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SOLVED PROBLEMS-II

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1. If $a,b,c,d \in \mathbb{R}_+^*$ such that $a^3+b^3=u$, then prove that

$$\frac{a^{3m+3}}{c^m}+\frac{b^{3m+3}}{d^m} \geq \frac{u^{m+1}}{(c+d)^m}, \forall m \in \mathbb{R}_+$$

Solution

$$U=\frac{a^{3m+3}}{c^m}+\frac{b^{3m+3}}{d^m}=\frac{\left(a^3\right)^{m+1}}{c^m}+\frac{\left(b^3\right)^{m+1}}{d^m} \text{ and by J. Radon inequality}$$

$$U \geq \frac{(a^3+b^3)^{m+1}}{(c+d)^m}=\frac{u^{m+1}}{(c+d)^m}$$

2. If $m,n,x,y,z>0$ then show that

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{xy}{my+nz}+\frac{yz}{mz+nx}+\frac{zx}{mx+ny}\right) \geq$$

$$\geq \frac{3}{m+n}+3\sqrt[3]{\frac{(x+y)(y+z)(z+x)}{(mx+ny)(my+nz)(mz+nx)}}$$

Solution

We have:

$$U=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\sum_{cyc}\frac{xy}{my+nz}=\sum_{cyc}\frac{y}{my+nz}+\sum_{cyc}\frac{x}{my+nz}+\sum_{cyc}\frac{xy}{z(my+nz)}=$$

$$=\sum_{cyc}\frac{x}{my+nz}+\sum_{cyc}\frac{xy+yz}{z(my+nz)}$$

Also, we have:

$$\sum_{cyc}\frac{x}{my+nz}=\sum_{cyc}\frac{x^2}{mxy+nxz} \overset{Bergstrom}{\geq} \frac{(x+y+z)^2}{\sum_{cyc}(mxy+nxz)}=\frac{(x+y+z)^2}{(m+n)(xy+yz+zx)} \geq$$

$$\geq \frac{3(xy+yz+zx)}{(m+n)(xy+yz+zx)}=\frac{3}{m+n}$$

So, by above and AM-GM inequality yields that:

$$U \geq \frac{3}{m+n} + 3\sqrt[3]{\frac{(x+y)(y+z)(z+x)}{(mx+ny)(my+nz)(mz+nx)}}$$

3. Show that in any triangle ABC with the usual notations, holds the following inequality:

$$\boldsymbol{r_a \cdot a^4 + r_b \cdot b^4 + r_c \cdot c^4 \geq 1296r^5}$$

Solution

By Bergstrom's inequality and well-known $\sum \frac{1}{r_a} = \frac{1}{r}$ we have:

$$V = \sum a^4 r_a = \sum \frac{(a^2)^2}{\frac{1}{r_a}} \overset{Bergstrom}{\geq} \frac{(a^2+b^2+c^2)^2}{\sum \frac{1}{r_a}} = \frac{(\sum a^2)^2}{\frac{1}{r}} = \left(\sum a^2\right)^2 r$$

By Ionescu-Weitzenböck inequality, i.e.

$a^2 + b^2 + c^2 \geq 4\sqrt{3}S = 4\sqrt{3}sr$, where we used $S = sr$.

So, $V \geq 48s^2r^3$, and by Mitrinovic's inequality, i.e. $s \geq 3\sqrt{3}r \Leftrightarrow s^2 \geq 27r^2$ we obtain:

$V \geq 48 \cdot 27r^5 = 1296r^5$, Q.E.D.

4. Prove that in any triangle ABC holds the inequality

$$\boldsymbol{\frac{a^2}{r_b r_c} + \frac{b^2}{r_c r_a} + \frac{c^2}{r_a r_b} \geq 4}$$

Solution

Bergström $\sum \frac{a^2}{r_b r_c} \geq \frac{(\sum a)^2}{\sum r_a r_b}$ and taking account by $\sum a = 2s$ and $\sum r_a r_b = s^2$, Q.E.D.

