left(\sqrt{n_a} - \sqrt{g_a}\right)}{s\sqrt{2}}$$

Proposed by Bogdan Fuştei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof : } r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc = an_a^2 + a(as - s^2)$$

$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc)$$

$$= as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \Rightarrow n_a^2 \overset{(I)}{=} s^2 - 2h_ar_a$$

$$an_a^2 \cdot ag_a^2 \geq a^2s^2(s-a)^2$$

$$\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\}\{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \overset{(a)}{\geq} a^2s^2(s-a)^2$$

Let $s-a = x$, $s-b = y$ and $s-c = z$ $\therefore s = x+y+z \Rightarrow a = y+z$, $b = z+x$ and $c = x+y$ and via these substitutions,

$$(a) \Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \to \text{true}$$

$$\Rightarrow (a) \text{ is true} \Rightarrow n_ag_a \overset{(b)}{\geq} s(s-a)$$

Again, Stewart's theorem

$$\Rightarrow b^2(s-c)+c^2(s-b)\overset{(m)}{\triangleq} an_a^2+a(s-b)(s-c) \text{ and } b^2(s-b)$$

$$+c^2(s-c)\overset{(n)}{\triangleq} ag_a^2+a(s-b)(s-c)$$

$$\text{and } (m)+(n)\Rightarrow (b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$

$$\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2$$

$$=2(n_a^2+g_a^2)\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)$$

$$\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)\Rightarrow n_a^2+g_a^2=(b-c)^2+2s(s-a)$$

$$\Rightarrow n_a^2+g_a^2+2n_ag_a \overset{\text{via (b)}}{\geq} (b-c)^2+4s(s-a)$$

$$\Rightarrow (n_a+g_a)^2\geq 4m_a^2\Rightarrow 2m_a\leq n_a+g_a$$

$$\Rightarrow \frac{m_a+\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{s\sqrt{2}} \overset{A-G}{\leq} \frac{m_a-g_a}{s\sqrt{2}}+\frac{n_a+g_a}{2\sqrt{2}s}$$

$$=\frac{2m_a-2g_a+n_a+g_a}{2\sqrt{2}s}$$

$$\leq \frac{n_a+g_a+n_a-g_a}{2\sqrt{2}s}=\frac{n_a}{\sqrt{2}s}\Rightarrow \sqrt{\frac{2r}{r_b+r_c}}+\frac{m_a+\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{s\sqrt{2}} \overset{\text{CBS and via (i)}}{\leq}$$

$$\sqrt{2}\sqrt{\frac{2r}{4R\cos^2\frac{A}{2}}+\frac{n_a^2}{2s^2}} \overset{\text{via }(l)}{\triangleq} \sqrt{\frac{r}{R\cos^2\frac{A}{2}}+\frac{s^2-2h_ar_a}{s^2}}$$

$$=\sqrt{1+\frac{r}{R\cos^2\frac{A}{2}}-\frac{4rs^2\tan\frac{A}{2}}{4Rs^2\tan\frac{A}{2}\cos^2\frac{A}{2}}}\Rightarrow \sqrt{\frac{2r}{r_b+r_c}}+\frac{m_a+\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{s\sqrt{2}}$$

$$\leq 1 \text{ and analogs}$$

$$\overset{\text{summing up}}{\Rightarrow} 3\geq \sum\sqrt{\frac{2r}{r_b+r_c}}+\sum\frac{m_a+\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{s\sqrt{2}} \text{ (QED)}$$

2389. In ΔABC the following relationship holds:

$$27r^2 \le h_a r_a + h_b r_b + h_c r_c \le \frac{27R^3}{8r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

For LHS: $h_a r_a + h_b r_b + h_c r_c \ge 27r^2$

$$2F^2\left(\frac{1}{a(s-a)}+\frac{1}{b(s-b)}+\frac{1}{c(s-c)}\right) \ge 27r^2; (1)$$

$$But \sum_{cyc}\frac{1}{a(s-a)} = \frac{s^2+(4R+r)^2}{4Rrs^2}; (2)$$

From $(1), (2)$ *we must to prove that:*

$$2s^2r^2 \cdot \frac{s^2+(4R+r)^2}{4Rrs^2} \ge 27r^2 \Leftrightarrow s^2+(4R+r)^2 \ge 54Rr$$

$$\Leftrightarrow s^2+16R^2+8Rr+r^2 \ge 54Rr; (3)$$

From $s^2 \ge 16Rr - 5r^2$ $(Gerretsen); (4)$

From $(3), (4)$ *we must to prove that:*

$$16R^2+24Rr-4r62 \ge 54Rr \Leftrightarrow 16R^2 \ge 30Rr+4r^2$$

$$\Leftrightarrow 8R^2 \ge 15Rr+2r^2; (5)$$

From $R \ge 2r$ $(Euler) \to r \le \frac{R}{r} \to 15Rr+4r^2 \le \frac{15R^2}{2}+\frac{R^2}{2} = \frac{16R^2}{2} = 8R^2 \to (5)$ *is true.*

For RHS, we have: $h_a r_a + h_b r_b + h_c r_c \le \frac{27R^3}{8r}; (6)$

Let $a \le b \le c \to h_a \ge h_b \ge h_c$ and $r_a \le r_b \le r_c$. *From Chebyshev's inequality, we have:*

$$h_a r_a + h_b r_b + h_c r_c \le \frac{1}{3}(h_a+h_b+h_c)(r_a+r_b+r_c); (7)$$

From $(6), (7)$ *ewe must show that:*

$$(h_a+h_b+h_c)(r_a+r_b+r_c) \le \frac{81R^3}{8r}; (8)$$

But $h_a+h_b+h_c = \frac{s^2+r^2+4Rr}{2R}$ and $r_a+r_b+r_c = 4R+r; (9)$

From $(8), (9)$ *we must to prove:*

$$\frac{s^2+r^2+4Rr}{2R}\cdot(4R+r)\le\frac{81R^3}{8r}\Leftrightarrow(s^2+r^2+4Rr)(4R+r)\le\frac{162R^4}{8r};(10)$$

From Gerretsen inequality we have: $s^2\le 4R^2+4Rr+3r^2;(11)$

From $(10),(11)$ *we must show that:*

$$4(R+r)^2(4R+r)\le\frac{162R^4}{8r}\Leftrightarrow 2r(R+r)^2(4R+r)\le\frac{81R^4}{8};(12)$$

