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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-XVI

By Marin Chirciu-Romania

1) In acute ΔABC:

$$\sum \frac{\cos^5 A}{\cos^3 B} \geq 1 - \left(\frac{r}{R}\right)^2$$

Proposed by Marian Ursărescu – Romania

Solution: We prove.**Lemma:**

2) If $x, y, z > 0$ then:

$$\sum \frac{x^5}{y^3} \geq \sum x^2$$

Proof: Using Holder's inequality we obtain:

$$LHS = \sum \frac{x^5}{y^3} = \sum \frac{x^6}{xy^3} = \sum \frac{x^6}{xy \cdot y^2} \overset{Holder}{\geq} \frac{(\sum x^2)^3}{\sum xy \sum y^2} \overset{(1)}{\geq} \frac{(\sum x^2)^3}{\sum x^2 \sum x^2} = \sum x^2 = RHS$$

where (1) $\Leftrightarrow \sum x^2 \geq \sum xy \Leftrightarrow \sum (x-y)^2 \geq 0$.We've used Holder's inequality:

$\sum \frac{x^3}{a_1 \cdot a_2} \geq \frac{(\sum x)^3}{\sum a_1 \sum a_2}$, where $x, a_1, a_2 > 0$.Let's get back to the main problem.

Using Lemma for $x = \cos A, y = \cos B, z = \cos C$ we obtain:

$$LHS = \sum \frac{\cos^5 A}{\cos^3 B} = \sum \frac{x^5}{y^3} \overset{Lemma}{\geq} \sum x^2 = \sum \cos^2 A = \frac{6R^2 + 4Rr + r^2 - p^2}{2R^2} \overset{Gerretsen}{\geq}$$

$$\geq \frac{6R^2 + 4Rr + r^2 - (4R^2 + 4Rr + 3r^2)}{2R^2} = \frac{2R^2 - 2r^2}{2R^2} = \frac{R^2 - r^2}{R^2} = 1 - \left(\frac{r}{R}\right)^2$$

We've used Gerretsen inequality in triangle: $p^2 \leq 4R^2 + 4Rr + 3r^2$.

Equality holds if and only if the triangle is equilateral.**Remark:** The problem can be developed.

3) In acute ΔABC:

$$\sum \frac{\cos^{2n+1} A}{\cos^{2n-1} B} \geq 1 - \left(\frac{r}{R}\right)^2, n \in \mathbb{N}^*$$

Marin Chirciu

Solution: We prove. **Lemma:**

4) If $x, y, z > 0$ and $n \in \mathbb{N}^*$ then:

$$\sum \frac{x^{2n+1}}{y^{2n-1}} \geq \sum x^2$$

Proof: Using Holder's inequality we obtain:

$$LHS=\sum\frac{x^{2n+1}}{y^{2n-1}}=\sum\frac{x^{2n+2}}{xy^{2n-1}}=\sum\frac{x^{2n+2}}{xy\cdot y^{2n-2}}\overset{Holder}{\geq}\frac{(\sum x^2)^{n+1}}{\sum xy(\sum y^2)^{n-1}}\overset{(1)}{\geq}$$

$\geq\frac{(\sum x^2)^{n+1}}{\sum x^2(\sum y^2)^{n-1}}=\sum x^2=RHS$, where $(1)\Leftrightarrow\sum x^2\geq\sum xy\Leftrightarrow\sum(x-y)^2\geq 0$.

We've used Holder's inequality:

$\sum\frac{x^{n+1}}{a_1\cdot a_2\cdot\ldots\cdot a_n}\geq\frac{(\sum x)^{n+1}}{\sum a_1\sum a_2\ldots\sum a_n}$, where $x,a_1,a_2,\ldots a_n>0$

Let's get back to the main problem.

Using the Lemma for $x=\cos A, y=\cos B, z=\cos C$ we obtain:

$$LHS=\sum\frac{\cos^{2n+1}A}{\cos^{2n-1}B}=\sum\frac{x^{2n+1}}{y^{2n-1}}\overset{Lemma}{\geq}\sum x^2=\sum\cos^2 A=\frac{6R^2+4Rr+r^2-p^2}{2R^2}\geq$$

$$\overset{Gerretsen}{\geq}\frac{6R^2+4Rr+r^2-(4R^2+4Rr+3r^2)}{2R^2}=\frac{2R^2-2r^2}{2R^2}=\frac{R^2-r^2}{R^2}=1-\left(\frac{r}{R}\right)^2$$

We've used Gerretsen's inequality in triangle: $p^2\leq 4R^2+4Rr+3r^2$.