5. Prove that in all triangle ABC, with usual notations, holds:

$$\boldsymbol{\frac{m_a^3}{R \cdot m_b + r \cdot m_c} + \frac{m_b^3}{R \cdot m_c + r \cdot m_a} + \frac{m_c^3}{R \cdot m_a + r \cdot m_b} \geq \frac{3\sqrt{3}}{R+r} S}$$

Solution

We have that:

$$U = \sum_{cyc} \frac{m_a^3}{R \cdot m_b + r \cdot m_c} = \sum_{cyc} \frac{(m_a^2)^2}{R \cdot m_a \cdot m_b + r \cdot m_a \cdot m_c} \geq$$

$$\geq 2 \cdot \sum_{cyc} \frac{(m_a^2)^2}{R(m_a^2 + m_b^2) + r(m_a^2 + m_c^2)}$$

where we apply Bergström's inequality and well-known formula:

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

We obtain that:

$$U \geq 2 \cdot \frac{\left(\sum_{cyc} m_a^2\right)^2}{R \cdot \sum(m_a^2 + m_b^2) + r \cdot \sum(m_a^2 + m_c^2)} = \frac{2 \cdot \left(\sum_{cyc} m_a^2\right)^2}{2R \cdot \sum m_a^2 + 2r \cdot \sum m_a^2}$$

$$= \frac{\sum_{cyc} m_a^2}{R + r} = \frac{3}{4} \cdot \frac{a^2 + b^2 + c^2}{R + r}$$

where we use the Ionescu-Weitzenböck inequality, i.e. $a^2 + b^2 + c^2 \geq 4S\sqrt{3}$, and we deduce that:

$$U \geq \frac{3}{4} \cdot \frac{1}{R + r} \cdot 4S\sqrt{3} = \frac{3\sqrt{3}}{R + r} S$$

and we are done.

6. If $x, y \in \mathbb{R}_+^*$ such that $xy = 1$, then prove that in any triangle ABC holds the inequality:

$$(xr_a r_b + r_b r_c + yr_c r_a)(yr_a r_b + r_b r_c + xr_c r_a) \geq s^4$$

Solution

$$(xr_a r_b + r_b r_c + yr_c r_a)(yr_a r_b + r_b r_c + xr_c r_a) = r_a^2 r_b^2 + r_b^2 r_c^2 + r_c^2 r_a^2 +$$
$$+r_a r_b^2 r_c(x + y) + r_a r_b r_c^2(x + y) + r_a^2 r_b r_c(x^2 + y^2) \geq r_a^2 r_b^2 + r_b^2 r_c^2 + r_c^2 r_a^2 +$$
$$+2r_a r_b^2 r_c\sqrt{xy} + 2r_a r_b r_c^2\sqrt{xy} + 2r_a^2 r_b r_c xy = (r_a r_b + r_b r_c + r_c r_a)^2$$

where we used $r_a r_b + r_b r_c + r_c r_a = p^2$ and we obtain:

$(xr_a r_b + r_b r_c + yr_c r_a)(yr_a r_b + r_b r_c + xr_c r_a) \geq s^4$, Q.E.D.

7. If $x, y \in \mathbb{R}_+^*$ and $xy = 1$, then in any triangle ABC is true the following inequality:

$$(x^2 r_a r_b + r_b r_c + y^2 r_c r_a)(y^2 r_a r_b + r_b r_c + x^2 r_c r_a) \geq s^4$$

Solution

From $xy = 1$ and AM-GM inequality we get:

$$(x^2 r_a r_b + r_b r_c + y^2 r_c r_a)(y^2 r_a r_b + r_b r_c + x^2 r_c r_a) = r_a^2 r_b^2 + r_b^2 r_c^2 + r_c^2 r_a^2 +$$
$$+r_a r_b^2 r_c(x^2 + y^2) + r_a r_b r_c^2(x^2 + y^2) + r_a^2 r_b r_c(x^4 + y^4) \geq r_a^2 r_b^2 + r_b^2 r_c^2 + r_c^2 r_a^2 +$$
$$+2r_a r_b^2 r_c xy + 2r_a r_b r_c^2 xy + 2r_a^2 r_b r_c x^2 y^2 = (r_a r_b + r_b r_c + r_c r_a)^2$$

and applying $r_ar_b + r_br_c + r_cr_a = s^2$ we obtain:

$(x^2r_ar_b + r_br_c + y^2r_cr_a)(y^2r_ar_b + r_br_c + x^2r_cr_a) \geq s^4$, Q.E.D.

