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ABOUT NAGEL'S AND GERGONNE'S CEVIAN-(VII)

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In ΔABC the following relationship holds:

$$s_a = \frac{2bc}{b^2+c^2}\ (and\ analogs)$$

$$m_a - s_a = \frac{m_a(b-c)^2}{b^2+c^2} \le \frac{1}{2}|b-c|.$$

If $b = c$ we have equality.

If $b \ne c \Rightarrow \frac{m_a(b-c)^2}{b^2+c^2} < \frac{1}{2}|b-c| \Leftrightarrow \frac{m_a|b-c|}{b^2+c^2} < \frac{1}{2} \Leftrightarrow 2m_a|b-c| < b^2+c^2$

$\Leftrightarrow 2m_a|b-c| < |b^2-c^2|$ true from $|b^2-c^2| < b^2+c^2$.

So, we have a new inequality:

$$\frac{1}{2}|b-c| \ge m_a - s_a\ (and\ analogs);(1)$$

$$\frac{1}{2}\sum_{cyc}|b-c| = \max\{a,b,c\} - \min\{a,b,c\} \Rightarrow$$

$$\max\{a,b,c\} - \min\{a,b,c\} \ge \sum_{cyc}(m_a - s_a);(2)$$

$$\text{But}\begin{cases}|b-c| \ge n_a - g_a\\ \frac{1}{2}|b-c| \ge m_a - s_a\end{cases} \Rightarrow$$

$$\frac{3}{2}|b-c| \ge n_a + m_a - g_a - s_a\ (and\ analogs);(3)$$

Adding these up relations, we get:

$$\max\{a,b,c\} - \min\{a,b,c\} \ge \frac{1}{3}\cdot\sum_{cyc}(n_a + m_a - g_a - s_s);(4)$$

$$\frac{3}{2}|b-c| \ge n_a + m_a - g_a - s_a; n_a + g_a \ge 2m_a \Rightarrow n_a \ge 2m_a - g_a$$

$$\frac{3}{2}|b-c| \ge 2m_a - g_a + m_a - g_a - s_a = 3m_a - 2g_a - s_a$$

$$\Rightarrow \frac{3}{2}|b-c| \ge \frac{3}{2}m_a - 2g_a - s_a\ (and\ analogs);(5)$$

Adding these up relations, we get:

$$\max\{a,b,c\} - \min\{a,b,c\} \ge \frac{1}{3}\cdot\sum_{cyc}(3m_a - 2g_a - s_a);(6)$$

$$\max\{a,b,c\}-\min\{a,b,c\}\geq \frac{1}{3}\cdot(n_a+n_b+n_c)+\frac{1}{3}\cdot\sum_{cyc}(m_a-g_a-s_a)$$

But $n_a+n_b+n_c\geq s\sqrt{4-\frac{2r}{R}}$ then,

$$\max\{a,b,c\}-\min\{a,b,c\}\geq \frac{1}{3}\cdot s\sqrt{4-\frac{2r}{R}}+\frac{1}{3}\cdot\sum_{cyc}(m_a-g_a-s_a)$$

So, it follows that:

$$\mathbf{3(\max\{a,b,c\}-\min\{a,b,c\})\geq s\sqrt{4-\frac{2r}{R}}+\sum_{cyc}(m_a-g_a-s_a)\,;(7)}$$

$$s^2=n_a^2+2r_ah_a\Rightarrow\frac{s^2}{h_a^2}=\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a};\ a\cdot h_a=2sr\Rightarrow\frac{a}{2r}=\frac{s}{h_a}$$

$$\Rightarrow\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a}(and\ analogs)$$

$$r_br_c=s(s-a)=\frac{(a+b+c)(b+c-a)}{4}=\frac{(b+c)^2-a^2}{4}$$

$$a^2=(b+c)^2-4r_br_c;\ \frac{a^2}{4r^2}=\frac{(b+c)^2}{4r^2}-\frac{r_br_c}{r^2}$$