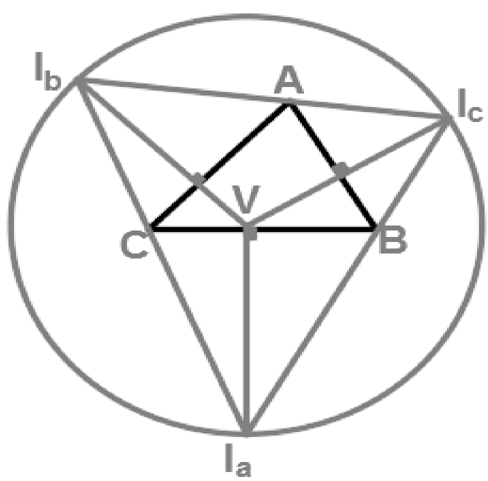
$$But\ r\le\frac{R}{2}\to 2r\cdot\frac{9}{4}R^2\cdot\frac{9}{2}R\le\frac{81R^4}{8}\Leftrightarrow R\ge 2r\ (Euler).$$

2390. V –Bevan point in ΔABC, R_a, R_bR_c –circumradii in $\Delta VI_bI_c, \Delta VI_cI_a, \Delta VI_aI_b$. Prove that :

$$\frac{w_a}{R_a}+\frac{w_b}{R_b}+\frac{w_c}{R_c}\ge\frac{9r}{2R}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have:\ \mu(I_bAC)=\frac{\pi-A}{2}\to\mu(I_cI_bV)=\frac{\pi}{2}-\mu(I_bAC)=\frac{A}{2}\ and\ I_cV=2R.$$

$$In\ \Delta VI_bI_c, we\ have:\ \sin I_cI_bV=\frac{I_cV}{2R_a}\to R_a=R\csc\frac{A}{2}\ (and\ analogs)$$

$$\sum\frac{w_a}{R_a}=\frac{1}{R}\sum\frac{2bc}{b+c}\cos\frac{A}{2}.\sin\frac{A}{2}=\frac{1}{R}\sum\frac{bc}{b+c}.\frac{a}{2R}=\frac{abc}{2R^2}\sum\frac{1}{b+c}\overset{CBS}{\geq}\frac{4sRr}{2R^2}.\frac{9}{2\sum a}$$
$$=\frac{9r}{2R}$$

$$Therefore,\quad \sum\frac{w_a}{R_a}\geq\frac{9r}{2R}.$$

2391. In any $\Delta\, ABC$ the following relationship holds:

$$3\sum a^6 tan\frac{A}{2}\geq\sum a^6 cot\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$Proof: Let\ s-a=x, s-b=y\ and\ s-c=z\ \therefore\sum(s-a)^2=\left(\sum x\right)^2-2\sum xy$$
$$=s^2-2(4Rr+r^2)\Rightarrow\sum x^2\overset{(1)}{=}s^2-8Rr-2r^2$$
$$Now,\sum(s-a)^3=\left(\sum(s-a)\right)^3-3\prod\big((s-a)+(s-b)\big)=s^3-12Rrs$$
$$\Rightarrow\sum x^3\overset{(2)}{=}s^3-12Rrs\ and$$
$$\sum(s-a)^4=\sum(s^4+a^4-4a^3s-4as^3+6s^2a^2)$$
$$=3s^4+2(s^2+4Rr+r^2)^2-32Rrs^2-16s^2r^2-8s^2(s^2-6Rr-3r^2)-8s^4$$
$$+12s^2(s^2-4Rr-4r^2)=s^4-16Rrs^2+2r^2(4R+r)^2$$
$$\Rightarrow\sum x^4\overset{(3)}{=}s^4-16Rrs^2+2r^2(4R+r)^2$$
$$\sum(s-a)^2(s-b)^2=\sum x^2y^2=\left(\sum xy\right)^2-2xyz\left(\sum x\right)$$
$$=\left(\sum(s-a)(s-b)\right)^2-2r^2s^2\Rightarrow\sum x^2y^2\overset{(4)}{=}(4Rr+r^2)^2-2r^2s^2$$
$$\left(\sum x^2\right)\left(\sum x^3\right)=\sum x^5+\sum\left(x^2y^2\left(\sum x-z\right)\right)$$
$$=\sum x^5+\left(\sum x\right)\left(\sum x^2y^2\right)-xyz\sum xy$$

$$\Rightarrow \sum x^5 = \left(\sum x^2\right)\left(\sum x^3\right) - \left(\sum x\right)\left(\sum x^2y^2\right) + xyz\sum xy$$

$$\overset{via\ (1),(2),(4)}{\triangleq} (s^2 - 8Rr - 2r^2)(s^3 - 12Rrs) - s((4Rr + r^2)^2 - 2r^2s^2) + r^2s(4Rr + r^2)$$

$$\Rightarrow \sum x^5 \overset{(5)}{\triangleq} s[s^4 - 20Rrs^2 + 20Rr^2(4R + r)]$$

$$\sum a^6 tan\frac{A}{2} = r\sum \frac{(a - s + s)^6}{s - a}$$

$$= r\left[\sum (s - a)^5 + \sum \frac{s^6}{s - a} - 3s^5 - 6s\sum (s - a)^4 + 15s^2\sum (s - a)^3 - 20s^3\sum (s - a)^2\right]$$

$$\overset{via\ (1),(2),(3),(5)}{\triangleq} r\left[s[s^4 - 20Rrs^2 + 20Rr^2(4R + r)] + \frac{s^6(4Rr + r^2)}{r^2s} - 3s^5 - 6s[s^4 - 16Rrs^2 + 2r^2(4R + r)^2] + 15s^2(s^3 - 12Rrs) - 20s^3(s^2 - 8Rr - 2r^2)\right]$$

$$= \boxed{4s[(R - 3r)s^4 + s^2r^2(14R + 10r) - r^3(28R^2 + 19Rr + 3r^2)] \overset{(l)}{\triangleq} \sum a^6 tan\frac{A}{2}}$$