8. Prove that in any triangle ABC is true the following inequality:

$$\frac{1}{2r_a^2 + 5r_br_c} + \frac{1}{2r_b^2 + 5r_cr_a} + \frac{1}{2r_c^2 + 5r_ar_b} \geq \frac{9}{2(4R+r)^2 + s^2}$$

Solution

By Bergström inequality we have $\sum \frac{1}{2r_a^2+5r_br_c} \geq \frac{9}{2\sum r_a^2 + 5\sum r_ar_b}$, and by

$\sum r_a^2 = (4R+r)^2 - 2s^2$ respectively $\sum r_ar_b = s^2$, Q.E.D.

9. Prove that in any triangle ABC the following inequality is true:

$$a^2 \cot\frac{B}{2}\cot\frac{C}{2} + b^2 \cot\frac{C}{2}\cot\frac{A}{2} + c^2 \cot\frac{A}{2}\cot\frac{B}{2} \geq 4s^2$$

Solution

Using Bergström inequality and $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$,

$\sum a^2 \cot\frac{B}{2}\cot\frac{C}{2} = \sum \frac{a^2}{\tan\frac{B}{2}\tan\frac{C}{2}} \geq \frac{(a+b+c)^2}{\sum \tan\frac{A}{2}\tan\frac{B}{2}}$, Q.E.D.

10. Prove that in any triangle ABC the following inequality is true:

$$a^2 \tan\frac{B}{2}\tan\frac{C}{2} + b^2 \tan\frac{C}{2}\tan\frac{A}{2} + c^2 \tan\frac{A}{2}\tan\frac{B}{2} \geq \frac{4p^2r}{4R+r}$$

Solution

Using Bergström inequality and $\sum \cot\frac{A}{2}\cot\frac{B}{2} = \frac{4R+r}{r}$

$U = \sum a^2 \tan\frac{B}{2}\tan\frac{C}{2} = \sum \frac{a^2}{\cot\frac{B}{2}\cot\frac{C}{2}} \geq \frac{(a+b+c)^2}{\sum \cot\frac{A}{2}\cot\frac{B}{2}}$, Q.E.D.

11. Prove that in any triangle the following inequalities holds:

i) $\frac{\sin^6 A}{\sin^2 B} + \frac{\sin^6 B}{\sin^2 C} + \frac{\sin^6 C}{\sin^2 A} \geq \frac{(s^2-4Rr-r^2)^3}{8s^2R^4}$;

ii) $\sin^2\frac{A}{2}\sin^2 A + \sin^2\frac{B}{2}\sin^2 B + \sin^2\frac{C}{2}\sin^2 C \geq \left(\frac{s(2R-r)}{R}\right)^2 \cdot \frac{1}{s^2+(4R+r)^2}$

Solution

i) We have:

$$V=\frac{\sin^6 A}{\sin^2 B}+\frac{\sin^6 B}{\sin^2 C}+\frac{\sin^6 C}{\sin^2 A}=\frac{(\sin^2 A)^3}{\sin^2 B}+\frac{(\sin^2 B)^3}{\sin^2 C}+\frac{(\sin^2 C)^3}{\sin^2 A}$$

And by J. Radon's inequality we obtain that:

$$V\geq\frac{(\sin^2 A+\sin^2 B+\sin^2 C)^3}{(\sin A+\sin B+\sin C)^2}$$

Since,

$\sin^2 A+\sin^2 B+\sin^2 C=\frac{s^2-4Rr-r^2}{2R^2}$ and $\sin A+\sin B+\sin C=\frac{s}{R}$

we deduce the conclusion.

ii) $W=\sum\sin^2 A\sin^2\frac{A}{2}=4\sum\sin^2\frac{A}{2}\cos^2\frac{A}{2}=4\cdot\sum\frac{\sin^4\frac{A}{2}}{\frac{1}{\cos^2\frac{A}{2}}}=4\cdot\sum\frac{\left(\sin^2\frac{A}{2}\right)^2}{\frac{1}{\cos^2\frac{A}{2}}}$

By H. Bergström's inequality, we have:

$$W\geq 4\cdot\frac{\left(\sum\sin^2\frac{A}{2}\right)^2}{\sum\frac{1}{\cos^2\frac{A}{2}}}$$

We use,

$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}$ and $\sum\frac{1}{\cos^2\frac{A}{2}}=\frac{(4R+r)^2+s^2}{s^2}$,

and we obtain the conclusion.

12. Prove that in all triangles ABC the following relationship holds:

$$\sum\frac{\cot^3\frac{A}{2}}{x+y\tan\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}}\geq\frac{s^3}{\left((4R+r)x+sy+3zr\right)r^2}$$

for any positive real numbers x,y and z (the notations are usual and the sum is cyclic).

Solution

We apply Bergström's inequality (or the inequality of Cauchy-Schwarz) and we deduce that:

$$\sum\frac{\cot^3\frac{A}{2}}{x+y\tan\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}}=\sum\frac{\cot^2\frac{A}{2}}{x\tan\frac{A}{2}+y\tan\frac{A}{2}\tan\frac{B}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}\geq$$

$$\geq \frac{\left(\sum\cot\frac{A}{2}\right)^2}{x\sum\tan\frac{A}{2}+y\sum\tan\frac{A}{2}\tan\frac{B}{2}+3z\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$$

Using the well-known formulas

$\sum\cot\frac{A}{2}=\frac{s}{r}, \sum\tan\frac{A}{2}=\frac{4R+r}{s}, \sum\tan\frac{A}{2}\tan\frac{B}{2}=1$ and $\prod\tan\frac{A}{2}=\frac{r}{s}$

by the above we obtain the result.

13. Prove that in all ABC triangles the following relationship holds:

$$\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\geq 9$$

Solution

By Bergström's inequality we deduce that:

$U=\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\geq\frac{\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2}{3}$ (1)

It is well-known that:

$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{p}{r}$ (2)

By Mitrinovic's inequality we have that:

$p\geq 3\sqrt{3}\cdot r$ (3)

From (1), (2) and (3) we obtain:

$U\geq\frac{p^2}{3r^2}\geq\frac{(3\sqrt{3})^2r^2}{3r^2}=9$, Q.E.D.

The equality holds if the triangle is equilateral.

14. Show that if $m\in[0,\infty), x,y,z,t\in(0,\infty)$, then in any triangle ABC, with usual notations ($a=BC, b=CA, c=AB, m_b=$ the median from the vertex $B, w_c=$ the internal bisector from the vertex $C, S=$ area ABC) holds

$$\sum_{cyc}\frac{\left(xa^2+ym_b^2\right)^{m+1}}{(zb^2+tw_c^2)^m}\geq\frac{(4x+3y)^{m+1}}{(4z+3t)^m}\sqrt{3}S$$

Solution

We have:

$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \Rightarrow w_a \le \sqrt{s(s-a)} \Rightarrow w_a^2 \le s(s-a)$ and other two similar. So,

$$w_a^2 + w_b^2 + w_c^2 \le s(s-a+s-b+s-c) = s(3s-2s) = s^2 =$$

$$= \frac{(a+b+c)^2}{4} \le \frac{3}{4}(a^2+b^2+c^2) = m_a^2 + m_b^2 + m_c^2 \quad (1)$$

By J. Radon's inequality, the inequality (1), and $\sum_{cyc} m_a^2 = \frac{3}{4}\sum_{cyc} a^2$, we deduce

$$\sum_{cyc} \frac{(xa^2 + ym_b^2)^{m+1}}{(zb^2 + tw_c^2)^m} \ge \frac{\left(\sum_{cyc}(xa^2 + ym_b^2)\right)^{m+1}}{\left(\sum_{cyc}(zb^2 + tw_c^2)\right)^m} = \frac{\left(x\sum_{cyc} a^2 + y\sum_{cyc} m_a^2\right)^{m+1}}{\left(z\sum_{cyc} a^2 + t\sum_{cyc} w_a^2\right)^m} \ge$$

$$\ge \frac{\left(x\sum_{cyc} a^2 + y\sum_{cyc} m_a^2\right)^{m+1}}{\left(z\sum_{cyc} a^2 + t\sum_{cyc} m_a^2\right)^m} = \frac{\left(x\sum_{cyc} a^2 + \frac{3}{4}y\sum_{cyc} a^2\right)^{m+1}}{\left(z\sum_{cyc} a^2 + t\cdot\frac{3}{4}\cdot\sum_{cyc} a^2\right)^m} =$$

$$= \frac{(4x+3y)^{m+1}}{(4z+3t)^m}\cdot\frac{1}{4}\cdot\sum_{cyc} a^2 \quad (2)$$

