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ABOUT AN INEQUALITY BY BOGDAN FUȘTEI-VI

By Marin Chirciu-Romania

1) In ΔABC:

$$\frac{m_a^2+m_b^2+m_c^2}{m_a+m_b+m_c}\le\frac{3R}{2}$$

Proposed by Bogdan Fuștei – Romania

Solution: Using Tereshin inequality $m_a\ge\frac{b^2+c^2}{4R}$ we obtain:

$$m_a+m_b+m_c=\sum m_a\overset{Tereshin}{\ge}\sum\frac{b^2+c^2}{4R}=\frac{2\sum a^2}{4R}=\frac{\sum a^2}{2R}\overset{(1)}{=}\frac{\frac{4}{3}\sum m_a^2}{2R}=\frac{2}{3R}\sum m_a^2$$

where (1) follows from the identity in triangle $\sum m_a^2=\frac{3}{4}\sum a^2$

From $\sum m_a\ge\frac{2}{3R}\sum m_a^2$ it follows the conclusion $\frac{\sum m_a^2}{\sum m_a}\le\frac{3R}{2}$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way:

2) In ΔABC:

$$\frac{m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2}{m_am_b+m_bm_c+m_cm_a}\le\frac{9R^2}{4}$$

Marin Chirciu

Solution: Using Tereshin's inequality $m_a\ge\frac{b^2+c^2}{4R}$ we obtain:

$$m_am_b+m_bm_c+m_cm_a=\sum m_bm_c\overset{Tereshin}{\ge}\sum\frac{a^2+c^2}{4R}\frac{a^2+b^2}{4R}=$$

$$=\frac{\sum(a^4+a^2b^2+b^2c^2+c^2a^2)}{16R^2}=$$

$$=\frac{\sum a^4+3\sum b^2c^2}{16R^2}\overset{(1)}{\ge}\frac{\sum b^2c^2+3\sum b^2c^2}{16R^2}=\frac{4\sum b^2c^2}{16R^2}=\frac{\sum b^2c^2}{4R^2}=\frac{\frac{16}{9}\sum m_b^2m_c^2}{4R^2}=$$

$=\frac{4}{9R^2}\sum m_b^2m_c^2$, where (1) follows from the identity in triangle $\sum m_b^2m_c^2=\frac{9}{16}\sum b^2c^2$

From $\sum m_bm_c\ge\frac{4}{9R^2}\sum m_b^2m_c^2$ it follows the conclusion $\frac{\sum m_b^2m_c^2}{\sum m_bm_c}\le\frac{9R^2}{4}$.

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way:

3) In ΔABC:

$$\frac{m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2}{m_a^2+m_b^2+m_c^2}\le\frac{9R^2}{4}$$

Marin Chirciu

Solution: Using Tereshin's inequality $m_a\ge\frac{b^2+c^2}{4R}$ we obtain:

$$m_a^2+m_b^2+m_c^2=\sum m_a^2\overset{Tereshin}{\ge}\sum\left(\frac{b^2+c^2}{4R}\right)=\frac{\sum(b^4+c^4+2b^2c^2)}{16R^2}=$$

$$=\frac{2\sum a^4+2\sum b^2c^2}{16R^2}=\frac{\sum a^4+\sum b^2c^2}{8R^2}\overset{(1)}{=}\frac{\frac{16}{9}\sum m_a^4+\frac{16}{9}\sum m_b^2m_c^2}{8R^2}=$$

$$=\frac{2}{9R^2}\left(\sum m_a^4+\sum m_b^2m_c^2\right)\ge\frac{2}{9R^2}\left(\sum m_b^2m_c^2+\sum m_b^2m_c^2\right)=$$

$$=\frac{2}{9R^2}\left(2\sum m_b^2m_c^2\right)=\frac{4}{9R^2}\sum m_b^2m_c^2$$

where (1) it follows from the identities in triangle $\sum m_a^4=\frac{9}{16}\sum a^4$ and $\sum m_b^2m_c^2=\frac{9}{16}\sum b^2c^2$

From $\sum m_a^2\ge\frac{4}{9R^2}\sum m_b^2m_c^2$ it follows the conclusion $\frac{\sum m_b^2m_c^2}{\sum m_a^2}\le\frac{9R^2}{4}$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

4) In ΔABC:

$$\frac{\sum m_a^4+\sum m_b^2m_c^2}{m_a^2+m_b^2+m_c^2}\le\frac{9R^2}{2}$$

Marin Chirciu

Solution: Using Tereshin's inequality $m_a\ge\frac{b^2+c^2}{4R}$ we obtain:

$$m_a^2+m_b^2+m_c^2=\sum m_a^2\overset{Tereshin}{\ge}\sum\left(\frac{b^2+c^2}{4R}\right)^2=\frac{\sum(b^4+c^4+2b^2c^2)}{16R^2}=$$

$$=\frac{2\sum a^4+2\sum b^2c^2}{16R^2}=\frac{\sum a^4+\sum b^2c^2}{8R^2}\overset{(1)}{=}\frac{\frac{16}{9}\sum m_a^4+\frac{16}{9}\sum m_b^2m_c^2}{8R^2}=$$

$$=\frac{2}{9R^2}\left(\sum m_a^4+\sum m_b^2m_c^2\right)$$

where (1) follows from the identities in triangle $\sum m_a^4=\frac{9}{16}\sum a^4$ and $\sum m_b^2m_c^2=\frac{9}{16}\sum b^2c^2$.

From $\sum m_a^2 \geq \frac{2}{9R^2}\left(\sum m_a^4 + \sum m_b^2 m_c^2\right)$ it follows the conclusion $\frac{\sum m_a^4 + \sum m_b^2 m_c^2}{m_a^2+m_b^2+m_c^2} \leq \frac{9R^2}{2}$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

5) In ΔABC:

$$\frac{\left(m_a^2+m_b^2\right)^2+\left(m_b^2+m_c^2\right)^2+\left(m_c^2+m_a^2\right)^2}{m_a^2+m_b^2+m_c^2} \leq 9R^2$$

Marin Chirciu

Solution: Using Tereshin's inequality $m_a \geq \frac{b^2+c^2}{4R}$ we obtain:

$$m_a^2+m_b^2+m_c^2 = \sum m_a^2 \overset{Tereshin}{\geq} \sum \left(\frac{b^2+c^2}{4R}\right)^2 = \frac{\sum(b^4+c^4+2b^2c^2)}{16R^2} =$$

$$= \frac{2\sum a^4 + 2\sum b^2c^2}{16R^2} = \frac{\sum a^4 + \sum b^2c^2}{8R^2} \overset{(1)}{=} \frac{\frac{16}{9}\sum m_a^4 + \frac{16}{9}\sum m_b^2 m_c^2}{8R^2} =$$

$$= \frac{2}{9R^2}\left(\sum m_a^4 + \sum m_b^2 m_c^2\right)$$

where (1) it follows from the identities in triangle $\sum m_a^4 = \frac{9}{16}\sum a^4$ and $\sum m_b^2 m_c^2 = \frac{9}{16}\sum b^2c^2$

From $\sum m_a^2 \geq \frac{2}{9R^2}\left(\sum m_a^4 + \sum m_b^2 m_c^2\right)$ it follows $\frac{\sum m_a^4 + \sum m_b^2 m_c^2}{m_a^2+m_b^2+m_c^2} \leq \frac{9R^2}{2} \Leftrightarrow$

$$\Leftrightarrow \frac{(m_a^2+m_b^2)^2+(m_b^2+m_c^2)^2+(m_c^2+m_a^2)^2}{m_a^2+m_b^2+m_c^2} \leq 9R^2$$

Equality holds if and only if the triangle is equilateral.

Reference:

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