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ABOUT NAGEL'S AND GERGONNE'S CEVIANS-VIII

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In ΔABC the following relationship holds:

$$n_a g_a \ge r_b r_c, \qquad b^2 + c^2 = n_a^2 + g_a^2 + 2rr_a, \qquad 2bc = 2r_b r_c + 2rr_a$$

$$4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c$$

$$g_a^2 \ge \frac{(r_b r_c)^2}{n_a^2} \Rightarrow n_a^2 + g_a^2 + 2r_b r_c \ge n_a^2 + 2r_b r_c + \left(\frac{r_b r_c}{n_a}\right)^2$$

$$\Rightarrow 4m_a^2 \ge n_a^2 + \left(\frac{r_b r_c}{n_a}\right)^2 + 2n_a \cdot \frac{r_b r_c}{n_a} \Rightarrow 2m_a \ge n_a + \frac{r_b r_c}{n_a}; (1)$$

Adding these up relations, it follows:

$$2(m_a + m_b + m_c) \ge \sum_{cyc} n_a + \sum_{cyc} \frac{r_b r_c}{n_a}; (2)$$

$$8m_a m_b m_c \ge \prod_{cyc} \left(n_a + \frac{r_b r_c}{n_a}\right); (3)$$

$$2m_a - n_a \ge \frac{r_b r_c}{n_a} \Rightarrow \prod_{cyc} (2m_a - n_a) \ge \frac{(r_a r_b r_c)^2}{n_a n_c n_c}; (4)$$

$$n_a(2m_a - n_a) \ge r_b r_c \Rightarrow 2m_a n_a \ge n_a^2 + r_b r_c; (5)$$

$$\because r_a r_b + r_b r_c + r_c r_a = s^2 \Rightarrow 2\sum_{cyc} m_a n_a = s^2 + \sum_{cyc} n_a^2; (6)$$

$$\because n_a^2 + n_b^2 + n_c^2 \ge n_a n_b + n_b n_c + n_c n_a \Rightarrow 2\sum_{cyc} m_a n_a \ge s^2 + \sum_{cyc} n_a n_b; (7)$$

But $n_a n_b + n_b n_c + n_c n_a \ge s^2 \Rightarrow \sum_{cyc} m_a n_a \ge s^2; (8)$

$$\because n_a^2 + n_b^2 + n_c^2 = \frac{s^2(3R - r) - r(4R + r)^2}{R} \Rightarrow$$

$$2\sum_{cyc} m_a n_a \ge s^2 + \frac{s^2(3R - r) - r(4R + r)^2}{R} \Rightarrow$$

$$\sum_{cyc} m_a n_a \ge \frac{s^2(4R - r) - r(4R + r)^2}{2R}; (9)$$

$$8m_a n_a \ge 4n_a^2 + 4r_b r_c \Rightarrow 8m_a n_a - 2r_b r_c \ge 4n_a^2 + 2r_b r_c$$

$$\because (b-c)^2 = n_a^2 + g_a^2 - 2r_br_c$$

$$4(n_a-m_a)^2 = 4m_a^2 + 4n_a^2 - 8n_am_a = n_a^2 + g_a^2 + 2r_br_c + 4n_a^2 - 8n_am_a$$

$$\Rightarrow 8m_an_a - 2r_br_c \geq 4n_a^2 + 2r_br_c$$

$$\Rightarrow n_a^2 + g_a^2 + 8m_an_a - 2r_br_c \geq n_a^2 + g_a^2 4n_a^2 + 2r_br_c \Rightarrow (b-c)^2 \geq 4(n_a-m_a)^2$$

$$\frac{1}{4}(b-c)^2 \geq (n_a-m_a)^2 \Rightarrow \frac{1}{2}|b-c| \geq n_a - m_a$$

So, we get a new inequality:

$$\frac{1}{2}|b-c| \geq n_a - m_a; (10)$$

Adding, it follows $\frac{1}{2}\sum_{cyc}|b-c| \geq \sum_{cyc}(n_a-m_a)$

But $\frac{1}{2}\sum_{cyc}|b-c| = \max\{a,b,c\} - \min\{a,b,c\}$ hence,

$$\max\{a,b,c\} - \min\{a,b,c\} \geq \sum_{cyc}(n_a-m_a)\,; (11)$$

$$\because \begin{cases} \frac{1}{2}|b-c| \geq m_a - s_a \\ \frac{1}{2}|b-c| \geq n_a - m_a \end{cases} \Rightarrow |b-c| \geq n_a - s_a; (12)$$

$$\Rightarrow \sum_{cyc}|b-c| \geq \sum_{cyc}(n_a-s_a)\,; (13)$$

$$\Rightarrow \max\{a,b,c\} - \min\{a,b,c\} \geq \frac{1}{2}\sum_{cyc}(n_a-s_a)\,; (14)$$