$$\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a}=\frac{(b+c)^2}{4r^2}-\frac{r_br_c}{r^2}$$

$$\frac{r}{2R}\cdot\frac{r_a}{h_a}=\frac{r_a-r}{4R}=\sin^2\frac{A}{2}\Rightarrow\frac{r_a}{h_a}=\frac{r_a-r}{2r}\ (and\ analogs)$$

$$bc=r_br_c+rr_a;\ \frac{(b+c)^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{r_a-r}{r}+\frac{r_br_c}{r^2}$$

$$\frac{(b+c)^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{bc-r^2}{r^2}\Rightarrow 1+\frac{(b+c)^2-4bc}{4r^2}=\frac{n_a^2}{h_a^2}$$

So, it follows that:

$$\mathbf{\frac{n_a^2}{h_a^2}=1+\frac{(b-c)^2}{4r^2}\ (and\ analogs);(8)}$$

$$\frac{(b-c)^2}{4}\geq\frac{(n_a+m_a-g_a-s_a)^2}{9}\Bigg|\cdot\left(\frac{1}{r^2}+1\right)\Rightarrow$$

$$1+\frac{(b-c)^2}{4r^2}\geq\frac{9r^2+(n_a+m_a-g_a-s_a)^2}{9r^2}$$

$$\frac{n_a^2}{h_a^2}\geq\frac{9r^2+(n_a+m_a-g_a-s_a)^2}{9r^2}$$

So, it follows that:

$$\frac{n_a}{h_a}\geq\frac{\sqrt{9r^2+(n_a+m_a-g_a-s_a)^2}}{3r};\ \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Rightarrow$$

$$\mathbf{3 \geq \sum_{cyc} \frac{\sqrt{9r^2 + (n_a + m_a - g_a - s_a)^2}}{n_a}; (9)}$$

$$\frac{a^2}{4r^2} = \frac{n_a^2}{h_a^2} + \frac{2r_a}{h_a} \geq \frac{9r^2 + (n_a + m_a - g_a - s_a)^2}{9r^2} + \frac{r_a - r}{r}$$

$$\frac{a^2}{4r^2} \geq \frac{9r^2 + (n_a + m_a - g_a - s_a)^2 + 9rr_a - 9r^2}{9r^2}$$

$$\frac{a^2}{4r^2} \geq \frac{9rr_a + (m_a + n_a - g_a - s_a)^2}{9r^2}$$

$$\frac{9r^2}{4r^2} \geq \frac{9rr_a + (m_a + n_a - g_a - s_a)^2}{a^2}$$

$$\mathbf{\frac{3}{2} \geq \frac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{a} \ (and\ analogs); (10)}$$

Summing, we get:

$$\mathbf{\frac{9}{2} \geq \sum_{cyc} \frac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{a}; (11)}$$

$$\frac{3}{2}a \geq \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2} \Rightarrow$$

$$\frac{3}{2}(a + b + c) \geq \sum_{cyc} \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2}$$

$$\mathbf{3s \geq \sum_{cyc} \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2}; (12)}$$

$$s^2 = n_a^2 + 2r_a h_a \Rightarrow 2r_a h_a = s^2 - n_a^2 = (s + n_a)(s + n_a)$$

$$s - n_a = \frac{2r_a h_a}{s + n_a} \Rightarrow s = n_a + \frac{2r_a h_a}{s + n_a} \ (and\ analogs) \Rightarrow$$

$$3s = n_a + n_b + n_c + \sum_{cyc} \frac{2r_a h_a}{s + n_a}$$

So, it follows that:

$$\mathbf{n_a + n_b + n_c + \sum_{cyc} \frac{2r_a h_a}{s + n_a} \geq \sum_{cyc} \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2}; (13)}$$

$$\begin{cases} \dfrac{s}{h_a} = \dfrac{a}{2r} = \dfrac{n_a}{h_a} + \dfrac{2r_a}{s + n_a} \\ \dfrac{a}{2r} \geq \dfrac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{3r} \end{cases} \Rightarrow$$

$$\mathbf{\frac{n_a}{h_a} + \frac{r_a}{s + n_a} \geq \frac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{3r}; (14)}$$

