IMO TYPE INEQUALITIES WITH FIBONACCI AND LUCAS NUMBERS

D.M. BĂTINEŢU-GIURGIU, MIHÁLY BENCZE, DANIEL SITARU, NECULAI STANCIU

Abstract. In this paper we present some IMO type inequalities with Fibonacci and Lucas numbers.

Problem 1.
If $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$, is Lucas's sequence, and $a, b, c \in \mathbb{R}_+^*$ such that $a + b + c \leq 24$, then:

$$\frac{L_n}{\sqrt{L_n^2 + aL_{n+1}L_{n+2}}} + \frac{L_{n+1}}{\sqrt{L_{n+1}^2 + bL_{n+2}L_n}} + \frac{L_{n+2}}{\sqrt{L_{n+2}^2 + cL_nL_{n+1}}} \geq 1, \forall n \in \mathbb{N}$$

Proof. We have:

$$U_n = \frac{L_n}{\sqrt{L_n^2 + aL_{n+1}L_{n+2}}} + \frac{L_{n+1}}{\sqrt{L_{n+1}^2 + bL_{n+2}L_n}} + \frac{L_{n+2}}{\sqrt{L_{n+2}^2 + cL_nL_{n+1}}}$$

(1)

$$U_n = \sum \frac{L_n}{\sqrt{L_n^2 + aL_{n+1}L_{n+2}}} = \sum \frac{L_n^2}{\sqrt{L_n} \cdot \sqrt{L_n^3 + aL_nL_{n+1}L_{n+2}}} = \sum \frac{L_n^2}{\sqrt{L_n} \cdot \sqrt{L_n^3 \cdot ap_n}}, \forall n \in \mathbb{N}$$

where $p_n = L_nL_{n+1}L_{n+2}, \forall n \in \mathbb{N}$. We denote $v_n = \sqrt{L_n} \cdot \sqrt{L_n^3 + ap_n}$ and

$$V_n = v_n + v_{n+1} + v_{n+2} = \sqrt{L_n} \cdot \sqrt{L_n^3 + ap_n} + \sqrt{L_n} \cdot \sqrt{L_n^3 + bp_n} + \sqrt{L_n} \cdot \sqrt{L_n^3 \cdot cp_n}, \forall n \in \mathbb{N}$$

We have that:

$$V_n^2 = \left(\sum L_n \sqrt{L_n^3 + ap_n}\right)^2 \overset{\text{C-B-S}}{\leq} s_n(L_n^3 + L_{n+1}^3 + L_{n+2}^3 + (a+b+c)p_n), \forall n \in \mathbb{N},$$

where $s_n = L_n + L_{n+1} + L_{n+2}, \forall n \in \mathbb{N}$. Since

$$(x + y + z)^3 \geq x^3 + y^3 + z^3 + 24xyz, \forall x, y, z \in \mathbb{R}_+^* \text{ and}$$

$$V_n^2 \leq s_n(L_n^3 + L_{n+1}^3 + L_{n+2}^3 + (a+b+c)p_n) \leq s_n(L_n^3 + L_{n+1}^3 + L_{n+2}^3 + 24L_nL_{n+1}L_{n+2})$$

then

$$V_n^2 \leq s_n \cdot s_n^3 = s_n^4, \forall n \in \mathbb{N} \Leftrightarrow v_n \leq s_n^2, \forall n \in \mathbb{N} \tag{2}$$

By Bergström's inequality, and from (1) and (2) we obtain that:

$$U_n \geq \frac{(s_n)^2}{V_n} \geq \frac{s_n^2}{s_n^2} = 1, \forall n \in \mathbb{N},$$

so we are done. $\square$

Problem 2.
If $(F_n)_{n\geq 0}, F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$, is Fibonacci's sequences, and $a, b, c \in \mathbb{R}_+^*$ such that $a + b + c \leq 24$, then:

$$\frac{F_n}{\sqrt{F_n^2 + aF_{n+1}F_{n+2}}} + \frac{F_{n+1}}{\sqrt{F_{n+1}^2 + bF_{n+2}F_n}} + \frac{F_{n+2}}{\sqrt{F_{n+2}^2 + cF_nF_{n+1}}} \geq 1, \forall n \in \mathbb{N}.$$

