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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-X

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1) In ΔABC, the following inequality holds:

$$\frac{2r}{R^2} \le \sum \frac{\sqrt{r_b r_c}}{m_b^2 + m_c^2} \le \frac{R^2}{8r^3}$$

Proposed by George Apostolopoulos – Greece

Solution: LHS inequality. We prove. **Lemma:**

2) In ΔABC the following relationship holds:

$$m_b^2 + m_c^2 \ge 2p\sqrt{(p-b)(p-c)}$$

Proof: Using $m_a \ge \frac{b+c}{2}\cos\frac{A}{2}$, (Lascu Inequality) we obtain:

$$m_b^2 + m_c^2 \ge 2m_b m_c \overset{Lascu}{\ge} 2\cdot\frac{a+c}{2}\cos\frac{B}{2}\cdot\frac{a+b}{2}\cos\frac{C}{2} \overset{AM-GM}{\ge} 2\sqrt{ac}\sqrt{ab}\cos\frac{B}{2}\cos\frac{C}{2} =$$

$$= 2a\sqrt{bc}\sqrt{\frac{p(p-b)}{ac}}\sqrt{\frac{p(p-c)}{ab}} = 2p\sqrt{(p-b)(p-c)}$$

We obtain:

$$E = \sum \frac{\sqrt{r_b r_c}}{m_b^2 + m_c^2} \overset{Lemma}{\le} \sum \frac{\sqrt{\frac{S}{p-b}\cdot\frac{S}{p-c}}}{2p\sqrt{(p-b)(p-c)}} = \sum \frac{S}{2p(p-b)(p-c)} =$$

$$= \sum \frac{pr}{2p(p-b)(p-c)} = \frac{r}{2}\sum \frac{1}{(p-b)(p-c)} = \frac{r}{2}\cdot\frac{1}{r^2} = \frac{1}{2r} \overset{Euler}{\le} \frac{R^2}{8r^3} = RHS$$

Equality holds if and only if the triangle is equilateral.LHS inequality.We prove **Lemma:**

3) In ΔABC the following inequality holds:

$$\sqrt[3]{\prod (m_b^2 + m_c^2)} \le 2(2R^2 + r^2)$$

Proof: Using means inequality we obtain:

$$\sqrt[3]{\prod (m_b^2 + m_c^2)} \le \frac{\sum (m_b^2 + m_c^2)}{3} = \frac{2}{3}\sum m_a^2 = \frac{2}{3}\cdot\frac{3}{4}\sum a^2 = \frac{1}{2}\cdot 2(p^2 - r^2 - 4Rr) =$$

$$= p^2 - r^2 - 4Rr \overset{Gerretsen}{\le} 4R^2 + 4Rr + 3r^2 - r^2 - 4Rr = 4R^2 + 2r^2 = 2(2R^2 + r^2)$$

Using Lemma and the means inequality we obtain:

$$E=\sum\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}\ge 3\sqrt[3]{\prod\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}}=3\sqrt[3]{\prod\frac{r_a}{m_b^2+m_c^2}}=\frac{3\sqrt[3]{r_ar_br_c}}{\sqrt[3]{\prod(m_b^2+m_c^2)}}\ge$$

$$\ge\frac{3\sqrt[3]{rp^2}}{2(2R^2+r^2)}\ge\frac{3\sqrt[3]{r\cdot 27r^2}}{2(2R^2+r^2)}=\frac{9r}{2(2R^2+r^2)}\overset{(1)}{\ge}\frac{2r}{R^2}=LHS$$

where (1) $\Leftrightarrow\frac{9r}{2(2R^2+r^2)}\ge\frac{2r}{R^2}\Leftrightarrow R\ge 2r$, (Euler's inequality).Equality holds if and only if the triangle is equilateral.**Remark:** The inequality can be strengthened.

