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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-XII

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\frac{8r}{R^2} \le \sum \frac{r_b + r_c}{m_b m_c} \le \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution: RHS inequality. Using $h_a \le m_a$ we obtain:

$$E = \sum \frac{r_b + r_c}{m_b m_c} \overset{h_a \le m_a}{\le} \sum \frac{r_b + r_c}{h_b h_c} = \frac{R}{r^2} \overset{(1)}{\le} \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right) = RHS$$

where (1) $\Leftrightarrow \frac{R}{r^2} \le \frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4 - 1\right) \Leftrightarrow (R-2r)(3R^3 + 3R^2 r + 3Rr^2 + r^3) \ge 0$, which follows from Euler's inequality $R \ge 2r$.

Above, we've used the identity $\sum \frac{r_b + r_c}{h_b h_c} = \frac{R}{r^2}$, which follows from:

$$\sum \frac{r_b + r_c}{h_b h_c} = \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\frac{2S}{b} \cdot \frac{2S}{c}} = \frac{1}{4S}\sum \frac{abc}{(s-b)(s-c)} = \frac{abc}{4S}\sum \frac{1}{(s-b)(s-c)} =$$

$$= \frac{4Rrs}{4sr} \cdot \frac{1}{r^2} = \frac{R}{r^2}$$

Equality holds if and only if the triangle is equilateral.LHS inequality.

Using Panaitopol Inequality: $m_a \le \frac{R}{2r} h_a$ we obtain:

$$E = \sum \frac{r_b + r_c}{m_b m_c} \overset{Panaitopol}{\ge} \sum \frac{r_b + r_c}{\frac{R}{r} h_b \cdot \frac{R}{2r} h_c} \ge \left(\frac{2r}{R}\right)^2 \sum \frac{r_b + r_c}{h_b h_c} =$$

$$= \left(\frac{2r}{R}\right)^2 \cdot \frac{R}{r^2} = \frac{4}{R} \ge \frac{8r}{R^2} = LHS$$

$$\sum \frac{r_b + r_c}{h_b h_c} = \frac{R}{r^2}$$

Equality holds if and only if the triangle is equilateral.**Remark:** The inequality can be strengthened.

2) In ΔABC the following relationship holds:

$$\frac{4}{R} \le \sum \frac{r_b + r_c}{m_b m_c} \le \frac{R}{r^2}$$

Marin Chirciu

Solution: Let's get back to the main problem. RHS inequality.

Using $h_a \le m_a$ we obtain:

$$E=\sum\frac{r_b+r_c}{m_bm_c}\overset{h_a\le m_a}{\le}\sum\frac{r_b+r_c}{h_bh_c}=\frac{R}{r^2}=RHS$$

Above, we've used the identity $\sum\frac{r_b+r_c}{h_bh_c}=\frac{R}{r^2}$, which follows from:

$$\sum\frac{r_b+r_c}{h_bh_c}=\sum\frac{\frac{S}{s-b}+\frac{S}{s-c}}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{4S}\sum\frac{abc}{(s-b)(s-c)}=\frac{abc}{4S}\sum\frac{1}{(s-b)(s-c)}=$$

$$=\frac{4Rrs}{4sr}\cdot\frac{1}{r^2}=\frac{R}{r^2}$$

Equality holds if and only if the triangle is equilateral.LHS inequality. Using Panaitopol Inequality: $m_a\le\frac{R}{2r}h_a$ we obtain:

$$E=\sum\frac{r_b+r_c}{m_bm_c}\overset{Panaitopol}{\ge}\sum\frac{r_b+r_c}{\frac{R}{2r}h_b\cdot\frac{R}{2r}h_c}\ge\left(\frac{2r}{R}\right)^2\sum\frac{r_b+r_c}{h_bh_c}=$$

$$=\left(\frac{2R}{r}\right)^2\cdot\frac{R}{r^2}=\frac{4}{R}=LHS$$

We've used the identity in triangle: $\sum\frac{r_b+r_c}{h_bh_c}=\frac{R}{r^2}$.

Equality holds if and only if the triangle is equilateral.

Note: The inequality strengthen Problem SP.367 from RMM no. 25, Summer Edition 2022, proposed by George Apostolopoulos, Greece.

In ΔABC the following relationship holds:

$$\frac{8r}{R^2}\le\sum\frac{r_b+r_c}{m_bm_c}\le\frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4-1\right)$$

George Apostolopoulos, Greece

Remark: Inequality 2) is stronger than inequality 1)

3) In ΔABC the following relationship holds:

$$\frac{8r}{R^2}\le\frac{4}{R}\le\sum\frac{r_b+r_c}{m_bm_c}\le\frac{R}{r^2}\le\frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4-1\right)$$

Solution: See inequality 2) and

i) $\frac{R}{r^2}\le\frac{1}{r}\left(3\left(\frac{2R}{r}\right)^4-1\right)\Leftrightarrow(R-2r)(3R^3+3R^2r+3Rr^2+r^3)\ge 0$,

which follows from Euler's inequality $R\ge 2r$.

ii) $\frac{8r}{R^2}\le\frac{4}{R}\Leftrightarrow R\ge 2r$, (Euler's inequality). Equality holds if and only if the triangle is equilateral.

Reference:

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