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2};\sin^2\frac{A}{2}+\cos^2\frac{A}{2}=1;\tan\frac{A}{2}=\frac{r_a}{s}$$

$$\sin A=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}\cdot\frac{\frac{1}{\cos^2\frac{A}{2}}}{\frac{1}{\cos^2\frac{A}{2}}}=\frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\sin A=\frac{2sr_a}{s^2+r_a^2};s^2=n_a^2+2r_ah_a$$

$$\frac{1}{\sin A}=\frac{n_a^2+r_a^2+2r_ah_a}{2sr_a}\ge\frac{2n_ar_a+2r_ah_a}{2sr_a}=\frac{n_a+h_a}{s}$$

So, we have:

$$\frac{1}{\sin A}\ge\frac{n_a+h_a}{s}\ (and\ analogs);2F=bc\cdot\sin A$$

$$\Rightarrow bc=\frac{2F}{\sin A}\ge\frac{2F(n_a+h_a)}{s}\ge 2r(n_a+h_a)$$

$$bc=2Rh_a\ge 2r(n_a+h_a)\Rightarrow\frac{R}{r}\ge\frac{n_a+h_a}{h_a}$$

$$\Rightarrow\frac{R-r}{r}\ge\frac{\sqrt{9r^2+(m_a+n_a-g_a-s_a)^2}}{3r}$$

$$\mathbf{3(R-r)\ge\sqrt{9r^2+(m_a+n_a-g_a-s_a)^2};(15)}$$

$$9R^2-18Rr+9r^2\ge 9r^2+(n_a+m_a-g_a-s_a)^2$$

$$9R(R-2r)\ge(n_a+m_a-g_a-s_a)^2$$

So, we get:

$$\mathbf{9R(R-2r)\ge(n_a+m_a-g_a-s_a)^2;(16)}$$

Now, using : $m_a-h_a\ge\frac{(b-c)^2}{2a}$ $(and\ analogs)$

$$\frac{a(m_a-h_a)}{2r^2}\ge\frac{(b-c)^2}{4r^2};\frac{(b-c)^2}{4r^2}=\frac{n_a^2}{h_a^2}-1\ (and\ analogs)$$

$$\frac{a(m_a-h_a)}{2r^2}\ge\frac{n_a^2-h_a^2}{h_a^2}\Rightarrow\frac{h_a^2}{2r^2}(m_a-h_a)\ge\frac{n_a^2-h_a^2}{a}$$

$$\frac{h_a}{2r^2}(m_a-h_a)\ge\frac{n_a^2-h_a^2}{2F}=\frac{n_a^2-h_a^2}{2sr}$$

$$\frac{s}{r}(m_a-h_a)\ge\frac{(n_a-h_a)(n_a+h_a)}{h_a}$$

$$\frac{s}{r}\cdot\frac{m_a-h_a}{n_a+h_a}\ge\frac{n_a-h_a}{h_a}=\frac{n_a}{h_a}-1\ \Rightarrow\frac{s}{r}\cdot\frac{m_a-h_a}{n_a+h_a}\ge\frac{n_a}{h_a}-1$$

So, we get:

$$\frac{s}{r}\cdot\frac{m_a-h_a}{n_a+h_a}\ge\frac{\sqrt{9r^2+(m_a+n_a-g_a-s_a)^2}-3r}{3r}$$

$$\frac{m_a-h_a}{n_a+h_a}\ge\frac{\sqrt{9r^2+(m_a+n_a-g_a-s_a)^2}-3r}{3s};(17)$$

$$\sum_{cyc}\frac{m_a-h_a}{n_a+h_a}\ge\sum_{cyc}\frac{\sqrt{9r^2+(m_a+n_a-g_a-s_a)^2}-3r}{3s};(18)$$

$$n_a^2=s^2-2r_ah_a;\ \frac{n_a^2}{h_a}=\frac{s^2}{h_a}-2r_a\Rightarrow\sum_{cyc}\frac{n_a^2}{h_a}=\frac{s^2}{r}-2(4R+r)$$

$$\sum_{cyc}\frac{n_a^2}{h_a}=\frac{s^2-2r(4R+r)}{r};\frac{n_a}{h_a}=\frac{n_a}{\sqrt{h_a}}\cdot\frac{1}{\sqrt{h_a}}$$

$$\sum_{cyc}\frac{n_a}{h_a}\overset{CBS}{\le}\sqrt{\left(\frac{n_a^2}{h_a}+\frac{n_b^2}{h_b}+\frac{n_c^2}{h_c}\right)\left(\frac{1}{h_a}+\frac{1}{h_c}+\frac{1}{h_c}\right)}$$

$$\sum_{cyc}\frac{n_a}{h_a}\ge\frac{1}{3r}\cdot\sum_{cyc}\sqrt{9r^2+(n_a+m_a-g_a-s_a)^2};(19)$$

$$\sqrt{s^2-2r(4R+r)}\ge\frac{1}{3}\cdot\sum_{cyc}\sqrt{9r^2+(n_a+m_a-g_a-s_a)^2};(20)$$

But: $s^2\le 4R^2+4Rr+3r^2\ (Gerretsen)\Rightarrow$

$$\sum_{cyc}\frac{n_a}{h_a}\le\sqrt{\frac{4R^2+4Rr+3r^2-8Rr-2r^2}{r^2}}=\sqrt{\frac{(2R-r)^2}{r^2}}$$

$$\sum_{cyc}\frac{n_a}{h_a}\le\frac{2R-r}{r}\Rightarrow 3(2R-r)\ge\sum_{cyc}\sqrt{9r^2+(n_a+m_a-g_a-s_a)^2};(21)$$

We known that: $n_ag_a\ge m_aw_a$, $n_a+g_a\ge 2m_a$ and $\frac{n_ag_a(n_a+g_a)}{2w_a}\ge m_a^2$.

But: $m_a^2=r_br_c+\frac{1}{4}(b-c)^2\Rightarrow\frac{n_ag_a(n_a+g_a)}{2w_a}-r_br_c\ge\frac{1}{4}(b-c)^2$

$$\sqrt{\frac{n_ag_a(n_a+g_a)}{2w_a}-r_br_c}\ge\frac{1}{2}|b-c|$$

Summing, we get:

$$\sum_{cyc}\sqrt{\frac{n_ag_a(n_a+g_a)}{2w_a}-r_br_c}\ge\max\{a,b,c\}-\min\{a,b,c\};(22)$$

$$\sum_{cyc}\sqrt{\frac{n_ag_a(n_a+g_a)}{2w_a}-r_br_c}\ge\sum_{cyc}(m_a-s_a);(23)$$

$$\sum_{cyc}\sqrt{\frac{n_a g_a(n_a+g_a)}{2w_a}-r_b r_c} \ge \frac{1}{3}\cdot\sum_{cyc}(n_a+m_a-g_a-s_a);(24)$$

$$\sum_{cyc}\sqrt{\frac{n_a g_a(n_a+g_a)}{2w_a}-r_b r_c} \ge \frac{1}{3}\cdot\sum_{cyc}(3m_a-2g_a-s_a);(25)$$

$$\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}=\frac{2R-r}{r}\ge\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}$$

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge\frac{1}{3r}\cdot\sum_{cyc}\sqrt{9r^2+(n_a+m_a-g_a-s_a)^2};(26)$$

Reference:

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