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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-XVII

By Marin Chirciu-Romania

1) *In ΔABC the following relationship holds*:

$$\sum \frac{w_b + w_c}{h_a^2} \ge \frac{2}{r}$$

By Marian Ursărescu – Romania

Solution:

Using $w_a \ge h_a$ we have $LHS = \sum \frac{w_b+w_c}{h_a^2} \ge \sum \frac{h_b+h_c}{h_a^2} \overset{(1)}{\ge} RHS$, where (1) it follows from

Lemma 1:

2) In ΔABC the following relationship holds:

$$\sum \frac{h_b + h_c}{h_a^2} \ge \frac{2}{r}$$

Proof: We have $\sum \frac{h_b+h_c}{h_a^2} \overset{(2)}{=} \frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2} \overset{(3)}{\ge} \frac{2}{r}$, where (2) it follows from

Lemma 2:

3) In ΔABC the following relationship holds:

$$\sum \frac{h_b + h_c}{h_a^2} = \frac{s^4 - 4s^2Rr - r^2(4Rr + r)^2}{4Rr^2s^2}$$

Proof: We have $\sum \frac{h_b+h_c}{h_a^2} = \sum \frac{\frac{2S}{b}+\frac{2S}{c}}{\left(\frac{2S}{a}\right)^2} = \frac{1}{2S}\sum \frac{a^2(b+c)}{bc} = \frac{\sum a^3(b+c)}{2sr\cdot abc} = \frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}$, which follows from $\sum a^3(b+c) = 2[s^4 - 4s^2Rr - r^2(4R+r)^2]$

$$\sum \frac{a^2(b+c)}{bc} = \frac{s^4 - 4s^2Rr - r^2(4R+r)^2}{2Rrs}$$

Let's get back to the main problem. Using the above Lemmas it suffices to prove that inequality 3) holds:

$$\frac{s^4 - 4s^2Rr - r^2(4R+r)^2}{4Rr^2s^2} \ge \frac{2}{r} \Leftrightarrow s^4 - 12s^2Rr - r^2(4R+r)^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow s^2(s^2 - 12Rr) \ge r^2(4R+r)^2$, which follows from Gerretsen's inequality:

$$s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$$

It remains to prove that: $\frac{r(4R+r)^2}{R+r}(16Rr - 5r^2 - 12Rr) \ge r^2(4R+r)^2 \Leftrightarrow R \ge 2r$, (Euler's inequality). Equality holds if and only if the triangle is equilateral.

Remark: The inequality can be strengthened.

4) In ΔABC the following inequality holds:

$$\sum \frac{w_b + w_c}{h_a^2} \geq \frac{1}{4r}\left(11 - \frac{6r}{R}\right) \geq \frac{2}{r}$$

Marin Chirciu

Solution: Using $w_a \geq h_a$ we have $LHS = \sum \frac{w_b+w_c}{h_a^2} \geq \sum \frac{h_b+h_c}{h_a^2} \geq \frac{2}{r} = RHS.$

We have $\sum \frac{h_b+h_c}{h_a^2} = \frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2} = \frac{s^2(s^2-4Rr)-r^2(4R+r)^2}{4Rr^2s^2} =$

$$= \frac{1}{4Rr^2}\left[(s^2 - 4Rr) - \frac{r^2(4R+r)^2}{s^2}\right] \overset{Gerretsen}{\geq} \frac{1}{4Rr^2}\left[(16Rr - 5r^2 - 4Rr) - \frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]$$

$$= \frac{1}{4Rr^2}[(12Rr - 5r^2) - r(R+r)] = \frac{11Rr - 6r^2}{4Rr^2} = \frac{11R - 6r}{4Rr} = \frac{1}{4r}\left(11 - \frac{6r}{R}\right) \overset{Euler}{\geq} \frac{2}{r}$$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

5) In ΔABC the following relationship holds:

$$\sum \frac{m_b + m_c}{h_a^2} \geq \frac{2}{r}$$

Marin Chirciu

Solution: We use $m_a \geq w_a \geq h_a$ and see above. Equality holds if and only if the triangle is equilateral.

