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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-XI

By Marin Chirciu-Romania

1) In ΔABC the following inequality holds:

$$\sum r_a h_a \tan\frac{A}{2} \le F\left(\frac{2R}{r} - 1\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution: We prove: **Lemma:**

$$\sum r_a h_a \tan\frac{A}{2} = \frac{p^2(r^2 - 8Rr) + r(4R + r)^3}{2pR}$$

Proof: We have:

$$\sum r_a h_a \tan\frac{A}{2} = \sum \frac{S}{s-a} \cdot \frac{2S}{a} \cdot \sqrt{\frac{(p-b)(p-c)}{p(p-a)}} =$$

$$= 2S^2 \sum \frac{1}{a(p-a)} \cdot \frac{\sqrt{p(p-a)(p-b)(p-c)}}{p(p-a)} =$$

$$= 2S^2 \cdot \frac{S}{p} \sum \frac{1}{a(p-a)} \cdot \frac{1}{(p-a)} = 2p^2r^2 \cdot \frac{pr}{p} \sum \frac{1}{a(p-a)^2} = 2p^2r^3 \sum \frac{1}{a(p-a)^2} =$$

$= 2p^2r^3 \cdot \frac{p^2(r-8R)+(4R+r)^3}{4Rr^2p^3} = \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR}$, which follows from:

$$\sum \frac{1}{a(p-a)^2} = \frac{\sum bc(p-b)^2(p-c)^2}{abc \cdot \prod(p-a)^2} = \frac{r^3[p^2(r-8R) + (4R+r)^3]}{4Rrp \cdot (r^2p)^2} =$$

$= \frac{p^2(r-8R)+(4R+r)^3}{4Rr^2p^3}$, true from:

$$\sum bc(p-b)^2(p-c)^2 = r^3[p^2(r-8R) + (4R+r)^3]$$

Let's get to the main problem.Using Lemma the inequality can be written:

$$\frac{p^2(r^2 - 8Rr) + r(4R + r)^3}{2pR} \le F\left(\frac{2R}{r} - 1\right) \Leftrightarrow \frac{p^2(r^2 - 8Rr) + r(4R + r)^3}{2pR} \le$$

$$\le pr\left(\frac{2R - r}{r}\right) \Leftrightarrow \frac{p^2(r^2 - 8Rr) + r(4R + r)^3}{2pR} \le pr\left(\frac{2R - r}{r}\right) \Leftrightarrow$$

$$\Leftrightarrow p^2(r^2 - 8Rr) + r(4R + r)^3 \le 2p^2(2R - r) \Leftrightarrow$$

$\Leftrightarrow p^2(4R^2 + 6Rr - r^2) \ge r(4R + r)^3$, which follows from Gerretesen's inequality:

$$p^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(4R^2+6Rr-r^2) \geq r(4R+r)^3 \Leftrightarrow (4R^2+6Rr-r^2) \geq$$
$$\geq (R+r)(4R+r) \Leftrightarrow R \geq 2r, \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark: The inequality can be strengthened.

2) In ΔABC the following inequality holds:

$$\sum r_a h_a \tan\frac{A}{2} \leq F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right)$$

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Proof: Using the Lemma the inequality can be written:

$$\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \leq F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right) \Leftrightarrow$$
$$\Leftrightarrow \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \leq pr\left(\frac{4R^2-3Rr+2r^2}{2Rr}\right) \Leftrightarrow$$
$$\Leftrightarrow p^2(r^2-8Rr)+r(4R+r)^3 \leq p^2(4R^2-3Rr+2r^2) \Leftrightarrow$$
$$\Leftrightarrow p^2(4R^2+5Rr+r^2) \geq r(4R+r)^3 \Leftrightarrow$$
$$\Leftrightarrow p^2(4R+r)(R+r) \geq r(4R+r)^3 \Leftrightarrow p^2(R+r) \geq r(4R+r)^2 \Leftrightarrow p^2 \geq \frac{r(4R+r)^2}{R+r},$$

which follows from Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$

Equality holds if and only if the triangle is equilateral. **Remark:**
Inequality 2) is stronger than inequality 1)

3) In ΔABC the following relationship holds:

$$\sum r_a h_a \tan\frac{A}{2} \leq F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right) \leq F\left(\frac{2R}{r}-1\right)$$

Solution: See inequality 1) and $F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right) \leq F\left(\frac{2R}{r}-1\right) \Leftrightarrow R \geq 2r$, (Euler's inequality).
Equality holds if and only if the triangle is equilateral.**Remark:**
Let's find an inequality having an opposite sense.

4) In ΔABC the following relationship holds:

$$\sum r_a h_a \tan\frac{A}{2} \geq 3F$$

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Proof: Using the Lemma we obtain:

$$\sum r_a h_a \tan\frac{A}{2} = \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} = \frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{p^2}\right] \ge$$

$$\overset{Gerretsen}{\ge} \frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right] = \frac{p}{2R}\left[(r^2-8Rr)+\frac{2r(2R-r)(4R+r)}{R}\right] =$$

$$= \frac{pr}{2R}\left[\frac{R(r-8R)+2(2R-r)(4R+r)}{R}\right] = \frac{S}{2R}\cdot\frac{8R^2-3Rr-2r^2}{R} \overset{Euler}{\ge} \frac{S}{2R}\cdot 6R = 3S$$

Equality holds if and only if the triangle is equilateral.**Remark:**

We can write the double inequality:

5) In ΔABC the following inequality holds:

$$3F \le \sum r_a h_a \tan\frac{A}{2} \le F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right)$$

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Proof:RHS inequality.Using the Lemma the inequality can be written:

$$\frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \le F\left(\frac{2R}{r}+\frac{r}{R}-\frac{3}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} \le pr\left(\frac{4R^2-3Rr+2r^2}{2Rr}\right) \Leftrightarrow$$

$$\Leftrightarrow p^2(r^2-8Rr)+r(4R+r)^3 \le p^2(4R^2-3Rr+2r^2) \Leftrightarrow$$

$$\Leftrightarrow p^2(4R^2+5Rr+r^2) \ge r(4R+r)^3 \Leftrightarrow$$

$$\Leftrightarrow p^2(4R+r)(R+r) \ge r(4R+r)^3 \Leftrightarrow p^2(R+r) \ge r(4R+r)^2 \Leftrightarrow p^2 \ge \frac{r(4R+r)^2}{R+r},$$

which follows from Gerretsen's inequality: $p^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r}$.

Equality holds if and only if the triangle is equilateral. LHS inequality.

Using the Lemma we obtain:

$$\sum r_a h_a \tan\frac{A}{2} = \frac{p^2(r^2-8Rr)+r(4R+r)^3}{2pR} = \frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{p^2}\right] \ge$$

$$\overset{Gerretsen}{\ge} \frac{p}{2R}\left[(r^2-8Rr)+\frac{r(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$= \frac{p}{2R}\left[(r^2-8Rr)+\frac{2r(2R-r)(4R+r)}{R}\right] =$$

$$= \frac{pr}{2R}\left[\frac{R(r-8R)+2(2R-r)(4R+r)}{R}\right] = \frac{S}{2R}\cdot\frac{8R^2-3Rr-2r^2}{R} \overset{Euler}{\ge} \frac{S}{2R}\cdot 6R = 3S$$

Equality holds if and only if the triangle is equilateral.

Reference:

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