ABOUT AN INEQUALITY BY BOGDAN FUȘTEI-V

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1) In ΔABC:

$$\sum \frac{m_a}{h_a} \ge \frac{1}{2}\sum \sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$$

Proposed by Bogdan Fuștei – Romania

Solution: We prove: **Lemma:**

2) In ΔABC:

$$\frac{m_a}{h_a} \ge \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Proof: Using $h_a = \frac{2S}{a} = \frac{bc}{2R}$ and Tereshin's inequality $m_a \ge \frac{b^2+c^2}{4R}$ we obtain:
$m_a \ge \frac{b^2+c^2}{4R} = \frac{b^2+c^2}{\frac{2bc}{h_a}} = h_a \cdot \frac{b^2+c^2}{2bc}$, wherefrom $m_a \ge h_a \cdot \frac{b^2+c^2}{2bc} \Leftrightarrow \frac{m_a}{h_a} \ge \frac{b^2+c^2}{2bc} = \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$.
Let's get back to the main problem.Using the Lemma and the inequality $\frac{m_a}{h_a} \ge \frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$, (Adil Abdullayev Inequality) we obtain:

$$\left(\frac{m_a}{h_a}\right)^2 = \frac{m_a}{h_a}\cdot\frac{m_a}{h_a} \ge \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\cdot\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right) = \frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$$

wherefrom it follows that: $\left(\frac{m_a}{h_a}\right)^2 \ge \frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right) \Leftrightarrow \frac{m_a}{h_a} \ge \frac{1}{2}\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$
Adding we deduce the conclusion.Equality holds if and only if the triangle is equilateral.
Remark: In the same way:

3) In ΔABC:

$$\sum \frac{m_a}{h_a} \ge \frac{27R}{2(4R+r)}$$

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Solution: We prove **Lemma:**

4) In ΔABC:

$$\frac{m_a}{h_a} \ge \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Proof: Using $h_a = \frac{2S}{a} = \frac{bc}{2R}$ and Tereshin's inequality $m_a \ge \frac{b^2+c^2}{4R}$ we obtain:

$m_a \geq \frac{b^2+c^2}{4R} = \frac{b^2+c^2}{\frac{2bc}{h_a}} = h_a \cdot \frac{b^2+c^2}{2bc}$, wherefrom $m_a \geq h_a \cdot \frac{b^2+c^2}{2bc} \Leftrightarrow \frac{m_a}{h_a} \geq \frac{b^2+c^2}{2bc} = \frac{1}{2}\left(\frac{b}{c} + \frac{c}{b}\right)$

Let's get back to the main problem.Using the Lemma we obtain:

$$LHS = \sum \frac{m_a}{h_a} \geq \sum \frac{1}{2}\left(\frac{b}{c} + \frac{c}{b}\right) = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + \frac{1}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \overset{(1)}{\geq}$$

$\overset{(1)}{\geq} \frac{1}{2} \cdot \frac{27R}{2(4R+r)} + \frac{1}{2} \cdot \frac{27R}{2(4R+r)} = \frac{27R}{2(4R+r)} = RHS$, where (1) follows from inequality:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{27R}{2(4R+r)}$$

Let's prove the inequality: $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{27R}{2(4R+r)}$

5) In ΔABC:

$$\boldsymbol{\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{27R}{2(4R+r)}}$$

Proof: We use the algebraic inequality:

6) If $a, b, c > 0$

$$\boldsymbol{\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{9(a^2+b^2+c^2)}{(a+b+c)^2}}$$

Indeed: The inequality can be written equivalently: $\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)(a+b+c)^2 \geq 9\sum a^2 \Leftrightarrow$

$$\Leftrightarrow \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)(a+b+c)^2 = \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\sum a^2 + 2\sum ab\right) =$$

$$= \sum \frac{a^3}{b} + \sum \frac{ac^2}{b} + 2\sum \frac{a^2c}{b} + 2\sum a^2 + 3\sum ab$$

The inequality can be written:

$$\sum \frac{a^3}{b} + \sum \frac{ac^2}{b} + 2\sum \frac{a^2c}{b} + 2\sum a^2 + 3\sum ab \geq 9\sum a^2 \Leftrightarrow$$

$$\Leftrightarrow \sum\left(\frac{a^3}{b} - \frac{2a^2c}{b} + \frac{ac^2}{b}\right) + \sum\left(\frac{4a^2c}{b} - 8ac + 4bc\right) \geq 7\sum a^2 - 7\sum ab \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{a(a-c)^2}{b} + \sum \frac{4c(a-b)^2}{b} \geq \frac{7}{2}\sum (a-b)^2 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{b(b-a)^2}{c} + \sum \frac{4c(a-b)^2}{b} \geq \frac{7}{2}\sum (a-b)^2 \Leftrightarrow \sum (a-b)^2\left(\frac{b}{c} + \frac{4c}{b} - \frac{7}{2}\right) \geq 0 \Leftrightarrow$$

$\Leftrightarrow \sum (a-b)^2\left[\frac{(b-2c)^2}{bc} + \frac{1}{2}\right] \geq 0$, obviously with equality for $a = b = c$.