6) In ΔABC the following inequality holds:

$$\frac{2}{r} \leq \sum \frac{r_b + r_c}{h_a^2} \leq \frac{R}{r^2}$$

Marin Chirciu

Solution: We prove: **Lemma:**

7) In ΔABC the following relationship holds:

$$\sum \frac{r_b + r_c}{h_a^2} = \frac{s^2(2R + 3r) - r(4R + r)^2}{2s^2r^2}$$

Proof: We have:

$$\sum \frac{r_b + r_c}{h_a^2} = \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\left(\frac{2S}{a}\right)^2} = \frac{1}{4S}\sum \frac{a^3}{(s-b)(s-c)} =$$

$$= \frac{1}{4sr} \cdot \frac{2[s^2(2R+3r) - r(4R+r)^2]}{sr} =$$

$=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}$, which follows from:

$$\sum\frac{a^3}{(s-b)(s-c)}=\frac{2[s^2(2R+3r)-r(4R+r)^2]}{sr}$$

$$\sum\frac{a^3}{(s-b)(s-c)}=\frac{\sum a^3(s-a)}{\prod(s-a)}=\frac{2r[s^2(2R+3r)-r(4R+r)^2]}{sr^2}=$$

$$=\frac{2[s^2(2R+3r)-r(4R+r)^2]}{sr}$$

$$\sum a^3(s-a)=2r[s^2(2R+3r)-r(4R+r)^2]$$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{h_a^2}=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}=\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\le}$$

$$\le\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=\frac{1}{2r^2}\left[(2R+3r)-\frac{2r(2R-r)}{R}\right]=$$

$$=\frac{1}{2r^2}\cdot\frac{R(2R+3r)-2r(2R-r)}{R}=\frac{2R^2-Rr+2r^2}{2Rr^2}\overset{Euler}{\le}\frac{2R^2}{2Rr^2}=\frac{R}{r^2}$$

Equality holds if and only if the triangle is equilateral.LHS inequality.
Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{h_a^2}=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}=\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\ge}$$

$$\ge\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{1}{2r^2}[(2R+3r)-(R+r)]=\frac{R+2r}{2r^2}\overset{Euler}{\ge}\frac{2}{r}$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

8) In ΔABC the following relationship holds:

$$\frac{2}{r}\le\sum\frac{h_b+h_c}{h_a^2}\le\frac{R}{r^2}$$

Marin Chirciu

Proof: We prove: **Lemma:**

9) In ΔABC the following relationship holds:

$$\sum\frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}$$

Proof: We have $\sum\frac{h_b+h_c}{h_a^2}=\sum\frac{\frac{2S}{b}+\frac{2S}{c}}{\left(\frac{2S}{a}\right)^2}=\frac{1}{2S}\sum\frac{a^2(b+c)}{bc}=\frac{\sum a^3(b+c)}{2sr\cdot abc}=\frac{s^4-4s^2Rr-r^2(4Rr+r)^2}{4Rr^2s^2}$

which follows from $\sum a^3(b+c)=2[s^4-4s^2Rr-r^2(4R+r)^2]$

$$\sum\frac{a^2(b+c)}{bc}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{2Rrs}$$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$\sum\frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}=\frac{s^2(s^2-4Rr)-r^2(4R+r)^2}{4Rr^2s^2}=$$

$$=\frac{1}{4Rr^2}\left[(s^2-4Rr)-\frac{r^2(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{1}{4Rr^2}\left[(4R^2+4Rr+3r^2-4Rr)-\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{1}{4Rr^2}\left[(4R^2+3r^2)-\frac{2r^2(2R-r)}{R}\right]=\frac{1}{4Rr^2}\cdot\frac{R(4R^2+3r^2)-2r^2(2R-r)}{R}=$$

$$=\frac{1}{4Rr^2}\cdot\frac{R(4R^2+3r^2)-2r^2(2R-r)}{R}=\frac{4R^3-Rr^2+2r^3}{4R^2r^2}\overset{Euler}{\leq}\frac{4R^3}{4R^2r^2}=\frac{R}{r^2}$$