$$\because \begin{cases} |b-c| \geq n_a - g_a \\ |b-c| \geq n_a - s_a \end{cases} \Rightarrow 2|b-c| \geq 2n_a - g_a - s_a$$

$$\Rightarrow |b-c| \geq \frac{1}{2}(2n_a - g_a - s_a); (15)$$

$$\Rightarrow \sum_{cyc}|b-c| \geq \frac{1}{2}\sum_{cyc}(2n_a - s_a - g_a)\,; (16)$$

$$\Rightarrow \max\{a,b,c\} - \min\{a,b,c\} \geq \frac{1}{4}\sum_{cyc}(2n_a - g_a - s_a)\,; (17)$$

$$\because n_a g_a \ge m_a w_a \Rightarrow n_a \ge \frac{m_a w_a}{g_a} \Rightarrow \frac{1}{2}|b-c| \ge \frac{m_a w_a}{g_a} - m_a$$

$$\Rightarrow \frac{1}{2}|b-c| \ge \frac{m_a(w_a - g_a)}{g_a} \Rightarrow \frac{1}{2}\cdot\frac{|b-c|}{m_a} \ge \frac{w_a}{g_a} - 1; (18)$$

$$\frac{1}{2}\sum_{cyc}\frac{|b-c|}{m_a} \ge \frac{w_a}{g_a} + \frac{w_b}{g_b} + \frac{w_c}{g_c} - 3; (19)$$

$$|b-c| \ge \frac{m_a w_a}{g_a} - s_a; (20)$$

$$\max\{a,b,c\} - \min\{a,b,c\} \ge \frac{1}{2}\sum_{cyc}\left(\frac{m_a w_a}{g_a} - s_a\right); (21)$$

$$\because \frac{n_a^2}{h_a^2} = 1 + \frac{(b-c)^2}{4R^2}$$

$$\frac{1}{2}|b-c| \ge n_a - m_a \Rightarrow \frac{|b-c|}{2r} \ge \frac{n_a - m_a}{r} \Rightarrow \frac{(b-c)^2}{4r^2} \ge \frac{(n_a - m_a)^2}{r^2} \Rightarrow$$

$$\frac{n_a^2}{h_a^2} \ge \frac{r^2 + (n_a - m_a)^2}{r^2} \Rightarrow \frac{n_a}{m_a} \ge \frac{\sqrt{(n_a - m_a)^2 + r^2}}{r}; (22)$$

$$\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \ge \sum_{cyc}\frac{\sqrt{(n_a - m_a)^2 + r^2}}{r}; (23)$$

$$\because \frac{r}{h_a} \ge \frac{\sqrt{(n_a - m_a)^2 + r^2}}{n_a^2}, \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \Rightarrow$$

$$1 \ge \sum_{cyc}\frac{\sqrt{(n_a - m_a)^2 + r^2}}{n_a^2}; (24)$$

From $|b-c| \ge n_a - s_a \Rightarrow (b-c)^2 \ge (n_a - s_a)^2 = n_a^2 + s_a^2 - 2n_a s_a$

$n_a^2 + g_a^2 - 2r_b r_c = n_a^2 + s_a^2 - 2n_a s_a \Rightarrow s_a(2n_a - s_a) \ge 2r_b r_c - g_a^2 \Rightarrow$

$$s_a(2n_a - s_a) \ge 2r_b r_c - g_a^2; (25)$$

$$\because s_a \ge \frac{2r_b r_c - g_a^2}{2n_a - s_a} \Rightarrow s_a + s_b + s_c \ge \sum_{cyc}\frac{2r_b r_c - g_a^2}{2n_a - s_a}; (26)$$

$$\because r_b r_c \ge w_a^2 \Leftrightarrow s(s-a) \ge w_a^2 \Rightarrow$$

$$s_a + s_b + s_c \ge \sum_{cyc} \frac{2w_a^2 - g_a^2}{2n_a - s_a}; (27)$$

$$s_a s_b s_c \ge \prod_{cyc} \frac{2r_b r_c - g_a^2}{2n_a - s_a}; (28)$$

$$s_a s_b s_c \ge \prod_{cyc} \frac{2w_a^2 - g_a^2}{2n_a - s_a}; (29)$$

$$\because 2n_a - s_a \ge \frac{2r_b r_c - g_a^2}{s_a} \Rightarrow 2n_a \ge s_a + \frac{2r_b r_c - g_a^2}{s_a}; (30)$$

$$\because 2n_a \ge s_a + \frac{2r_b r_c - g_a^2}{s_a}, \quad g_a^2 = (s-a)^2 + 2rh_a$$

