TSINTSIFAS-ŞAHIN'S INEQUALITY

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Abstract. In this paper we connect two famous relationships in a triangle, both published in American Mathematical Monthly

Main result:

If $x, y, z > 0$ then in acute ΔABC the following relationship holds:

$$\frac{x}{y+z}\cdot a+\frac{y}{z+x}\cdot b+\frac{z}{x+y}\cdot c\geq\sqrt{3r(4R+r)}$$

Lemma 1 (TSINTSIFAS' INEQUALITY)

If $x, y, z > 0$ then in acute ΔABC holds:

$$\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2\geq 2F\sqrt{3}$$

Proof.

$$\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2=$$

$$=\frac{(x+y+z)a^2-(y+z)a^2}{y+z}+\frac{(x+y+z)b^2-(z+x)b^2}{z+x}+\frac{(x+y+z)c^2-(x+y)c^2}{x+y}=$$

$$=(x+y+z)\left(\frac{a^2}{y+z}+\frac{b^2}{z+x}+\frac{c^2}{x+y}\right)-(a^2+b^2+c^2)=$$

$$=\left(\frac{x+y}{2}+\frac{y+z}{2}+\frac{z+x}{2}\right)\left(\frac{a^2}{y+z}+\frac{b^2}{z+x}+\frac{c^2}{x+y}\right)-(a^2+b^2+c^2)\geq$$

$$\overset{\text{CBS}}{\geq}\left(\sqrt{\frac{x+y}{2}\cdot\frac{a^2}{x+y}}+\sqrt{\frac{y+z}{2}\cdot\frac{b^2}{y+z}}+\sqrt{\frac{z+x}{2}\cdot\frac{c^2}{z+x}}\right)^2-(a^2+b^2+c^2)=$$

$$=\left(\frac{a}{\sqrt{2}}+\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}\right)^2-(a^2+b^2+c^2)=$$

$$=\frac{1}{2}(a+b+c)^2-(a^2+b^2+c^2)=$$

$$=\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{2}=$$

$$=\frac{2(s^2+r^2+4Rr)-2(s^2-r^2-4Rr)}{2}=$$

$$=s^2+r^2+4Rr-s^2+r^2+4Rr=$$

$$=2r^2+8Rr=2r(r+4R)\overset{\text{DOUCET}}{\geq}$$

$$\geq 2r\cdot s\sqrt{3}=2F\sqrt{3}$$

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Equality hold for $a = b = c$ and $x = y = z$.
Observation:

$$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} > a + b > c = (\sqrt{c})^2$$

$$(\sqrt{a} + \sqrt{b})^2 > (\sqrt{c})^2 \Rightarrow \sqrt{a} + \sqrt{b} > \sqrt{c} \text{ and analogous:}$$

$$\sqrt{b} + \sqrt{c} > \sqrt{a}; \sqrt{c} + \sqrt{a} > \sqrt{b}$$

hence: $\sqrt{a}, \sqrt{b}, \sqrt{c}$ can be sides in a triangle □

Lemma 2
(MEHMET ŞAHIN'S IDENTITY)
Let a, b, c - be sides in a triangle. The triangle formed with sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$ has area $\Delta = \frac{1}{2}\sqrt{r(4R + r)}$.

Proof.

$$\Delta \overset{\text{HERON}}{=} \sqrt{\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{2} \cdot \frac{\sqrt{a} + \sqrt{b} - \sqrt{c}}{2} \cdot \frac{\sqrt{b} + \sqrt{c} - \sqrt{a}}{2} \cdot \frac{\sqrt{c} + \sqrt{a} - \sqrt{b}}{2}} =$$

$$= \frac{1}{4}\sqrt{((\sqrt{a} + \sqrt{b})^2 - (\sqrt{c})^2)((\sqrt{c})^2 - (\sqrt{a} - \sqrt{b})^2)} =$$

$$= \frac{1}{4}\sqrt{(a + b + 2\sqrt{ab} - c)(c - a - b + 2\sqrt{ab})} =$$

$$= \frac{1}{4}\sqrt{(2\sqrt{ab} + (a + b - c)(2\sqrt{ab} - (a + b - c))} =$$

$$= \frac{1}{4}\sqrt{4ab - (a + b - c)^2} =$$

$$= \frac{1}{4}\sqrt{4ab - a^2 - b^2 - c^2 - 2ab + 2bc + 2ca} =$$

$$= \frac{1}{4}\sqrt{2(ab + bc + ca) - (a^2 + b^2 + c^2)} =$$

$$= \frac{1}{4}\sqrt{2s^2 + 2r^2 + 8Rr - 2s^2 + 2r^2 + 8Rr} =$$

$$= \frac{1}{4}\sqrt{4r^2 + 16Rr} = \frac{1}{2}\sqrt{r(4R + r)}$$

□

Back to the main problem:
We apply Tsintsifas' inequality for the triangle with sides: $\sqrt{a}, \sqrt{b}, \sqrt{c}$:

$$\frac{x}{y + z} \cdot (\sqrt{a})^2 + \frac{y}{z + x} \cdot (\sqrt{b})^2 + \frac{z}{x + y} \cdot (\sqrt{c})^2 \geq 2\sqrt{3}\Delta$$

$$\frac{x}{y + z} \cdot a + \frac{y}{z + x} \cdot b + \frac{z}{x + y} \cdot c \geq 2\sqrt{3} \cdot \frac{1}{2}\sqrt{r(4R + r)}$$

$$\frac{x}{y + z} \cdot a + \frac{y}{z + x} \cdot b + \frac{z}{x + y} \cdot c \geq \sqrt{3r(4R + r)}$$

Equality holds for $a = b = c$ and $x = y = z$.

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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