$$\sum a^6 cot\frac{A}{2} = s\sum \frac{a^6}{r_a} = \sum \frac{a^6(s - a)}{r} = \frac{1}{r}\left(s\sum a^6 - \sum a^7\right) \overset{(a)}{\triangleq} \sum a^6 cot\frac{A}{2}$$

$$\left(\sum a^2\right)\left(\sum a^4\right) = \sum a^6 + \left(\sum a^2\right)\left(\sum a^2b^2\right) - 3a^2b^2c^2 \Rightarrow s\sum a^6$$

$$= s\left(\sum a^2\right)\left(2\sum a^2b^2 - 16s^2r^2 - \sum a^2b^2\right) + 48R^2r^2s^3$$

$$\Rightarrow s\sum a^6 \overset{(i)}{\triangleq} s\left(\sum a^2\right)[(s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 16s^2r^2] + 48R^2r^2s^3$$

$$\left(\sum a^3\right)\left(\sum a^4\right) = \sum a^7 + 2s\left(\sum a^3b^3\right) - abc\left(\sum a^2b^2\right) \Rightarrow -\sum a^7$$

$$= -\left(\sum a^3\right)\left(2\sum a^2b^2 - 16s^2r^2\right) - abc\left(\sum a^2b^2\right)$$

$$+2s[(s^2 + 4Rr + r^2)^3 - 3.4Rrs.2s(s^2 + 2Rr + r^2)] = 32r^2s^3(s^2 - 6Rr - 3r^2)$$

$$-[(s^2 + 4Rr + r^2)^2 - 16Rrs^2]\left(4Rrs + 4s(s^2 - 6Rr - 3r^2)\right) + 2s(s^2 + 4Rr + r^2)^3$$

$$- 48Rrs^3(s^2 + 2Rr + r^2)$$

$$\Rightarrow -\sum a^7 \overset{(ii)}{=} 32r^2s^3(s^2-6Rr-3r^2)$$
$$-[(s^2+4Rr+r^2)^2-16Rrs^2]\left(4Rrs+4s(s^2-6Rr-3r^2)\right)$$
$$+2s(s^2+4Rr+r^2)^3-48Rrs^3(s^2+2Rr+r^2)$$

$$(i),(ii) \Rightarrow s\sum a^6-\sum a^7$$
$$=s\left(\sum a^2\right)[(s^2+4Rr+r^2)^2-16Rrs^2-16s^2r^2]+48R^2r^2s^3$$
$$+32r^2s^3(s^2-6Rr-3r^2)$$
$$-[(s^2+4Rr+r^2)^2-16Rrs^2]\left(4Rrs+4s(s^2-6Rr-3r^2)\right)$$
$$+2s(s^2+4Rr+r^2)^3-48Rrs^3(s^2+2Rr+r^2)$$
$$=4rs[(R+3r)s^4-rs^2(20R^2+34Rr+10r^2)+r^2(80R^3+88R^2r+29Rr^2+3r^3)]$$

$$\overset{via\ (a)}{\Rightarrow} \boxed{\sum a^6\cot\frac{A}{2} \overset{(m)}{=} 4s\left[(R+3r)s^4-rs^2\left(20R^2+34Rr+10r^2\right)+r^2\left(80R^3+88R^2r+29Rr^2+3r^3\right)\right]}$$

$$(l),(m) \Rightarrow 3\sum a^6\tan\frac{A}{2} \geq \sum a^6\cot\frac{A}{2}$$
$$\Leftrightarrow 3(R-3r)s^4+3s^2r^2(14R+10r)-3r^3(28R^2+19Rr+3r^2)$$
$$\geq (R+3r)s^4-rs^2(20R^2+34Rr+10r^2)+r^2(80R^3+88R^2r+29Rr^2+3r^3)$$
$$\Leftrightarrow (2R-4r)s^4-8rs^4+3s^2r^2(14R+10r)$$
$$+rs^2(20R^2+34Rr+10r^2) \overset{(iii)}{\geq} 3r^3(28R^2+19Rr+3r^2)$$
$$+r^2(80R^3+88R^2r+29Rr^2+3r^3)$$

$$Now,\ LHS\ of\ (iii) \overset{Gerretsen}{\geq} rs^2[(2R-4r)(16R-5r)-8(4R^2+4Rr+3r^2)$$
$$+3r(14R+10r)+20R^2+34Rr+10r^2]$$
$$=rs^2(20R^2-30Rr+36r^2) \overset{Gerretsen}{\geq} r^2(20R^2-30Rr+36r^2)(16R-5r)$$
$$\overset{?}{\geq} 3r^3(28R^2+19Rr+3r^2)+r^2(80R^3+88R^2r+29Rr^2+3r^3)$$
$$\Leftrightarrow 15t^3-47t^2+40t-12 \overset{?}{\geq} 0 \left(where\ t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(15t+13)+32\right)\overset{?}{\gtreqless} 0 \to true \because t \overset{Euler}{\gtreqless} 2 \Rightarrow (iii) is\ true$$

$$\therefore 3\sum a^6 tan\frac{A}{2} \geq \sum a^6 cot\frac{A}{2}\ (QED)$$

2392. In ΔABC the following relationship holds:

$$\frac{2r}{R^2} \leq \frac{\sqrt{r_a r_b}}{m_a^2+m_b^2}+\frac{\sqrt{r_b r_c}}{m_b^2+m_c^2}+\frac{\sqrt{r_c r_a}}{m_c^2+m_a^2} \leq \frac{R^2}{8r^3}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$For\ LHS: \sum_{cyc}\frac{\sqrt{r_a r_b}}{m_a^2+m_b^2} \geq 3\sqrt[3]{\frac{r_a r_b r_c}{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}}; (1)$$

$$\sqrt[3]{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)} \leq \frac{2(m_a^2+m_b^2+m_c^2)}{3} = \frac{a^2+b^2+c^2}{2}; (2)$$

From (1), (2) we have:

$$\sum_{cyc}\frac{\sqrt{r_a r_b}}{m_a^2+m_b^2} \geq \frac{6\sqrt[3]{r_a r_b r_c}}{a^2+b^2+c^2}; (3)$$

$$But\ r_a r_b r_c = rs^2\ and\ s^2 \geq 27r^2 \to r_a r_b r_c \geq 27r^3; (4)$$

$$From\ (4), (5) it\ follows\ \sum_{cyc}\frac{\sqrt{r_a r_b}}{m_a^2+m_b^2} \geq \frac{18r}{a^2+b^2+c^2}; (5)$$