4) In ΔABC the following inequality holds:

$$\frac{9r}{2(2R^2+r^2)}\le\sum\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}\le\frac{1}{2r}$$

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Solution RHS inequality.We prove.**Lemma:**

5) In ΔABC the following inequality holds:

$$m_b^2+m_c^2\ge 2p\sqrt{(p-b)(p-c)}$$

Proof: Using $m_a\ge\frac{b+c}{2}\cos\frac{A}{2}$, (Lascu Inequality) we obtain:

$$m_b^2+m_c^2\ge 2m_bm_c\overset{Lascu}{\ge}2\cdot\frac{a+c}{2}\cos\frac{B}{2}\cdot\frac{a+b}{2}\cos\frac{C}{2}\overset{AM-GM}{\ge}$$

$$\ge 2\sqrt{ac}\sqrt{ab}\cos\frac{B}{2}\cos\frac{C}{2}=2a\sqrt{bc}\sqrt{\frac{p(p-b)}{ac}}\sqrt{\frac{p(p-c)}{ab}}=2p\sqrt{(p-b)(p-c)}$$

We obtain:

$$E=\sum\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}\overset{Lemma}{\le}\sum\frac{\sqrt{\frac{S}{p-b}\cdot\frac{S}{p-c}}}{2p\sqrt{(p-b)(p-c)}}=\sum\frac{S}{2p(p-b)(p-c)}=$$

$$=\sum\frac{pr}{2p(p-b)(p-c)}=\frac{r}{2}\sum\frac{1}{(p-b)(p-c)}=\frac{r}{2}\cdot\frac{1}{r^2}=\frac{1}{2r}=RHS$$

Equality holds if and only if the triangle is equilateral.LHS inequality.We prove **Lemma:**

6) In ΔABC the following inequality holds:

$$\sqrt[3]{\prod(m_b^2+m_c^2)}\le 2(2R^2+r^2)$$

Proof: Using the means inequality we obtain:

$$\sqrt[3]{\prod(m_b^2+m_c^2)}\le\frac{\sum(m_b^2+m_c^2)}{3}=\frac{2}{3}\sum m_a^2=\frac{2}{3}\cdot\frac{3}{4}\sum a^2=\frac{1}{2}\cdot 2(p^2-r^2-4Rr)=$$

$$=p^2-r^2-4Rr\overset{Gerretsen}{\le}4R^2+4Rr+3r^2-r^2-4Rr=4R^2+2r^2=2(2R^2+r^2)$$

Using Lemma and means inequality we obtain:

$$E=\sum\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}\ge 3\sqrt[3]{\prod\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}}=3\sqrt[3]{\prod\frac{r_a}{m_b^2+m_c^2}}=\frac{3\sqrt[3]{r_ar_br_c}}{\sqrt[3]{\prod(m_b^2+m_c^2)}}\ge$$

$$\ge\frac{3\sqrt[3]{rp^2}}{2(2R^2+r^2)}\ge\frac{3\sqrt[3]{r\cdot 27r^2}}{2(2R^2+r^2)}=\frac{9r}{2(2R^2+r^2)}=LHS$$

Equality holds if and only if the triangle is equilateral.

Note: The inequality strengthen Inequality in triangle 2300, proposed by George Apostolopoulos, Greece, in RMM 11/2020.

Remark: Inequality 4) is stronger than inequality 1).

7) In ΔABC the following inequality holds:

$$\frac{2r}{R^2}\le\frac{9r}{2(2R^2+r^2)}\le\sum\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}\le\frac{1}{2r}\le\frac{R^2}{8r^3}$$

Solution: See inequality 4) and Euler's inequality $R\ge 2r$.Equality holds if and only if the triangle is equilateral.**Remark:**In the same way:

8) In ΔABC the following inequality holds:

$$\frac{9r}{2(2R^2+r^2)}\le\sum\frac{\sqrt{h_bh_c}}{m_b^2+m_c^2}\le\frac{1}{2r}$$

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Solution: RHS inequality. We prove.**Lemma:**

9) In ΔABC the following inequality holds:

$$m_b^2+m_c^2\ge 2p\sqrt{(p-b)(p-c)}$$

Proof: Using $m_a\ge\frac{b+c}{2}\cos\frac{A}{2}$, (Lascu Inequality) we obtain:

$$m_b^2+m_c^2\ge 2m_bm_c\overset{Lascu}{\ge}2\cdot\frac{a+c}{2}\cos\frac{B}{2}\cdot\frac{a+b}{2}\cos\frac{C}{2}\overset{AM-GM}{\ge}2\sqrt{ac}\sqrt{ab}\cos\frac{B}{2}\cos\frac{C}{2}=$$

$$=2a\sqrt{bc}\sqrt{\frac{p(p-b)}{ac}}\sqrt{\frac{p(p-c)}{ab}}=2p\sqrt{(p-b)(p-c)}$$