Application in triangle.

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{9(a^2+b^2+c^2)}{(a+b+c)^2} = \frac{9 \cdot 2(s^2-r^2-4Rr)}{4s^2} = \frac{9(s^2-r^2-4Rr)}{2s^2} \overset{Gerretsen}{\geq} \frac{27R}{2(4R+r)}.$$

We obtain $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{27R}{2(4R+r)}$

Equality holds if and only if the triangle is equilateral.
Remark: The inequality can be strengthened.

7) In ΔABC:

$$\sum\frac{m_a}{h_a}\geq\sqrt{\frac{3s^2}{r(4R+r)}}$$

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Solution: We prove **Lemma:**

8) In ΔABC:

$$\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Proof: Using $h_a=\frac{2S}{a}=\frac{bc}{2R}$ and Tereshin's inequality $m_a\geq\frac{b^2+c^2}{4R}$ we obtain:
$m_a\geq\frac{b^2+c^2}{4R}=\frac{b^2+c^2}{\frac{2bc}{h_a}}=h_a\cdot\frac{b^2+c^2}{2bc}$, wherefrom $m_a\geq h_a\cdot\frac{b^2+c^2}{2bc}\Leftrightarrow\frac{m_a}{h_a}\geq\frac{b^2+c^2}{2bc}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$
Let's get back to the main problem. Using the Lemma we obtain:

$$LHS=\sum\frac{m_a}{h_a}\geq\sum\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\overset{(1)}{\geq}$$

$\overset{(1)}{\geq}\frac{1}{2}\sqrt{\frac{3s^2}{r(4R+r)}}+\frac{1}{2}\sqrt{\frac{3s^2}{r(4R+r)}}=\sqrt{\frac{3s^2}{r(4R+r)}}=RHS$, where (1) follows from:

$$\sqrt{\frac{3s^2}{r(4R+r)}}\leq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq\frac{s^2}{r(4r+r)},\ \text{(Mateescu-2016)}$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way:

9) In ΔABC:

$$\sum\frac{h_a}{w_a}\geq 3\left(\frac{2r}{R}\right)^{\frac{2}{3}}$$

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Solution: We prove **Lemma:**

10) In ΔABC:

$$\frac{h_a}{w_a}=\frac{b+c}{a}\sin\frac{A}{2}$$

Proof: We have:

$$\frac{h_a}{w_a}=\cos\frac{B-C}{2}=\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}=$$

$$=\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}+\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}=$$

$$=\left(\frac{s}{a}+\frac{s-a}{a}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}=\frac{b+c}{a}\sqrt{\frac{(s-b)(s-c)}{bc}}=\frac{b+c}{a}\sin\frac{A}{2}$$

Let's get back to the main problem. Using the Lemma and the means inequality we obtain:

$$\sum\frac{h_a}{w_a}=\sum\frac{b+c}{a}\sin\frac{A}{2}\geq 3\sqrt[3]{\prod\frac{b+c}{a}\sin\frac{A}{2}}=3\sqrt[3]{\frac{\prod(b+c)\prod\sin\frac{A}{2}}{abc}}=$$

$$=3\sqrt[3]{\frac{2s(s^2+r^2+2Rr)\cdot\frac{r}{4R}}{4Rrs}}=\frac{3}{2}\sqrt[3]{\frac{s^2+r^2+2Rr}{R^2}}\overset{Gerretsen}{\geq}$$

$$\geq\frac{3}{2}\sqrt[3]{\frac{16Rr-5r^2+r^2+2Rr}{R^2}}=\frac{3}{2}\sqrt[3]{\frac{18Rr-4r^2}{R^2}}\overset{Euler}{\geq}\frac{3}{2}\sqrt[3]{\frac{32r^2}{R^2}}=$$

$$=3\sqrt[3]{\frac{4r^2}{R^2}}=3\left(\frac{2r}{R}\right)^{\frac{2}{3}}$$

Equality holds if and only if the triangle is equilateral.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro