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ROMANIAN MATHEMATICAL MAGAZINE
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ABOUT A RMM INEQUALITY-XII

By Marin Chirciu-Romania

1) If $a,b,c>0$ then:

$$\sum\frac{a}{3b+\sqrt[7]{ab^6}}\ge\frac{3}{4}$$

Proposed by Daniel Sitaru-Romania

Solution:

Using the means inequality we obtain:

$$LHS=\sum\frac{a}{3b+\sqrt[7]{ab^6}}\overset{AM-GM}{\ge}\sum\frac{a}{3b+\frac{a+6b}{7}}=\sum\frac{7a}{a+27b}=7\sum\frac{a^2}{a^2+27ab}\overset{Bergstrom}{\ge}$$

$$\ge 7\cdot\frac{(\sum a)^2}{\sum(a^2+27ab)}\overset{(1)}{\ge}\frac{3}{4}=RHS,\text{ where (1)}\Leftrightarrow 7\cdot\frac{(\sum a)^2}{\sum(a^2+27ab)}\ge\frac{3}{4}\Leftrightarrow$$

$$\Leftrightarrow 28\left(\sum a\right)^2\ge 3\sum(a^2+27ab)\Leftrightarrow$$

$$\Leftrightarrow 28\left(\sum a^2+2\sum bc\right)\ge 3\sum a^2+81\sum bc\Leftrightarrow 28\sum a^2+56\sum bc\ge$$

$$\ge 3\sum a^2+81\sum bc\Leftrightarrow 25\sum a^2\ge 25\sum bc\Leftrightarrow\sum a^2\ge\sum bc\Leftrightarrow\sum(b-c)^2\ge 0$$

Equality holds if and only if $a=b=c$.**Remark:** The problem can be developed.

2) If $a,b,c>0$ and $\lambda\ge\frac{1}{2}, n\in\mathbb{N}, n\ge 2$ then:

$$\sum\frac{a}{\lambda b+\sqrt[n]{ab^{n-1}}}\ge\frac{3}{\lambda+1}$$

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Solution:

Using the means inequality we obtain:

$$LHS=\sum\frac{a}{\lambda b+\sqrt[n]{ab^{n-1}}}\overset{AM-GM}{\ge}\sum\frac{a}{\lambda b+\frac{a+(n-1)b}{n}}=\sum\frac{na}{a+(\lambda n+n-1)b}=$$

$$=n\sum\frac{a^2}{a^2+(\lambda n+n-1)ab}\overset{Bergstrom}{\ge}n\cdot\frac{(\sum a)^2}{\sum(a^2+(\lambda n+n-1)ab)}\overset{(1)}{\ge}\frac{3}{\lambda+1}=RHS,\text{ where}$$

$$(1) \Leftrightarrow n \cdot \frac{(\sum a)^2}{\sum(a^2+(\lambda+n-1)ab)} \geq \frac{3}{\lambda+1}$$

$$\Leftrightarrow n(\lambda+1)\left(\sum a\right)^2 \geq 3\sum(a^2+(\lambda n+n-1)ab) \Leftrightarrow$$

$$\Leftrightarrow n(\lambda+1)\left(\sum a^2+2\sum bc\right) \geq 3\sum a^2+3(\lambda n+n-1)\sum bc \Leftrightarrow$$

$$\Leftrightarrow n(\lambda+1)\sum a^2+2n(\lambda+1)\sum bc \geq 3\sum a^2+3(\lambda n+n-1)\sum bc \Leftrightarrow$$

$\Leftrightarrow (\lambda n+n-3)\sum a^2 \geq (\lambda n+n-3)\sum bc$, which follows from $(\lambda n+n-3) \geq 0$, true from $\lambda \geq \frac{1}{2}, n \in \mathbb{N}, n \geq 2$ and $\sum a^2 \geq \sum bc \Leftrightarrow \sum(b-c)^2 \geq 0$

Equality holds if and only if $a=b=c$.

Note:

For $\lambda=3, n=7$ we obtain the problem proposed by Daniel Sitaru in RMM 12/2020

Reference:

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