Equality holds if and only if the triangle is equilateral.

Note: For $n=2$ we obtain Inequality in triangle 2252, proposed by Marian Ursărescu, Romania, in RMM 11/2020.**Remark:** The problem can be developed.

5) In ΔABC:

$$\sum\frac{\cos^5\frac{A}{2}}{\cos^3\frac{B}{2}}\geq 2+\frac{r}{2R}$$

Marin Chirciu

Solution. We prove. **Lemma:**

6) If $x,y,z>0$ then:

$$\sum\frac{x^5}{y^3}\geq\sum x^2$$

Proof: Using Holder's inequality we obtain:

$$LHS=\sum\frac{x^5}{y^3}=\sum\frac{x^6}{xy^3}=\sum\frac{x^6}{xy\cdot y^2}\overset{Holder}{\geq}\frac{(\sum x^2)^3}{\sum xy\sum y^2}\overset{(1)}{\geq}\frac{(\sum x^2)^3}{\sum x^2\sum x^2}=\sum x^2=RHS$$

where $(1)\Leftrightarrow\sum x^2\geq\sum xy\Leftrightarrow\sum(x-y)^2\geq 0$.

We've used Holder's inequality: $\sum\frac{x^3}{a_1\cdot a_2}\geq\frac{(\sum x)^3}{\sum a_1\sum a_2}$, where $x,a_1,a_2>0$

Let's get back to the main problem.

Using the Lemma for $x=\cos\frac{A}{2}, y=\cos\frac{B}{2}, z=\cos\frac{C}{2}$ we obtain:

$$LHS=\sum\frac{\cos^5\frac{A}{2}}{\cos^3\frac{B}{2}}=\sum\frac{x^5}{y^3}\overset{Lemma}{\geq}\sum x^2=\sum\cos^2\frac{A}{2}=2+\frac{r}{2R}$$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

7) In ΔABC:

$$\sum \frac{\cos^{2n+1}\frac{A}{2}}{\cos^{2n-1}\frac{B}{2}} \geq 2+\frac{r}{2R}, n\in\mathbb{N}^*$$

Marin Chirciu

Solution: We prove **Lemma:**

8) If $x,y,z>0$ and $n\in\mathbb{N}^*$ then:

$$\sum \frac{x^{2n+1}}{y^{2n-1}} \geq \sum x^2$$

Proof: Using Holder's inequality we obtain:

$$LHS=\sum \frac{x^{2n+1}}{y^{2n-1}}=\sum \frac{x^{2n+2}}{xy^{2n-1}}=\sum \frac{x^{2n+2}}{xy\cdot y^{2n-2}} \overset{Holder}{\geq} \frac{(\sum x^2)^{n+1}}{\sum xy(y^2)^{n-1}} \overset{(1)}{\geq}$$

$\geq \frac{(\sum x^2)^{n+1}}{\sum x^2(\sum y^2)^{n-1}}=\sum x^2=RHS$, where (1) $\Leftrightarrow \sum x^2 \geq \sum xy \Leftrightarrow \sum(x-y)^2 \geq 0$

We've used Holder's inequality:

$\sum \frac{x^{n+1}}{a_1\cdot a_2\cdot\ldots\cdot a_n} \geq \frac{(\sum x)^{n+1}}{\sum a_1 \sum a_2 \ldots \sum a_n}$, where $x, a_1, a_2, \ldots, a_n > 0$.

Let's get back to the main problem.

Using the Lemma for $x=\cos\frac{A}{2}, y=\cos\frac{B}{2}, z=\cos\frac{C}{2}$ we obtain:

$$LHS=\sum \frac{\cos^{2n+1}A}{\cos^{2n-1}B}=\sum \frac{x^{2n+1}}{y^{2n-1}} \overset{Lemma}{\geq} \sum x^2=\sum \cos^2\frac{A}{2}=2+\frac{r}{2R}$$

Equality holds if and only if the triangle is equilateral.

Reference:

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