By Ion Ionescu – Weitzenböck inequality we have

$$a^2 + b^2 + c^2 \ge 4\sqrt{3}S \quad (3)$$

From (2) and (3) we obtain:

$$\sum_{cyc} \frac{(xa^2 + ym_b^2)^{m+1}}{(zb^2 + tw_c^2)^m} \ge \frac{(4x+3y)^{m+1}}{(4z+3t)^m}\sqrt{3}S$$

15. If $n \in \mathbb{N}^*$, then in any triangle ABC occurs

$$a^n h_a^{n-1} + b^n h_b^{n-1} + c^n h_c^{n-1} = 2^n s^n r^{n-1}$$

Solution

$$\sum a^n h_a^{n-1} = \sum \frac{(ah_a)^n}{h_a} = 2^n S^n\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) = 2^n S^{n-1}\left(\frac{S}{h_a} + \frac{S}{h_b} + \frac{S}{h_c}\right) =$$

$$= 2^n S^{n-1}\left(\frac{ah_a}{2h_a} + \frac{bh_b}{2h_b} + \frac{ch_c}{2h_c}\right) = 2^{n-1}S^{n-1}(a+b+c) = 2^{n-1}\cdot S^{n-1}\cdot 2\cdot s =$$

$$= 2^n r^{n-1} s = 2^n s^n r^{n-1}$$

16. If $m, n \in \mathbb{R}_+^*, m \ge n$, then prove that in any triangle ABC holds:

$$\frac{a}{m(b+c)-na} + \frac{b}{m(c+a)-nb} + \frac{c}{m(a+b)-nc} \ge \frac{3}{2m-n}$$

Solution

$U=\sum\frac{a}{m(b+c)-na}=\sum\frac{a^2}{m(ab+ac)-na^2}$ and by Bergström inequality:

$$U\geq\frac{(\sum a)^2}{m\sum(ab+ac)-n\sum a^2}=\frac{(\sum a)^2}{2m\sum ab-n\sum a^2}$$

Since,

$(\sum a)^2\geq 3\sum ab$ and $\sum a^2\geq\sum ab$, Q.E.D.

Observation. For $m=1$ and $n=0$ results the inequality of Nesbitt.

17. If $m,n\in\mathbb{R}_+^*$, then prove that in any triangle ABC is true the following inequality:

$$\frac{ma^2+nb^2}{a+b-c}+\frac{mb^2+nc^2}{b+c-a}+\frac{mc^2+na^2}{c+a-b}\geq 2(m+n)s$$

Solution

$U=\sum\frac{ma^2+nb^2}{a+b-c}=m\sum\frac{a^2}{a+b-c}+n\sum\frac{b^2}{a+b-c}$ and from Bergström's inequality

$$U\geq m\cdot\frac{(\sum a)^2}{\sum(a+b-c)}+n\cdot\frac{(\sum b)^2}{\sum(a+b-c)}=(m+n)\cdot\frac{(\sum a)^2}{\sum(a+b-c)}=$$

$=(m+n)\cdot\frac{(a+b+c)^2}{a+b+c}=2(m+n)s$, Q.E.D.

18. Prove that in any triangle ABC the following inequality holds:

$$\frac{a^2}{h_b\cdot h_c}+\frac{b^2}{h_c\cdot h_a}+\frac{c^2}{h_a\cdot h_b}\geq 4$$

Solution

$$U=\sum\frac{a^2}{h_bh_c}=\sum\frac{a^2bc}{(bh_b)(ch_c)}=\frac{abc}{4S^2}\sum a=\frac{2pabc}{4S^2}=\frac{pabc}{2prS}=\frac{abc}{2rS}$$

Since $S=\frac{abc}{4R}$ we get $U=\frac{2R}{r}$. From $R\geq 2r$ yields the conclusion.

19. If $a,b\in\mathbb{R}_+^*$, $x\in\mathbb{R}$, then prove that

$$\frac{\sin^{2m+2}x}{a^m}+\frac{\cos^{2m+2}x}{b^m}\geq\frac{1}{(a+b)^m},\ \forall m\in\mathbb{R}_+.$$

Solution

$E=\frac{\sin^{2m+2}x}{a^m}+\frac{\cos^{2m+2}x}{b^m}=\frac{(\sin^2x)^{m+1}}{a^m}+\frac{(\cos^2x)^{m+1}}{b^m}$, and by J. Radon inequality

$E\geq\frac{(\sin^2x+\cos^2x)^{m+1}}{(a+b)^m}=\frac{1}{(a+b)^m}$, Q.E.D.