But $a^2+b^2+c^2 = 2(s^2-r^2-4Rr); (6)$. From (5), (6) we must to prove that:

$$\sum_{cyc}\frac{\sqrt{r_a r_b}}{m_a^2+m_b^2} \geq \frac{9r}{s^2-r^2-4Rr}; (7)$$

From (7) we must to prove:

$$\frac{9r}{s^2-r^2-4Rr} \geq \frac{2r}{R^2} \Leftrightarrow 9R^2 \geq 2(s^2-r^2-4Rr); (8)$$

From Gerretsen inequality we have: $s^2 \leq 4R^2+4Rr+3r^2; (9)$

From (8), (9) we must to show that:

$$9R^2 \geq 2(4R^2+2r^2) \Leftrightarrow 9R^2 \geq 8R^2+4r^2 \Leftrightarrow R^2 \geq 4r^2 \Leftrightarrow R \geq 2r\ (Euler).$$

$$For\ RHS, use: m_a \geq \frac{b+c}{2}\cos\frac{A}{2}, we\ have:$$

$$m_a^2 + m_b^2 \geq 2m_am_b \geq 2\cdot\frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{a+c}{2}\cos\frac{B}{2} \geq 2\sqrt{bc}\cos\frac{A}{2}\cdot\sqrt{ac}\cos\frac{B}{2} =$$

$$= 2\sqrt{ab}\cdot c\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ac}} = 2s\sqrt{(s-a)(s-b)} \to$$

$$\frac{1}{m_a^2+m_b^2} \leq \frac{1}{2s\sqrt{(s-a)(s-b)}} \to \frac{\sqrt{r_ar_b}}{m_a^2+m_b^2} <= \frac{\sqrt{\frac{F^2}{(s-a)(s-b)}}}{2s\sqrt{(s-a)(s-b)}}$$

$$\to \frac{\sqrt{r_ar_b}}{m_a^2+m_b^2} \leq \frac{F}{2s\sqrt{(s-a)(s-b)}}$$

We must show that:

$$\frac{F}{2s}\cdot\sum_{cyc}\frac{1}{(s-a)(s-b)} \leq \frac{R^2}{8r^3} \Leftrightarrow \frac{sr}{s}\cdot\sum_{cyc}\frac{1}{(s-a)(s-b)} \leq \frac{R^3}{4r^3}$$

$$4r^4\cdot\sum_{cyc}\frac{1}{(s-a)(s-b)} \leq R^2; (10)$$

$$But \sum_{cyc}\frac{1}{(s-a)(s-b)} = \frac{1}{r^2}; (11). From\ (10),(11) we\ must\ show:$$

$$4r^2 \leq R^2 \Leftrightarrow 2r \leq R\ (Euler).$$

2393. In acute ΔABC the following relationship holds:

$$\sum\frac{\tan^5 A}{\tan^3 B} \geq \frac{1}{3}\left(\frac{p}{r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\Delta ABC\ is\ acute\ triangle \to \tan A, \tan B, \tan C > 0$$

$$\left(\sum\frac{\tan^5 A}{\tan^3 B}\right)\left(\sum\tan B\right)^3\left(\sum 1\right) \overset{Hölder}{\geq} \left(\sum\tan A\right)^5 \to \sum\frac{\tan^5 A}{\tan^3 B} \geq \frac{1}{3}\left(\sum\tan A\right)^2$$

$$We\ have:\sum\tan A=\prod\tan A=\frac{\prod\sin A}{\prod\cos A}=\frac{pr}{2R^2}\cdot\frac{4R^2}{p^2-(2R+r)^2}=\frac{2pr}{p^2-(2R+r)^2}$$

$$\ge\overset{Gerresten}{\overbrace{\ge}}\frac{2pr}{(4R^2+4Rr+3r^2)-(2R+r)^2}=\frac{2pr}{2r^2}=\frac{p}{r}$$

$$Therefore,\quad \sum\frac{\tan^5 A}{\tan^3 B}\ge\frac{1}{3}\left(\frac{p}{r}\right)^2$$

2394. In acute ΔABC the following relationship holds:

$$\sum\cos A\left(\frac{\cos B}{\cos C}\right)^3\ge\frac{27}{8}\left(\frac{R}{R+r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$ABC\ is\ an\ acute\ triangle\to\cos A,\cos B,\cos C>0.$$

$$\sum\cos A\left(\frac{\cos B}{\cos C}\right)^3\cdot\left(\sum\cos A\right)^2\overset{Hölder}{\overbrace{\ge}}\left(\sum\frac{\cos A\cdot\cos B}{\cos C}\right)^3\to\sum\cos A\left(\frac{\cos B}{\cos C}\right)^3$$

$$\ge\left(\frac{R}{R+r}\right)^2\left(\sum\frac{\cos A\cdot\cos B}{\cos C}\right)^3$$

$$So,we\ need\ to\ prove:\ \sum\frac{\cos A\cdot\cos B}{\cos C}\overset{(*)}{\overbrace{\ge}}\frac{3}{2},\forall\Delta ABC\ acute.$$

$$Using\ the\ substitutions:\ A=\frac{\pi-X}{2},B=\frac{\pi-Y}{2},C=\frac{\pi-Z}{2},X,Y,Z\in(0,\pi),\sum X=\pi$$

$$\to(*)\leftrightarrow\sum\frac{\sin\frac{X}{2}\cdot\sin\frac{Y}{2}}{\sin\frac{Z}{2}}\ge\frac{3}{2},\forall\Delta XYZ\qquad\leftrightarrow\left(\prod\sin\frac{X}{2}\right)\sum\frac{1}{\sin^2\frac{Z}{2}}\ge\frac{3}{2}$$

$$We\ know\ that:\ \prod\sin\frac{X}{2}=\frac{r}{4R}\ and\ \sum\frac{1}{\sin^2\frac{X}{2}}=\frac{s^2+r^2-8Rr}{r^2}$$

$$\to(*)\leftrightarrow\frac{s^2+r^2-8Rr}{4Rr}\ge\frac{3}{2}\leftrightarrow s^2\ge14Rr-r^2$$

$$Which\ is\ true\ from\ Gerretsen,s^2\ge16Rr-5r^2\overset{?}{\overbrace{\ge}}14Rr-r^2\leftrightarrow R\ge2r\ (Euler)$$

$$\rightarrow (*) \text{is true} \rightarrow \sum \cos A \left(\frac{\cos B}{\cos C}\right)^3 \geq \frac{27}{8}\left(\frac{R}{R+r}\right)^2.$$

2395. In any Δ ABC holds:

$$32r^3(4R+r)^2 \leq \sum a^5 \cot\frac{A}{2} \leq \frac{2R^4}{r}(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof}: \sum a^5 \cot\frac{A}{2} = s\sum \frac{a^5}{r_a} = \sum \frac{a^5(s-a)}{r}$$

$$= \frac{1}{2r}\left(2s\sum a^5 - 2\sum a^6\right) \stackrel{(1)}{\triangleq} \sum a^5 \cot\frac{A}{2}$$

$$\text{Now, } \left(\sum a\right)\left(\sum a^5\right) = \sum a^6 + \sum ab(a^4+b^4) \Rightarrow 2s\sum a^5 - \sum a^6$$

$$= \sum \left\{ab\left(\sum a^4 - c^4\right)\right\} \stackrel{(i)}{\triangleq} \left(\sum a^4\right)\left(\sum ab\right) - abc\sum a^3$$

$$\text{Now, } \left(\sum a^2\right)\left(\sum a^4\right) = \sum a^6 + \sum\left\{a^2b^2\left(\sum a^2 - c^2\right)\right\}$$

$$= \left(\sum a^2\right)\left(\sum a^2b^2\right) + \sum a^6 - 3a^2b^2c^2$$

$$\Rightarrow -\sum a^6 \stackrel{(ii)}{\triangleq} \left(\sum a^2\right)\left(\sum a^2b^2\right) - \left(\sum a^2\right)\left(\sum a^4\right) - 3a^2b^2c^2$$

$$(i) + (ii) \Rightarrow 2s\sum a^5 - 2\sum a^6$$

$$= \left(\sum a^4\right)\left(\sum ab\right) + \left(\sum a^2\right)\left(\sum a^2b^2\right) - \left(\sum a^2\right)\left(\sum a^4\right)$$

$$- abc\sum a^3 - 3a^2b^2c^2$$

$$= \left(\sum a^2\right)\left(\sum a^2b^2 - 2\sum a^2b^2 + 16r^2s^2\right) + \left(2\sum a^2b^2 - 16r^2s^2\right)\left(\sum ab\right)$$

$$- abc(2s(s^2 - 6Rr - 3r^2) + 12Rrs)$$

$$= 16r^2s^2\left(\sum a^2 - \sum ab\right) + \left(\sum a^2b^2\right)\left(2\sum ab - \sum a^2\right) - 8Rrs^2(s^2 - 3r^2)$$

$$= 16r^2s^2(s^2 - 12Rr - 3r^2) + ((s^2 + 4Rr + r^2)^2 - 16Rrs^2)(16Rr + 4r^2)$$

$$- 8Rrs^2(s^2 - 3r^2)$$

$$\Rightarrow \frac{1}{2r}\left(2s\sum a^5 - 2\sum a^6\right)$$

$$= \frac{16r^2s^2(s^2-12Rr-3r^2)+((s^2+4Rr+r^2)^2-16Rrs^2)(16Rr+4r^2)-8Rrs^2(s^2-3r^2)}{2r}$$

$$= (4R+10r)s^4 - 2rs^2(32R^2+42Rr+10r^2)+2r^2(4R+r)^3$$

$$\overset{\text{via (1)}}{\Rightarrow}\sum a^5\cot\frac{A}{2}\overset{(a)}{=}(4R+10r)s^4-2rs^2(32R^2+42Rr+10r^2)+2r^2(4R+r)^3 \therefore (a)$$

$$\Rightarrow \sum a^5\cot\frac{A}{2}\le\frac{2R^4}{r}(4R+r)^2$$

$$\Leftrightarrow r(2R+5r)s^4-r^2s^2(32R^2+42Rr+10r^2)+r^3(4R+r)^3\overset{(iii)}{\le}R^4(4R+r)^2$$

$$\text{Now, LHS of (iii)}\overset{\text{Gerretsen}}{\le} rs^2\Big((2R+5r)(4R^2+4Rr+3r^2)$$
$$-r(32R^2+42Rr+10r^2)\Big)+r^3(4R+r)^3$$
$$= rs^2\left((R-2r)(8R^2+12Rr+8r^2)+21r^3\right)+r^3(4R+r)^3$$

$$\overset{\text{Gerretsen}}{\le} r\left((R-2r)(8R^2+12Rr+8r^2)+21r^3\right)(4R^2+4Rr+3r^2)$$
$$+r^3(4R+r)^3\overset{?}{\le}R^4(4R+r)^2$$

$$\Leftrightarrow 16t^6-24t^5-15t^4-8t^3+8t^2+16t-16\overset{?}{\ge}0\ \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(16t^4+40t^3+81t^2+156t+308)+624\right)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\Rightarrow \text{(iii) is true}\therefore\sum a^5\cot\frac{A}{2}\le\frac{2R^4}{r}(4R+r)^2$$

$$\text{Now, }2s^2\overset{\text{Gerretsen}}{\ge}27Rr+5r(R-2r)\overset{\text{Euler}}{\ge}27Rr\Rightarrow 2s^2\overset{(*)}{\ge}27Rr$$

$$\text{Again, }\sum a^5\cot\frac{A}{2}$$

$$= s\sum\frac{a^4}{\left(\frac{r_a}{a}\right)}\overset{\text{Bergstrom}}{\ge}\frac{4s(s^2-4Rr-r^2)^2}{\sum\left(\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}\right)}\overset{\text{Gerretsen}}{\ge}\frac{16Rs^2(12Rr-6r^2)^2}{s^2+(4R+r)^2}\overset{\text{Trucht}}{\ge}\frac{16Rs^2(12Rr-6r^2)^2}{\frac{4}{3}(4R+r)^2}$$

$$\overset{?}{\gtreqless} 32r^3(4R+r)^2 \Leftrightarrow 27R(2R-r)^2(2s^2) \overset{?}{\gtreqless} 4r(4R+r)^4 \text{ and } \because 2s^2 \overset{\text{via }(*)}{\gtreqless} 27Rr$$