We obtain:

$$E=\sum\frac{\sqrt{h_bh_c}}{m_b^2+m_c^2}\overset{Lemma}{\le}\sum\frac{\sqrt{\frac{2S}{b}\cdot\frac{2S}{c}}}{2p\sqrt{(p-b)(p-c)}}=\sum\frac{2S}{2p\sqrt{bc(p-b)(p-c)}}=$$

$$=\sum\frac{2pr}{2p\sqrt{bc(p-b)(p-c)}}=\sum\frac{r}{\sqrt{bc(p-b)(p-c)}}\overset{(2)}{\le}\frac{1}{2r}=RHS$$

where (2) $\Leftrightarrow \sum \frac{r}{\sqrt{bc(p-b)(p-c)}} \leq \frac{1}{2r} \Leftrightarrow \sum \frac{1}{\sqrt{bc(p-b)(p-c)}} \leq \frac{1}{2r^2}$, which follows from CBS inequality.
Indeed:

$$\sum \frac{1}{\sqrt{bc(p-b)(p-c)}} \overset{CBS}{\leq} \sqrt{3\sum \frac{1}{bc(p-b)(p-c)}} = \sqrt{3 \cdot \frac{4R+r}{2Rr^2p^2}} \overset{(3)}{\leq} \frac{1}{2r^2}$$

where (3) $\Leftrightarrow \sqrt{3 \cdot \frac{4R+r}{2Rr^2p^2}} \leq \frac{1}{2r^2} \Leftrightarrow 3 \cdot \frac{4R+r}{2Rr^2p^2} \leq \frac{1}{4r^2} \Leftrightarrow Rp^2 \geq 6r^2(4R+r)$, which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.
It remains to prove that:

$$R(16Rr - 5r^2) \geq 6r^2(4R+r) \Leftrightarrow 16R^2 - 29Rr - 6r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(16R+3r) \geq 0$, obviously from Euler's inequality $R \geq 2r$.
We've used $\sum \frac{1}{bc(p-b)(p-c)} = \frac{4R+r}{2Rr^2p^2}$.
Equality holds if and only if the triangle is equilateral.LHS inequality.We prove.**Lemma:**

10) In ΔABC the following relationship holds:

$$\sqrt[3]{\prod (m_b^2 + m_c^2)} \leq 2(2R^2 + r^2)$$

Proof: Using means inequality we obtain:

$$\sqrt[3]{\prod (m_b^2 + m_c^2)} \leq \frac{\sum (m_b^2 + m_c^2)}{3} = \frac{2}{3}\sum m_a^2 = \frac{2}{3} \cdot \frac{3}{4}\sum a^2 = \frac{1}{2} \cdot 2(p^2 - r^2 - 4Rr) =$$

$$= p^2 - r^2 - 4Rr \overset{Gerretsen}{\leq} 4R^2 + 4Rr + 3r^2 - r^2 - 4Rr = 4R^2 + 2r^2 = 2(2R^2 + r^2)$$

Using Lemma and the means inequality we obtain:

$$E = \sum \frac{\sqrt{h_b h_c}}{m_b^2 + m_c^2} \geq 3\sqrt[3]{\prod \frac{\sqrt{h_b r h}}{m_b^2 + m_c^2}} = 3\sqrt[3]{\prod \frac{h_a}{m_b^2 + m_c^2}} = \frac{3\sqrt[3]{h_a h_b h_c}}{\sqrt[3]{\prod (m_b^2 + m_c^2)}} \overset{Lemma}{\geq}$$

$$\overset{Lemma}{\geq} \frac{3\sqrt[3]{\frac{2r^2p^2}{R}}}{2(2R^2 + r^2)} \overset{(4)}{\geq} \frac{3\sqrt[3]{27r^3}}{2(2R^2 + r^2)} = \frac{9r}{2(2R^2 + r^2)} = LHS$$

where (4) $\Leftrightarrow \frac{2r^2p^2}{R} \overset{Gerretsen}{\geq} \frac{2r^2(16Rr - 5r^2)}{R} = \frac{2r^3(16R - 5r)}{R} \overset{Euler}{\geq} 2r^3 \cdot \frac{27}{2} = 27r^3$
Equality holds if and only if the triangle is equilateral.

Reference:

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