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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-XIX

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1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{a^2}\ge\frac{27}{4(4R+r)}$$

Proposed by Marian Ursărescu-Romania

Solution. **Lemma 2) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{m_a}{a^2}\ge\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Proof. Using Tereshin inequality, we get:

$$\sum_{cyc}\frac{m_a}{a^2}\ge\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{a^2}=\frac{1}{4R}\sum_{cyc}\frac{b^2+c^2}{a^2}\ge\frac{1}{4R}\cdot\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)=$$

$$=\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right),\quad \text{which follows from}$$

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(24R^2+8Rr-r^2)-r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^4+s^2(r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\ge}$$

$$\ge\frac{1}{8R^2r^2}\left[(16Rr-5r^2)(16Rr-5r^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{8R^2r^2}[(16Rr-5r^2)(4Rr-4r^2)+r^2(24R^2+8Rr-r^2)$$
$$-r^2(R+r)(4R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[(16R-5r)(4R-4r)+(24R^2+8R-r^2)-(R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[64R^2-64Rr-20Rr+20r^2+(24R^2+8Rr-r^2)-4R^2-5Rr-r^2]=$$

$$=\frac{84R^2-81Rr+18r^2}{8R^2}=\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Let's get back to the main problem. Using Lemma , it follows that:

$$E=\sum_{cyc}\frac{m_a}{a^2}\geq\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\overset{(1)}{\geq}\frac{27}{4(4R+r)}=RHS$$

$$(1)\Leftrightarrow\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\geq\frac{27}{4(4R+r)}\Leftrightarrow 40R^3-80R^2r-3Rr^2+6r^3\geq 0\Leftrightarrow$$

$(R-2r)(40R^2-3r^2)\geq 0$, which is obviously true from $R\geq 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. Inequality can be much stronger.

3) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{a^2}\geq\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

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Solution. Using Tereshin inequality, we get:

$$\sum_{cyc}\frac{m_a}{a^2}\geq\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{a^2}=\frac{1}{4R}\sum_{cyc}\frac{b^2+c^2}{a^2}\geq\frac{1}{4R}\cdot\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)=$$

$$=\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right),\qquad \text{which follows from}$$

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(24R^2+8Rr-r^2)-r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^4+s^2(r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\geq\frac{1}{8R^2r^2}\left[(16Rr-5r^2)(16Rr-5r^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{8R^2r^2}[(16Rr-5r^2)(4Rr-4r^2)+r^2(24R^2+8Rr-r^2)$$
$$-r^2(R+r)(4R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[(16R-5r)(4R-4r)+(24R^2+8R-r^2)-(R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[64R^2-64Rr-20Rr+20r^2+(24R^2+8Rr-r^2)-4R^2-5Rr-r^2]=$$

$$= \frac{84R^2 - 81Rr + 18r^2}{8R^2} = \frac{3}{8}\left(28 - \frac{27r}{R} + \frac{6r^2}{R^2}\right)$$

Equality holds if and only if triangle is equilateral.

Remark. Inequality 3) is much stronger such inequality 1).

4) In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a}{a^2} \geq \frac{3}{32R}\left(28 - \frac{27r}{R} + \frac{6r^2}{R^2}\right) \geq \frac{27}{4(4R + r)}$$

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Solution. See inequality 3) and $\frac{3}{32R}\left(28 - \frac{27r}{R} + \frac{6r^2}{R^2}\right) \geq \frac{27}{4(4R+r)} \Leftrightarrow$ $40R^3 - 80R^2r - 3Rr^2 + 6r^3 \geq 0 \Leftrightarrow (R - 2r)(40R^2 - 3r^2) \geq 0$ which is true from $R \geq 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

Remark. Let's find an opposite inequality.

5) In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a}{a^2} \leq \frac{3R}{8r^2}$$

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Solution. Using Panaitopol's inequality $m_a \leq \frac{Rs}{a}$ we get:

$$\sum_{cyc} \frac{m_a}{a^2} \leq \sum_{cyc} \frac{\frac{Rs}{a}}{a^2} \leq Rs \sum_{cyc} \frac{1}{a^3} \leq Rs \cdot \frac{1}{64RF}\left(68 - 47\frac{r}{R} + 14\frac{r^2}{R^2}\right) =$$

$$= \frac{1}{64r}\left(68 - 47\frac{r}{R} + 14\frac{r^2}{R^2}\right) \overset{(1)}{\leq} \frac{1}{64r} \cdot \frac{24R}{r} = \frac{3R}{8r^2}, \text{ where}$$

$$(1) \Leftrightarrow 68 - 47\frac{r}{R} + 14\frac{r^2}{R^2} \leq \frac{24R}{r} \Leftrightarrow 24R^3 - 68R^2r + 47Rr^2 - 14r^3 \geq 0 \Leftrightarrow$$

$(R - 2r)(24R^2 - 20Rr + 7r^2) \geq 0$, which is true from $R \geq 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

$$\sum_{cyc} \frac{1}{a^3} \leq \frac{3}{8sr^2}, \text{ which follows from:}$$

$$\sum_{cyc} \frac{1}{a^3} \leq \frac{1}{64RF}\left(68 - 47\frac{r}{R} + 14\frac{r^2}{R^2}\right) \overset{(1)}{\leq} \frac{1}{64RF} \cdot \frac{24R}{r} = \frac{3}{8sr^2}, \text{ which is true from:}$$

$$\sum_{cyc} \frac{1}{a^3} = \frac{\sum b^3c^3}{a^3b^3c^3} = \frac{s^6 + s^4(3r^2 - 12Rr) + 3s^2r^4 + r^3(4R + r)^3}{64R^3r^3s^3} =$$

$$=\frac{1}{64R^3r^3s}\left[s^4+s^2(3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{64R^3r^3s}\left[s^2(s^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{1}{64R^3r^3s}\left[(16Rr-5r^2)(16Rr-5r^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{64R^3rs}[(16R-5r)(4R-2r)+3r^2+(R+r)(4R+r)]=$$

$$=\frac{1}{64R^3rs}[64r^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{1}{64R^3F}[64R^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{68R^2-47Rr+14r^2}{64R^3F}=\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)$$

$$\sum_{cyc}b^3c^3=s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4R+r)^3$$

6) In ΔABC the following relationship holds:

$$\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\leq\sum_{cyc}\frac{m_a}{a^2}\leq\frac{3R}{8r^2}$$

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Solution. For LHS, using Tereshin inequality, we get:

$$\sum_{cyc}\frac{m_a}{a^2}\geq\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{a^2}=\frac{1}{4R}\sum_{cyc}\frac{b^2+c^2}{a^2}\geq\frac{1}{4R}\cdot\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)=$$

$$=\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$, which follows from:

Lemma 7) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{b^2+c^2}{a^2}\geq\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Proof.

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(24R^2+8Rr-r^2)-r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^4+s^2(r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\geq\frac{1}{8R^2r^2}\left[(16Rr-5r^2)(16Rr-5r^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{8R^2r^2}[(16Rr-5r^2)(4Rr-4r^2)+r^2(24R^2+8Rr-r^2)$$
$$-r^2(R+r)(4R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[(16R-5r)(4R-4r)+(24R^2+8R-r^2)-(R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[64R^2-64Rr-20Rr+20r^2+(24R^2+8Rr-r^2)-4R^2-5Rr-r^2]=$$

$$=\frac{84R^2-81Rr+18r^2}{8R^2}=\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Equality holds if and only if triangle is equilateral. For RHS, using Panaitopol's inequality $m_a\leq\frac{Rs}{a}$ we get:

$$\sum_{cyc}\frac{m_a}{a^2}\leq\sum_{cyc}\frac{\frac{Rs}{a}}{a^2}\leq Rs\sum_{cyc}\frac{1}{a^3}\leq Rs\cdot\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)=$$

$$=\frac{1}{64r}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\overset{(1)}{\leq}\frac{1}{64r}\cdot\frac{24R}{r}=\frac{3R}{8r^2}, \text{ where}$$

$$(1)\Leftrightarrow 68-47\frac{r}{R}+14\frac{r^2}{R^2}\leq\frac{24R}{r}\Leftrightarrow 24R^3-68R^2r+47Rr^2-14r^3\geq 0\Leftrightarrow$$
$$(R-2r)(24R^2-20Rr+7r^2)\geq 0, \text{ which is true from } R\geq 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

$$\sum_{cyc}\frac{1}{a^3}\leq\frac{3}{8sr^2}, \text{ which follows from:}$$

$$\sum_{cyc}\frac{1}{a^3}\leq\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\overset{(1)}{\leq}\frac{1}{64RF}\cdot\frac{24R}{r}=\frac{3}{8sr^2}, \text{ which is true from:}$$

$$\sum_{cyc}\frac{1}{a^3}=\frac{\sum b^3c^3}{a^3b^3c^3}=\frac{s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4R+r)^3}{64R^3r^3s^3}=$$

$$=\frac{1}{64R^3r^3s}\left[s^4+s^2(3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{64R^3r^3s}\left[s^2(s^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq \frac{1}{64R^3r^3s}\left[(16Rr-5r^2)(16Rr-5r^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{64R^3rs}[(16R-5r)(4R-2r)+3r^2+(R+r)(4R+r)]=$$

$$=\frac{1}{64R^3rs}[64r^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{1}{64R^3F}[64R^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{68R^2-47Rr+14r^2}{64R^3F}=\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)$$

$$\sum_{cyc} b^3c^3=s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4R+r)^3$$

Equality holds if and only if triangle is equilateral.

8) In ΔABC the following relationship holds:

$$\frac{27}{4(4R+r)}\leq\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\leq\sum_{cyc}\frac{m_a}{a^2}\leq\frac{1}{64r}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\leq\frac{3R}{8r^2}$$

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Solution. See up these inequalities. Equality holds if and only if triangle is equilateral.

Reference:

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