$\therefore$ **it suffices to prove :**

$$27R(2R-r)^2(2s^2)(27Rr) \overset{?}{\gtreqless} 4r(4R+r)^4 \Leftrightarrow 27R(2R-r) \overset{?}{\gtreqless} 2(4R+r)^2$$

$$\Leftrightarrow 22R^2 - 43Rr - 2r^2 \overset{?}{\gtreqless} 0 \Leftrightarrow (R-2r)(22R+r) \overset{?}{\gtreqless} 0 \rightarrow \text{true}$$

$$\therefore 32r^3(4R+r)^2 \le \sum a^5 \cot\frac{A}{2} \text{ (QED)}$$

2396. In ΔABC the following relationship holds:

$$12r \le \frac{r_a}{sin^2A} + \frac{r_b}{sin^2B} + \frac{r_c}{sin^2C} \le \frac{3R^2}{r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$We\ have: \frac{r_a}{sin^2A} + \frac{r_b}{sin^2B} + \frac{r_c}{sin^2C} \ge 3\sqrt[3]{\frac{r_ar_br_c}{sin^2Asin^2Bsin^2C}} \rightarrow$$

$$We\ must\ show\ that: \sqrt[3]{\frac{r_ar_br_c}{sin^2Asin^2Bsin^2C}} \ge 4r; (1)$$

$$But\ r_ar_br_c = s^2r\ and\ sinAsinBsinC = \frac{sr}{2R^2}; (2)$$

From (1), (2) we must to prove that:

$$\sqrt[3]{\frac{4R^4}{r}} \ge 4r \Leftrightarrow 4R^4 \ge 64r^4 \Leftrightarrow R \ge 2r\ (Euler).$$

$$r_a = \frac{F}{s-a}\ and\ sinA = \frac{a}{2R} \rightarrow we\ must\ to\ show:$$

$$4R^2F\left(\frac{1}{a^2(s-a)} + \frac{1}{b^2(s-b)} + \frac{1}{c^2(s-c)}\right) \le \frac{3R^2}{r} \Leftrightarrow \sum_{cyc} \frac{1}{a^2(s-a)} \le \frac{3}{4s^2r}; (3)$$

$$\sum_{cyc} \frac{b^2c^2(s-b)(s-c)}{a^2b^2c^2(s-a)(s-b)(s-c)} \le \frac{4}{4s^2r}; (4)$$

$$But: \sqrt{(s-b)(s-c)} \le \frac{s-b+s-c}{2} \rightarrow (s-b)9s-c) \le \frac{a^2}{4}; (5)$$

From $(4),(5)$ *we must to prove:*

$$\sum_{cyc}\frac{a^2b^2c^2}{4a^2b^2c^2(s-a)(s-b)(s-c)}\le\frac{3}{4r^2s}\Leftrightarrow\frac{3}{(s-a)(s-b)(s-c)}\le\frac{3}{sr^2}$$

$$\Leftrightarrow(s-a)(s-b)(s-c)=sr^2$$

2397. In acute ΔABC the following relationship holds:

$$\frac{8}{9R^3}\le\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\le\frac{2R^2-Rr}{54r^5}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$In\ acute\ \Delta ABC, we\ have: m_a\le 2R\cdot\cos^2\frac{A}{2}\to\frac{1}{m_a^3}\ge\frac{1}{8R^3\cdot\cos^6\frac{A}{2}}\to$$

$$\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\ge\frac{1}{8R^3}\left(\frac{1}{\cos^6\frac{A}{2}}+\frac{1}{\cos^6\frac{B}{2}}+\frac{1}{\cos^6\frac{C}{2}}\right). We\ must\ show\ that:$$

$$\frac{1}{8R^3}\left(\frac{1}{\cos^6\frac{A}{2}}+\frac{1}{\cos^6\frac{B}{2}}+\frac{1}{\cos^6\frac{C}{2}}\right)\ge\frac{8}{9R^3}\Leftrightarrow\frac{1}{\cos^6\frac{A}{2}}+\frac{1}{\cos^6\frac{B}{2}}+\frac{1}{\cos^6\frac{C}{2}}\ge\frac{64}{9};(1)$$

$$But:\frac{1}{\cos^6\frac{A}{2}}+\frac{1}{\cos^6\frac{B}{2}}+\frac{1}{\cos^6\frac{C}{2}}\ge 3\sqrt[3]{\frac{1}{\cos^6\frac{A}{2}\cos^6\frac{B}{2}\cos^6\frac{C}{2}}};(2)$$

From $(1),(2)$ *we must to prove that:*

$$\frac{1}{\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}\ge\frac{64}{27}\Leftrightarrow\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\le\frac{3\sqrt{3}}{8}, which\ is\ true\ because:$$

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{4R}\le\frac{3\sqrt{3}}{8}\Leftrightarrow s\le\frac{3\sqrt{3}}{2}R, which\ is\ true.$$

$$Now, m_a\ge\sqrt{s(s-a)}\to\frac{1}{m_a^3}\le\frac{1}{\sqrt{s^3(s-a)^3}}\to$$

$$\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\le\frac{1}{\sqrt{s^3}}\left(\frac{1}{\sqrt{(s-a)^3}}+\frac{1}{\sqrt{(s-b)^3}}+\frac{1}{\sqrt{(s-c)^3}}\right)\Leftrightarrow$$

$$\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\le\frac{1}{\sqrt{s^3}}\cdot\frac{\sum\left(\sqrt{(s-a)(s-b)}\right)^3}{\left(\sqrt{(s-a)(s-b)(s-c)}\right)^3};(3)$$

$$But:\sqrt{(s-a)(s-b)}\le\frac{s-a+s-b}{2}=\frac{c}{2}\to\left(\sqrt{(s-a)(s-b)}\right)^3\le\frac{c^3}{8};(4)$$

$$From\ (3),(4)\to\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\le\frac{1}{\sqrt{s^3}}\cdot\frac{a^3+b^3+c^3}{8\left(\sqrt{(s-a)(s-b)(s-c)}\right)^3};(5)$$

$$But:a^3+b^3+c^3=2s(s^2-3r^2-6Rr);(6)\ and\ (s-a)(s-b)(s-c)=sr^2;(7)$$

From $(5),(6),(7)$ **we must to prove:**

$$\frac{1}{m_a^3}+\frac{1}{m_b^3}+\frac{1}{m_c^3}\le\frac{2s(s^2-3r^2-6Rr)}{8\sqrt[3]{s^3}\cdot\sqrt{s^3r^6}}=\frac{s^2-3r^2-6Rr}{4s^2r^3};(8)$$

From (8) **we must show that:**

$$\frac{s^2-3r^2-6Rr}{4s^2r^3}\le\frac{2R^2-Rr}{54r^5}\Leftrightarrow\frac{s^2-3r^2-6Rr}{s^2}\le\frac{2(2R^2-Rr)}{27r^2};(9)$$

$$But:s^2\ge 27r^2\to\frac{1}{s^2}\le\frac{1}{27r^2};(10)$$

From $(9),(10)$ **we must to prove that:**

$$s^2-3r^2-6Rr\le 4R^2-2Rr\Leftrightarrow s^2\le 4R^2+4Rr+3r^2\ (Gerretsen).$$

2398. In ΔABC the following relationship holds:

$$R\sum(b\sin 3C-c\sin 3B)\ge 12\sqrt{3}r^2\sum\sin(B-C)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$R\sum(b\sin 3C-c\sin 3B)=2R^2\sum(\sin B\sin 3C-\sin C\sin 3B)$$

$$=2R^2\sum[\sin B(3\sin C-4\sin^3 C)-\sin C(3\sin B-4\sin^3 B)]=$$

$$=8R^2\sum\sin B\sin C(\sin^2 B-\sin^2 C)$$

$$=8R^2\sum\sin B\sin C(\sin B+\sin C)(\sin B-\sin C)$$

$$= 8R^2\sum \sin B \sin C\left(2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\right)\left(2\sin\frac{B-C}{2}\cos\frac{B+C}{2}\right) =$$

$$= 8R^2\sum \sin B\sin C\sin(B+C)\sin(B-C) = 8R^2\left(\prod \sin A\right)\left(\sum \sin(B-C)\right) =$$

$$= 8R^2.\frac{sr}{2R^2}\sum \sin(B-C) = 4sr\sum\sin(B-C) \overset{Mitrinovic}{\geq} 4.3\sqrt{3}r^2\sum\sin(B-C)$$

$$Therefore,\quad R\sum(b\sin 3C - c\sin 3B) \geq 12\sqrt{3}r^2\sum\sin(B-C)$$

2399. In ΔABC the following relationship holds:

$$3\sum_{cyc}\frac{1}{a^3}\tan\frac{A}{2} \leq \sum_{cyc}\frac{1}{a^3}\cot\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{1}{a^3}cot\frac{A}{2} = s\sum\left(\frac{1}{a^3}\left(\frac{s-a}{rs}\right)\right)$$

$$= \frac{1}{r}\left(\frac{s}{64R^3r^3s^3}\left(\left(\sum ab\right)^3 - 3.4Rrs.2s(s^2+2Rr+r^2)\right)\right.$$

$$\left. - \frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{16R^2r^2s^2}\right)$$

$$= \frac{(s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2) - 4Rr((s^2+4Rr+r^2)^2 - 16Rrs^2)}{64R^3r^4s^2}$$

$$= \frac{s^6 - (16Rr-3r^2)s^4 + r^2s^2(32R^2-8Rr+3r^2) + r^4(4R+r)^2}{64R^3r^4s^2} \overset{(i)}{=} \sum\frac{1}{a^3}cot\frac{A}{2}$$

$$\sum\frac{1}{a^3}tan\frac{A}{2} = \sum\left(\frac{1}{a^2}\left(\frac{tan\frac{A}{2}}{4Rcos^2\frac{A}{2}tan\frac{A}{2}}\right)\right) = \frac{1}{4R}\sum\left(\frac{1}{a^2}\left(1+tan^2\frac{A}{2}\right)\right)$$

$$= \frac{1}{4R}\left(\sum\frac{1}{a^2} + \sum\frac{tan^2\frac{A}{2}}{16R^2cos^4\frac{A}{2}tan^2\frac{A}{2}}\right)$$

$$= \frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{16R^2r^2s^2} + \frac{1}{16R^2}\sum\left(1+tan^2\frac{A}{2}\right)^2\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\sum\left(1+tan^4\frac{A}{2}+2tan^2\frac{A}{2}\right)\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}\right.$$

$$\left.+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(\left(\sum r_a^2\right)^2-2\sum r_a^2r_b^2\right)+\frac{2}{s^2}\left(\sum r_a^2\right)\right)\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}\right.$$

$$+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(((4R+r)^2-2s^2)^2-2\left(s^4-2rs^2(4R+r)\right)\right)\right.$$

$$\left.\left.+\frac{2}{s^2}((4R+r)^2-2s^2)\right)\right)$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{64R^3r^2s^2}$$

$$+\frac{3s^4+2s^2(4R+r)^2-4s^4+(4R+r)^4-4s^2(4R+r)^2+4s^4-2s^4+4rs^2(4R+r)}{64R^3s^4}$$

$$=\frac{s^6-(8Rr-3r^2)s^4-r^2s^2(16R^2-8Rr-3r^2)+r^2(4R+r)^4}{64R^3r^2s^2}\overset{(ii)}{=}\sum\frac{1}{a^3}tan\frac{A}{2}$$

$$\therefore (i),(ii)\Rightarrow 3\sum\frac{1}{a^3}tan\frac{A}{2}\le\sum\frac{1}{a^3}cot\frac{A}{2}$$

$$\Leftrightarrow\frac{3s^6-3(8Rr-3r^2)s^4-3r^2s^2(16R^2-8Rr-3r^2)+3r^2(4R+r)^4}{64R^3r^2s^2}$$

$$\le\frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{64R^3r^4s^2}$$

$$\Leftrightarrow s^8-16Rrs^6+(32R^2+16Rr-6r^2)r^2s^4+r^4(64R^2-16Rr-8r^2)s^2$$

$$-3r^4(4R+r)^4\overset{(1)}{\geq}0$$

$$Now, LHS\ of\ (1)\overset{Gerretsen}{\geq}-5r^2s^6+(32R^2+16Rr-6r^2)r^2s^4$$

$$+r^4(64R^2-16Rr-8r^2)s^2-3r^4(4R+r)^4$$

$$\overset{Gerretsen}{\geq} r^2s^4\left(32R^2+16Rr-6r^2-5(4R^2+4Rr+3r^2)\right)$$
$$+r^4(64R^2-16Rr-8r^2)s^2-3r^4(4R+r)^4$$
$$\overset{Gerretsen}{\geq} r^2s^2\left((12R^2-4Rr-21r^2)(16Rr-5r^2)+r^2(64R^2-16Rr-8r^2)\right)$$
$$-3r^4(4R+r)^4$$
$$=r^3s^2(192R^3-60R^2r-332Rr^2+97r^3)$$
$$-3r^4(4R+r)^4\overset{Gerretsen}{\geq} r^4\left((192R^3-60R^2r-332Rr^2+97r^3)(16R-5r)-3(4R+r)^4\right)$$
$$\overset{?}{\geq}0\Leftrightarrow 576t^4-672t^3-1325t^2+791t-122\overset{?}{\geq}0\ \left(where\ t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(576t^3+480t^2-365t+61)\overset{?}{\geq}0\to true\because t\overset{Euler}{\geq}2$$
$$\Rightarrow (1) is\ true\ \therefore 3\sum\frac{1}{a^3}tan\frac{A}{2}\leq\sum\frac{1}{a^3}cot\frac{A}{2}\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$3\sum_{cyc}\frac{1}{a^3}\tan\frac{A}{2}\leq\sum_{cyc}\frac{1}{a^3}\cot\frac{A}{2};(*)$$

$$(*)\Leftrightarrow\sum_{cyc}(bc)^3\left(\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}-\frac{3\sin\frac{A}{2}}{\cos\frac{A}{2}}\right)\geq 0\Leftrightarrow 4R\sum_{cyc}\frac{(bc)^3}{a}\left(\cos^2\frac{A}{2}-2\sin^2\frac{A}{2}\right)\geq 0$$

$$\Leftrightarrow\frac{1}{abc}\sum_{cyc}(bc)^4\left(\frac{1+\cos A}{2}-3\cdot\frac{1-\cos A}{2}\right)\geq 0\Leftrightarrow\sum_{cyc}(bc)^4(2\cos A-1)\geq 0$$

$$\Leftrightarrow\sum_{cyc}(bc)^4\left(\frac{b^2+c^2-a^2}{bc}-1\right)\geq 0\Leftrightarrow\sum_{cyc}(bc)^3(b^2+c^20a^2-bc)\geq 0$$

$$\Leftrightarrow\sum_{cyc}b^5c^3+\sum_{cyc}b^3c^5\geq\sum_{cyc}a^2(bc)^3+\sum_{cyc}(bc)^4;(**)$$

$$We\ have:\sum_{cyc}b^5c^3+\sum_{cyc}b^3c^5=\sum_{cyc}(bc)^3(b^2+c^2)\overset{AM-GM}{\geq}2\sum_{cyc}(bc)^4;(1)$$

$$and\sum_{cyc}b^5c^3+\sum_{cyc}a^3c^5+\sum_{cyc}(bc)^4=$$

$$=\sum_{cyc}\left(b^5c^3+a^3c^5+(bc)^4\right)\overset{AM-GM}{\geq}3\sum_{cyc}a^2(bc)^3;(2)$$

$$From\ 2\cdot(1)+(2)\to(**)$$

Therefore,

$$3\sum_{cyc}\frac{1}{a^3}\tan\frac{A}{2}\leq\sum_{cyc}\frac{1}{a^3}\cot\frac{A}{2}$$

2400. In any scalene ΔABC the following relationship holds:

$$\frac{(2s+a)bc}{(a-b)(a-c)}+\frac{(2s+b)ca}{(b-a)(b-c)}+\frac{(2s+c)ab}{(c-a)(c-b)}>6\sqrt{3}r$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Şerban-Romania

$$\sum_{cyc}\frac{(2s+a)bc}{(a-b)(a-c)}=\sum_{cyc}\frac{2sbc}{(a-b)(a-c)}+\sum_{cyc}\frac{abc}{(a-b)(a-c)}=$$

$$=2s\sum_{cyc}\frac{bc}{(a-b)(a-c)}+abc\sum_{cyc}\frac{1}{(a-b)(a-c)}$$

$$\sum_{cyc}\frac{1}{(a-b)(a-c)}=\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}=$$

$$=-\frac{1}{(a-b)(c-a)}-\frac{1}{(a-b)(b-c)}-\frac{1}{(c-a)(b-c)}=\frac{-b+c-c+a-a+b}{(a-b)(b-c)(c-a)}=0$$

$$\sum_{cyc}\frac{bc}{(a-b)(a-c)}=\frac{bc}{(a-b)(c-a)}+\frac{ca}{(a-b)(b-c)}+\frac{ab}{(c-a)(b-c)}$$

$$=-\frac{bc}{(a-b)(c-a)}-\frac{ca}{(a-b)(b-c)}-\frac{ab}{(c-a)(b-c)}=$$

$$=\frac{-bc(b-c)-ac(c-a)-ab(a-b)}{(a-b)(b-c)(c-a)}=$$

$$=\frac{-b^2c+bc^2-ac^2+a^2c-a^2b+ab^2}{(a-b)(bc-ab-c^2+ac)}=$$

$$=\frac{-b^2c+bc^2-ac^2+a^2c-a^2b+ab^2}{abc-b^2c+bc^2-ac^2+a^2c-a^2b+ab^2-abc}=1$$

$$\to\sum_{cyc}\frac{(2s+a)bc}{(a-b)(a-c)}=2s\sum_{cyc}\frac{bc}{(a-b)(a-c)}+abc\sum_{cyc}\frac{1}{(a-b)(a-c)}=2s>$$

$$\overset{Mitrinovic}{\gtrsim}\ 2\cdot 3\sqrt{3}=6\sqrt{3}r$$

Therefore,

$$\frac{(2s+a)bc}{(a-b)(a-c)}+\frac{(2s+b)ca}{(b-a)(b-c)}+\frac{(2s+c)ab}{(c-a)(c-b)}>6\sqrt{3}r$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru