R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-XVIII

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{a^2} \ge \frac{9}{4F}$$

Proposed by Marian Ursărescu – Romania

Solution: We prove: **Lemma:**

2) In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3}$$

Proof: Using $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ we obtain: $\sum\frac{\cot\frac{A}{2}}{a^2} = \sum\frac{\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}}{a^2} = \sum\frac{s(s-a)}{a^2\sqrt{s(s-a)(s-b)(s-c)}} =$

$$= \sum\frac{s(s-a)}{a^2S} = \frac{s}{S}\sum\frac{s-a}{a^2} =$$

$$= \frac{s}{sr}\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3}, \text{ true from:}$$

$$\sum\frac{s-a}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^2}$$

Let's get back to the main problem. Using the Lemma we obtain:

$$LHS = \sum\frac{\cot\frac{A}{2}}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} \overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge} \frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} =$$

$$= \frac{(16R-5r)(4R-3r)+r(4R+r)}{16sR^2r} = \frac{64R^2-64Rr+16r^2}{16sR^2r} = \frac{16(4R^2-4Rr+r^2)}{16sR^2r} =$$

$$= \frac{(2R-r)^2}{sR^2r} = \frac{1}{sr}\left(2-\frac{r}{R}\right)^2 = \frac{1}{F}\left(2-\frac{r}{R}\right)^2 \overset{Euler}{\ge} \frac{9}{4R} = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: The inequality can be strengthened.

3) In ΔABC the following relationship holds:

$$\sum\frac{\cot\frac{A}{2}}{a^2} \ge \frac{1}{F}\left(2-\frac{r}{R}\right)^2$$

Marin Chirciu

Solution: Using the Lemma we obtain:

$$LHS = \sum \frac{\cot\frac{A}{2}}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} \overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq} \frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} =$$

$$= \frac{(16R-5r)(4R-3r)+r(4R+r)}{16sR^2r} = \frac{64R^2-64Rr+16r^2}{16sR^2r} = \frac{16(4R^2-4Rr+r^2)}{16sR^2r} =$$

$$= \frac{(2R-r)^2}{sR^2r} = \frac{1}{sr}\left(2-\frac{r}{R}\right)^2 = \frac{1}{F}\left(2-\frac{r}{R}\right)^2 = RHS$$

Remark: Inequality 3) is stronger than inequality 1)

4) In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{a^2} \geq \frac{1}{F}\left(2-\frac{r}{R}\right)^2 \geq \frac{9}{4F}$$

Solution: See inequality 3) and $\frac{1}{F}\left(2-\frac{r}{R}\right)^2 \geq \frac{9}{4F} \Leftrightarrow R \geq 2r$, (Euler's inequality).

Equality holds if and only if the triangle is equilateral.**Remark:** Let's find an inequality of opposite sense.

5) In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{a^2} \leq \frac{1}{F}\left(\frac{R^2}{r^2}-\frac{R}{r}+\frac{r^2}{R^2}\right)$$

Marin Chirciu

Solution: Using the Lemma we obtain:

$$LHS = \sum \frac{\cot\frac{A}{2}}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} \overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\leq} \frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} =$$

$$= \frac{(4R^2+4Rr+3r^2)(4R^2-8Rr+5r^2)+r^3(4R+r)}{16sR^2r^3} = \frac{16R^4-16R^3r+16r^4}{16sR^2r^3} =$$

$$= \frac{R^4-R^3r+r^4}{sR^2r^3} = \frac{1}{sr}\cdot\frac{R^4-R^3r+r^4}{R^2r^2} = \frac{1}{F}\left(\frac{R^2}{r^2}-\frac{R}{r}+\frac{r^2}{R^2}\right) = RHS$$

Equality holds if and only if the triangle is equilateral.**Remark:** We can write the double inequality:

6) In ΔABC the following relationship holds:

$$\frac{1}{F}\left(2-\frac{r}{R}\right)^2 \leq \sum \frac{\cot\frac{A}{2}}{a^2} \leq \frac{1}{F}\left(\frac{R^2}{r^2}-\frac{R}{r}+\frac{r^2}{R^2}\right)$$

Marin Chirciu

Solution: We prove: **Lemma:**

7) In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3}$$

Proof: Using $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ we obtain:

$$\sum \frac{\cot\frac{A}{2}}{a^2} = \sum \frac{\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}}{a^2} = \sum \frac{s(s-a)}{a^2\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$= \sum \frac{s(s-a)}{a^2 S} = \frac{s}{S}\sum \frac{s-a}{a^2} =$$

$$= \frac{s}{sr}\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3}, \text{ true from:}$$

$$\sum \frac{s-a}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^2}$$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$E = \sum \frac{\cot\frac{A}{2}}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} \overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq} \frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} =$$

$$= \frac{(4R^2+4Rr+3r^2)(4R^2-8Rr+5r^2)+r^3(4R+r)}{16sR^2r^3} = \frac{16R^4-16R^3r+16r^4}{16sR^2r^3} =$$

$$= \frac{R^4-R^3r+r^4}{sR^2r^3} = \frac{1}{sr}\cdot\frac{R^4-R^3r+r^4}{R^2r^2} = \frac{1}{F}\left(\frac{R^2}{r^2}-\frac{R}{r}+\frac{r^2}{R^2}\right) = RHS$$

Equality holds if and only if the triangle is equilateral.LHS inequality. Using the Lemma we obtain:

$$E = \sum \frac{\cot\frac{A}{2}}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} \overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq} \frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r)}{16sR^2r^3} =$$

$$= \frac{(16R-5r)(4R-3r)+r(4R+r)}{16sR^2r} = \frac{64R^2-64Rr+16r^2}{16sR^2r} = \frac{16(4R^2-4Rr+r^2)}{16sR^2r} =$$

$$= \frac{(2R-r)^2}{sR^2r} = \frac{1}{sr}\left(2-\frac{r}{R}\right)^2 = \frac{1}{F}\left(2-\frac{r}{R}\right)^2 = LHS$$

Equality holds if and only if the triangle is equilateral.

Note: LHS inequality strengthen the proposed problem by Marian Ursărescu in RMM 1/2021:

In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{a^2} \geq \frac{9}{4F}$$

Marian Ursărescu

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro