R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ABOUT AN INEQUALITY BY BOGDAN FUȘTEI -VIII

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1) In ΔABC the following relationship holds:

$$\sum_{cyc} m_a \cos\frac{A}{2} \geq \frac{3s}{2}$$

Proposed by Bogdan Fuștei-Romania

Solution. Using Lascu's inequality: $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$ it follows,

$$LHS = \sum_{cyc} m_a \cos\frac{A}{2} \overset{Lascu}{\geq} \sum_{cyc} \frac{b+c}{2}\cos\frac{A}{2}\cdot\cos\frac{A}{2} = \sum_{cyc}\frac{b+c}{2}\cos^2\frac{A}{2} = \sum_{cyc}\frac{b+c}{2}\cdot\frac{s(s-a)}{bc}$$

$$= \frac{s}{2}\sum_{cyc}\frac{(b+c)(s-a)}{bc} \overset{(1)}{=} \frac{s}{2}\cdot 3 = \frac{3s}{2} = RHS, \text{where } \sum_{cyc}\frac{(b+c)(s-a)}{bc} = 3.$$

Equality holds if and only if triangle is equilateral.

Remark. In same class of the problem.

2) In ΔABC the following relationship holds:

$$\sum_{cyc} m_a \cos^3\frac{A}{2} \geq \frac{9s}{8}$$

Marin Chirciu

Solution. Using Lascu's inequality: $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$ it follows,

$$LHS = \sum_{cyc} m_a \cos^3\frac{A}{2} \overset{Lascu}{\geq} \sum_{cyc}\frac{b+c}{2}\cos\frac{A}{2}\cdot\cos^3\frac{A}{2} = \sum_{cyc}\frac{(b+c)}{2}\cos^4\frac{A}{2} =$$

$$= \sum_{cyc}\frac{b+c}{2}\cdot\left(\frac{s(s-a)}{bc}\right)^2 = \frac{s^2}{2}\sum_{cyc}\frac{(b+c)(s-a)^2}{b^2c^2} \overset{(1)}{=} \frac{s^2}{2}\cdot\frac{24R^2+2Rr-r^2-s^2}{8R^2s} \overset{Gerretsen}{\geq}$$

$$\geq s\cdot\frac{24R^2+2Rr-r^2-4R^2-4Rr-3r^2}{16R^2} = s\cdot\frac{20R^2-2Rr-4r^2}{16R^2} =$$

$$= s \cdot \frac{10R^2 - Rr - 2r^2}{8R^2} \overset{Euler}{\geq} s \cdot \frac{9}{8} = \frac{9s}{8} = RHD, \text{where}$$

$$(1) \Leftrightarrow \sum_{cyc} \frac{(b+c)(s-a)^2}{b^2c^2} = \frac{\sum(b+c)a^2(s-a)^2}{a^2b^2c^2} = \frac{2sr^2(24R^2 + 2Rr - r^2 - s^2)}{16R^2r^2s^2} =$$

$$= \frac{24R^2 + 2Rr - r^2 - s^2}{8R^2s}, \text{which follows from:}$$

$$\sum_{cyc}(b+c)a^2(s-a)^2 = 2sr^2(24R^2 + 2Rr - r^2 - s^2) \text{ true from}$$

$$\sum_{cyc}(b+c)a^2(s-a)^2 = 2s\sum_{cyc} a^2(s-a)^2 - \sum_{cyc} a^3(s-a)^2 =$$

$$= 2s \cdot 2r^2[(4R+r)^2 - s^2] - 2sr^2(8R^2 + 14Rr + 3r^2 - s^2)$$

$$= 2sr^2(24R^2 + 2Rr - r^2 - s^2)$$

It is well-known that sums: $\sum a^2(s-a)^2 = 2r^2[(4R+r)^2 - s^2]$ and $\sum a^3(s-a)^2 = 2sr^2(8R^2 + 14Rr + 3r^2 - s^2)$. Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sum_{cyc} m_a \cos^{2n+1}\frac{A}{2} \geq \frac{3s}{2} \cdot \left(\frac{2}{3} + \frac{r}{6R}\right)^n, n \in \mathbb{N}$$

Marin Chirciu

Solution. For $n = 0$ we get **Lemma.**

Lemma. 4) In ΔABC the following relationship holds:

$$\sum_{cyc} m_a \cos\frac{A}{2} \geq \frac{3s}{2}$$

Bogdan Fuștei

Solution. Using Lascu's inequality: $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$ it follows,

$$LHS = \sum_{cyc} m_a \cos\frac{A}{2} \overset{Lascu}{\geq} \sum_{cyc} \frac{b+c}{2}\cos\frac{A}{2} \cdot \cos\frac{A}{2} = \sum_{cyc} \frac{b+c}{2} \cdot \cos^2\frac{A}{2} =$$

$$=\sum_{cyc}\frac{b+c}{2}\cdot\frac{s(s-a)}{bc}=\frac{s}{2}\sum_{cyc}\frac{(b+c)(s-a)}{bc}\overset{(1)}{=}\frac{s}{2}\cdot 3=\frac{3s}{2}=RHS,\text{where}$$

$(1)\Leftrightarrow\sum_{cyc}\frac{(b+c)(s-a)}{bc}=3$. Equality holds if and only if triangle is equilateral.

For $n\geq 1$ triplets $\left(m_a\cos\frac{A}{2},m_b\cos\frac{B}{2},m_c\cos\frac{C}{2}\right)$ and $\left(\cos^{2n}\frac{A}{2},\cos^{2n}\frac{B}{2},\cos^{2n}\frac{C}{2}\right)$ has same ordered, using Chebychev's inequality it follows that,

$$LHS=\sum_{cyc}m_a\cos^{2n+1}\frac{A}{2}=\sum_{cyc}m_a\cos\frac{A}{2}\cdot\cos^{2n}\frac{A}{2}\overset{Chebychev's}{\geq}\frac{1}{3}\sum_{cyc}m_a\cos\frac{A}{2}\cdot\sum_{cyc}\cos^{2n}\frac{A}{2}$$

$$\overset{Lemma}{\geq}\frac{1}{3}\cdot\frac{3s}{2}\cdot\sum_{cyc}\cos^{2n}\frac{A}{2}=\frac{s}{2}\cdot\sum_{cyc}\left(\cos^2\frac{A}{2}\right)^n\overset{(1)}{\geq}\frac{s}{2}\cdot\frac{\left(2+\frac{r}{2R}\right)^n}{3^{n-1}}=\frac{3s}{2}\cdot\left(\frac{2}{3}+\frac{r}{6R}\right)^n=RHD,$$

which follows from $\sum\frac{\left(\cos^2\frac{A}{2}\right)^n}{1}\overset{Holder}{\geq}\frac{\left(\sum\cos^2\frac{A}{2}\right)^n}{3^{n-2}(1+1+1)}=\frac{\left(\sum\cos^2\frac{A}{2}\right)^n}{3^{n-1}}=\frac{\left(2+\frac{r}{2R}\right)^n}{3^{n-1}}$.

Equality holds if and only if triangle is equilateral.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-Interactive Journal-www.ssmrmh.ro