Equality holds if and only if the triangle is equilateral.LHS inequality.Using the Lemma we obtain:

$$\sum\frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}=\frac{s^2(s^2-4Rr)-r^2(4R+r)^2}{4Rr^2s^2}=$$

$$=\frac{1}{4Rr^2}\left[(s^2-4Rr)-\frac{r^2(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\geq}\frac{1}{4Rr^2}\left[(16Rr-5r^2-4Rr)-\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]$$

$$=\frac{1}{4Rr^2}[(12Rr-5r^2)-r(R+r)]=\frac{11Rr-6r^2}{4Rr^2}=\frac{11R-6r}{4Rr}\overset{Euler}{\geq}\frac{2}{r}$$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

10) In ΔABC the following inequality holds:

$$\frac{2}{r}\left(\frac{R}{r}-1\right)\leq\sum\frac{r_b+r_c}{r_a^2}\leq\frac{4}{r}\left(\frac{R}{r}-1\right)^2$$

Marin Chirciu

Solution: We prove: **Lemma:**

11) In ΔABC the following relationship holds:

$$\sum\frac{r_b+r_c}{r_a^2}=\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}$$

Proof: We have:

$$\sum\frac{r_b+r_c}{r_a^2}=\sum\frac{\frac{S}{s-b}+\frac{S}{s-c}}{\left(\frac{S}{s-a}\right)^2}=\frac{1}{S}\sum\frac{a(s-a)^2}{(s-b)(s-c)}=\frac{\sum a(s-a)^3}{sr\cdot\prod(s-a)}=$$

$$=\frac{4Rrs^2-2r^2(4R+r)^2}{sr\cdot sr^2}=$$

$$=\frac{4Rs^2-2r(4R+r)^2}{s\cdot sr^2}=\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}$$

which follows from $\sum a(s-a)^3=4Rrs^2-2r^2(4R+r)^2$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{r_a^2}=\frac{2[2Rrs^2-r(4R+r)^2]}{s^2r^2}=\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{2}{r^2}\left[2R-\frac{2r(2R-r)}{R}\right]=\frac{4}{r^2}\cdot\frac{R^2-r(2R-r)}{R}=4\frac{R^2-2Rr+r^2}{Rr^2}=$$

$$=4\frac{(R-r)^2}{Rr^2}=\frac{4}{R}\left(\frac{R}{r}-1\right)^2$$

Equality holds if and only if the triangle is equilateral.LHS inequality.Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{r_a^2}=\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}=\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\geq\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{2}{r^2}[2R-(R+r)]=\frac{2(R-r)}{r^2}=\frac{2}{r}\left(\frac{R}{r}-1\right)$$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

12) In ΔABC the following relationship holds:

$$\frac{2}{r}\leq\sum\frac{h_b+h_c}{r_a^2}\leq\frac{R}{r^2}$$

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Solution: We prove: **Lemma:**

13) In ΔABC the following relationship holds:

$$\sum\frac{h_b+h_c}{r_a^2}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}$$

Proof: We have $\sum\frac{h_b+h_c}{r_a^2}=\sum\frac{\frac{2S}{b}+\frac{2S}{c}}{\left(\frac{S}{s-a}\right)^2}=\frac{2}{S}\sum\frac{(b+c)(s-a)^2}{bc}=\frac{2}{sr}\cdot\frac{\sum a(b+c)(s-a)^2}{abc}=$

$$=\frac{2}{sr}\cdot\frac{2r[s^2(4R-r)-r(4R+r)^2]}{4Rrs}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}$$

which follows from:

$$\sum a(b+c)(s-a)^2 = 2r[s^2(4R-r)-r(4R+r)^2]$$

Let's get back to the main problem.RHS inequality. Using the Lemma we obtain:

$$\sum \frac{h_b+h_c}{r_a^2} = \frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2} = \frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{s^2}\right] \overset{Gerretsen}{\leq}$$

$$\leq \frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right] = \frac{1}{Rr}\left[(4R-r)-\frac{2r(2R-r)}{R}\right] =$$

$$= \frac{1}{Rr}\cdot\frac{R(4R-r)-2r(2R-r)}{R} = \frac{1}{Rr}\cdot\frac{4R^2-5Rr+2r^2}{R} =$$

$$= \frac{4R^2-5Rr+2r^2}{R^2r} \overset{(1)}{\leq} \frac{R}{r^2}, \text{ where } (1) \Leftrightarrow \frac{4R^2-5Rr+2r^2}{R^2r} \leq \frac{R}{r^2} \Leftrightarrow$$