Proof.

$$U_n = \frac{F_n}{\sqrt{F_n^2 + aF_{n+1}F_{n+2}}} + \frac{F_{n+1}}{\sqrt{F_{n+1}^2 + bF_{n+2}F_n}} + \frac{F_{n+2}}{\sqrt{F_{n+2}^2 + cF_nF_{n+1}}}$$

(1)

$$U_n = \sum \frac{F_n}{\sqrt{F_n^2 + aF_{n+1}F_{n+2}}} = \sum \frac{F_n^2}{\sqrt{F_n} \cdot \sqrt{F_n^3 + aF_nF_{n+1}F_{n+2}}} = \sum \frac{F_n^2}{\sqrt{F_n} \cdot \sqrt{F_n^3 + ap_n}}, \forall \in \mathbb{N}$$

where we denote $p_n = F_nF_{n+1}F_{n+2}, \forall n \in \mathbb{N}$. We denote $v_n = \sqrt{F_n} \cdot \sqrt{F_n^3 + ap_n}$, and

$$V_n = v_n + v_{n+1} + v_{n+2} = \sqrt{F_n} \cdot \sqrt{F_n^3 + ap_n} + \sqrt{F_{n+1}} \cdot \sqrt{F_{n+1}^3 + bp_n} + \sqrt{F_{n+1}} \cdot \sqrt{F_{n+1}^3 + cp_n}, \forall n \in \mathbb{N}.$$

We have:

$$V_n^2 = \left(\sum \sqrt{F_n} \cdot \sqrt{F_n^3 + ap_n}\right)^2 \overset{\text{C-B-S}}{\leq} s_n(F_n^3 + F_{n+1}^3 + F_{n+2}^3 + (a+b+c)p_n), \forall n \in \mathbb{N},$$

where we denote $s_n = F_n + F_{n+1} + F_{n+2}, \forall n \in \mathbb{N}$. Since

$$(x + y + z)^3 \geq x^3 + y^3 + z^3 + 24xyz, \forall x, y, z \in \mathbb{R}, \text{ and}$$

$$V_n^2 \leq s_n(F_n^3 + F_{n+1}^3 + F_{n+2}^3 + (a+b+c)p_n) \leq s_n(F_n^3 + F_{n+1}^3 + F_{n+2}^3 + 24F_nF_{n+1}F_{n+2})$$

then

$$V_n^2 \leq s_n \cdot s_n^3 = s_n^4, \forall n \in \mathbb{N} \Leftrightarrow V_n \leq s_n^2, \forall n \in \mathbb{N} \tag{2}$$

We apply Bergström's inequality, and from (1) and (2) we deduce that:

$$U_n \geq \frac{(s_n)^2}{V_n} \geq \frac{s_n^2}{s_n^2} = 1, \forall n \in \mathbb{N},$$

i.e. we obtain the given inequality. □

Problem 3.
If $(F_n)_{n\geq 0}, F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$, is Fibonacci's sequence and $a, b, c \in \mathbb{R}_+^*$ with $abc = 1$, then for any $m \in \mathbb{R}_+$ is true the inequality.

$$\frac{1}{a^{3m+3}(F_n + F_{n+1}c)^{m+1}} + \frac{1}{b^{3m+3}(F_nc + F_{n+1}a)^{m+1}} + \frac{1}{c^{3m+3}(F_na + F_{n+1}b)^{m+1}} \geq \frac{3}{F_{n+2}^{m+1}}, \forall n \in \mathbb{N}$$

Proof. By J. Radon's inequality we have:

$$A_n = \frac{1}{a^{3m+3}(F_nb + F_{n+1}c)^{m+1}} + \frac{1}{b^{3m+3}(F_nc + F_{n+1}a)^{m+1}} + \frac{1}{c^{3m+3}(F_na + F_{n+1}b)^{m+1}} \geq$$

$$\geq \frac{1}{3^m}\left(\frac{1}{a^3(F_nb + F_{n+1}c)} + \frac{1}{b^3(F_nc + F_{n+1}a)} + \frac{1}{c^3(F_na + F_{n+1}b)}\right)^{m+1}, \forall n \in \mathbb{N} \tag{1}$$