$$2r_b r_c - g_a^2 = 2s(s-a) - g_a^2 = 2s(s-a) - (s-a)^2 - 2rh_a$$

$$2r_b r_c - g_a^2 = (s-a)(2s - s + a) - 2rh_a$$

$$2r_b r_c - g_a^2 = (s-a)(s+a) - 2rh_a = s^2 - a^2 - 2rh_a$$

$$s^2 = n_a^2 + 2r_a h_a \Rightarrow 2r_b r_c - g_a^2 = n_a^2 - a^2 + 2h_a - a^2 - 2rh_a$$

$$\Rightarrow 2n_a \ge s_a + \frac{n_a^2 - a^2 + 2h_a(r_a - r)}{s_a}; (31)$$

$$2\sum_{cyc} n_a \ge \sum_{cyc} s_a + \sum_{cyc} \frac{n_a^2 - a^2 + 2h_a(r_a - r)}{s_a}; (32)$$

$$\because s^2 = n_a^2 + 2r_a h_a \Rightarrow s^2 - n_a^2 = 2r_a h_a \Rightarrow (s + n_a)(s - n_a) = 2r_a h_a$$

$$\Rightarrow \frac{s - n_a}{h_a} = \frac{2r_a}{s + n_a} \Rightarrow \frac{s}{h_a} = \frac{n_a}{h_a} + \frac{2r_a}{s + n_a}, \frac{s}{h_a} = \frac{a}{2r} \Rightarrow$$

$$\frac{a}{2r} = \frac{n_a}{h_a} + \frac{2r_a}{s + n_a}$$

$$\frac{n_a}{h_a} \ge \frac{\sqrt{(n_a - m_a)^2 + r^2}}{r} \Rightarrow \frac{a - \sqrt{(n_a - m_a)^2 + r^2}}{2r} \ge \frac{2r_a}{s + n_a}$$

$$a - 2\sqrt{(n_a - m_a)^2 + r^2} \ge \frac{4rr_a}{s + n_a} = \frac{4(s-b)(s-c)}{s + n_a}$$

$$a \ge 2\sqrt{(n_a - m_a)^2 + r^2} + \frac{4rr_a}{s + n_a} = \frac{4(s-b)(s-c)}{s + n_a}; (33)$$

$$s\ge\sum_{cyc}\left(\sqrt{(n_a-m_a)^2+r^2}+\frac{2(s-b)(s-c)}{s+n_a}\right);(34)$$

$$\because\frac{s}{s-a}=\frac{h_a}{h_a-2r}=\frac{r_a}{r}\Rightarrow\frac{s-a}{h_a-2r}=\frac{s}{h_a},\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r},\frac{s-a}{h_a-2r}=\frac{a}{2r}$$

$$\frac{s-a}{h_a-2r}\ge\frac{\sqrt{(n_a-m_a)^2+r^2}}{r}+\frac{2r_a}{s+n_a};(35)$$

$$\Rightarrow\frac{s}{r}=\sum_{cyc}\frac{s-a}{h_a-2r}\ge\sum_{cyc}\frac{\sqrt{(n_a-m_a)^2+r^2}}{r}+\frac{2r_a}{s+n_a};(36)$$

$$\because\frac{s^2}{h_a^2}=\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a}\Rightarrow\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a}$$

$$\frac{r}{2R}\cdot\frac{r_a}{ha}=\frac{r_a-r}{4R}=\sin^2\frac{A}{2}\Rightarrow\frac{r_a}{h_a}=\frac{r_a-r}{2r}$$

$$\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{r_a-r}{r}\ge\frac{(n_a-m_a)^2+r^2}{r^2}+\frac{r_a-r}{r}$$

$$\frac{a^2}{4R^2}\ge\frac{(n_a-m_a)^2}{r^2}+1-1+\frac{r_a}{r}=\frac{(n_a-m_a)^2+(s-b)(s-c)}{r^2}$$

$$\Rightarrow\frac{a^2}{4}\ge(n_a-m_a)^2+(s-b)(s-c),rr_a=(s-b)(s-c)\Rightarrow$$

$$\frac{a}{2}\ge\sqrt{(n_a-m_a)^2+(s-b)(s-c)};(37)$$

$$s\ge\sum_{cyc}\sqrt{(n_a-m_a)^2+(s-b)(s-c)};(38)$$

$$2\sum_{cyc}m_an_a\ge s^2+\sum_{cyc}n_a^2$$

$$2\sum_{cyc}m_an_a\ge\left[\sum_{cyc}\sqrt{(n_a-m_a)^2+(s-b)(s-c)}\right]^2+\sum_{cyc}n_a^2;(39)$$

Reference:

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