$$\Leftrightarrow R^3-4R^2r+5Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(R-r)^2 \geq 0, \text{ obviously from Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.LHS inequality .Using Lemma we obtain:

$$\sum \frac{h_b+h_c}{r_a^2} = \frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2} = \frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{s^2}\right] \overset{Gerretsen}{\geq}$$

$$\geq \frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] = \frac{1}{Rr}[(4R-r)-(R+r)] = \frac{1}{Rr}\cdot(3R-2r) =$$

$$= \frac{3R-2r}{Rr} \overset{Euler}{\geq} \frac{2}{r} = \frac{4R^2-5Rr+2r^2}{R^2r} \overset{(1)}{\leq} \frac{R}{r^2}, \text{ where } (1) \Leftrightarrow \frac{4R^2-5Rr+2r^2}{R^2r} \leq \frac{R}{r^2} \Leftrightarrow$$

$$\Leftrightarrow R^3-4R^2r+5Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(R-r)^2 \geq 0, \text{ obviously from Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

$$= \frac{2}{r^2}[2R-(R+r)] = \frac{2(R-r)}{r^2} = \frac{2}{r}\left(\frac{R}{r}-1\right)$$

Equality holds if and only if the triangle is equilateral.**Remark:** Between the sums $\sum \frac{h_b+h_c}{h_a^2}$ and $\sum \frac{r_b+r_c}{r_a^2}$ the following relationship exists:

14) In ΔABC:

$$\sum \frac{h_b+h_c}{h_a^2} \leq \sum \frac{r_b+r_c}{r_a^2}$$

Marin Chirciu

Solution: Using the above Lemmas we have the sums:

$\sum \frac{h_b+h_c}{h_a^2} = \frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}$ and $\sum \frac{r_b+r_c}{r_a^2} = \frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}$

The inequality can be written:

$$\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}\le\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}\Leftrightarrow$$
$\Leftrightarrow s^2(16R^2+4Rr-s^2)\ge r(8R-r)(4R+r)^2$, which follows from Gerretsen's inquality:
$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2$$
It remains to prove that:
$$(16Rr-5r^2)(16R^2+4Rr-4R^2-4Rr-3r^2)\ge r(8R-r)(4R+r)^2\Leftrightarrow$$
$\Leftrightarrow(16R-5r)(12R^2-3r^2)\ge(8R-r)(4R+r)^2\Leftrightarrow 16R^3-27R^2r-12Rr^2+4r^3\ge 0$
$\Leftrightarrow(R-2r)(16R^2+5Rr-2r^2)\ge 0$, obviously from Euler's inequality $R\ge 2r$.
Equality holds if and only if the triangle is equilateral.**Remark:** Between the sums $\sum\frac{h_b+h_c}{r_a^2}$ and $\sum\frac{r_b+r_c}{h_a^2}$ the following relationship exists:

15) In ΔABC:

$$\sum\frac{h_b+h_c}{r_a^2}\le\sum\frac{r_b+r_c}{r_a^2}$$

Marin Chirciu

Solution: Using the above Lemmas we have the sums:

$\sum\frac{h_b+h_c}{r_a^2}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}$ and $\sum\frac{r_b+r_c}{h_a^2}=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}$

The inequality can be written:
$$\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}\le\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}\Leftrightarrow$$
$$\Leftrightarrow s^2(2R^2-5Rr+2r^2)\ge(Rr-2r^2)(4R+r)^2\Leftrightarrow$$
$$\Leftrightarrow s^2(R-2r)(2R-r)\ge r(R-2r)(4R+r)^2\Leftrightarrow$$
$$\Leftrightarrow(R-2r)[s^2(2R-r)-r(4R+r)^2]\ge 0$$
Because $(R-2r)\ge 0$, from Euler's inequality $R\ge 2r$ and

$[s^2(2R-r)-r(4R+r)^2]\ge 0\Leftrightarrow s^2(2R-r)\ge r(4R+r)^2$, which follows from

Gerretsen's inequality: $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$. It remains to prove that:

$\frac{r(4R+r)^2}{R+r}(2R-r)\ge r(4R+r)^2\Leftrightarrow 2R-r\ge R+r\Leftrightarrow R\ge 2r$, (Euler's inequality).

Equality holds if and only if the triangle is equilateral.

Reference:

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