We also have

$$B_n = \frac{1}{a^3(F_nb + F_{n+1}c)} + \frac{1}{b^3(F_nc + F_{n+1}a)} + \frac{1}{c^3(F_na + F_{n+1}b)} =$$

$$= \frac{\frac{1}{a^2}}{F_n ab + F_{n+1}ac} + \frac{\frac{1}{b^2}}{F_n bc + F_{n+1}ab} + \frac{\frac{1}{c^2}}{F_n ac + F_{n+1}bc}, \forall n \in \mathbb{N},$$

where we apply Bergström's inequality, and we deduce that:

$$B_n \geq \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{(F_n + F_{n+1})(ab + bc + ca)} = \frac{(ab + bc + ca)^2}{F_{n+2}(abc)^2(ab + bc + ca)} = \frac{ab + bc + ca}{F_{n+2}} \overset{\text{MA-MG}}{\geq}$$

(2)
$$\overset{\text{MA-MG}}{\geq} \frac{3 \cdot \sqrt[3]{(abc)^2}}{F_{n+2}} = \frac{3}{F_{n+2}}, \forall n \in \mathbb{N}$$

From (1) and (2) yields that:

$$A_n \geq \frac{1}{3^m}\left(\frac{3}{F_{n+2}}\right)^{m+1} = \frac{3}{F_{n+2}^{m+1}}, \forall n \in \mathbb{N}$$

□

Observation. For $n = m = 0$, i.e. $F_0 = F_1 = 1, F_2 = 2$, the given inequality becomes
If $a, b, c \in \mathbb{R}_+^*$ such that $abc = 1$, then $\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$, i.e. the problem at IMO, Canada, 1995, proposed by Russia.

Problem 4.

If $(F_n)_{n\geq 0}, F_0 = F_1 = 1, F_{n+2} + F_n, \forall n \in \mathbb{N}$, and $a, b, c \in \mathbb{R}_+^*$ such that $abc = 1$ then the following inequality is true.

$$\frac{1}{a^3(F_n b + F_{n+1}c)} + \frac{1}{b^3(F_n c + F_{n+1}a)} + \frac{1}{c^3(F_n a + F_{n+1}b)} \geq \frac{3}{F_{n+2}}, \forall n \in \mathbb{N}$$

Proof. We denote:

$$B_n = \frac{1}{a^3(F_n b + F_{n+1}c)} + \frac{1}{b^3(F_n c + F_{n+1}a)} + \frac{1}{c^2(F_n a + F_{n+1}b)} =$$

$$= \frac{\frac{1}{a^2}}{F_n ab + F_{n+1}ac} + \frac{\frac{1}{b^2}}{F_n bc + F_{n+1}ab} + \frac{\frac{1}{c^2}}{F_n ac + F_{n+1}bc}, \forall n \in \mathbb{N}$$

and by Bergström's inequality we have:

$$B_n = \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{(F_n + F_{n+1})(ab + bc + ca)} = \frac{(ab + bc + ca)^2}{F_{n+2}(abc)^2(ab + bc + ca)} = \frac{ab + bc + ca}{F_{n+2}} \overset{\text{MA-MG}}{\geq}$$

$$\overset{\text{MA-MG}}{\geq} \frac{3 \cdot \sqrt[3]{(abc)^2}}{F_{n+2}} = \frac{3}{F_{n+2}}, \forall n \in \mathbb{N}$$

□

Observation. For $n = 0, F_0 = F_1 = 1, F_2 = 2$, and the inequality becomes:
If $a, b, c \in \mathbb{R}_+^*$ such that $abc = 1$, then:

$$\frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)} \geq \frac{3}{2}$$

i.e. the IMO Problem, Canada, 1995.

Problem 5.

If $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$, is the Lucas's sequences, and $a, b, c \in \mathbb{R}_+^*$ such that $abc = 1$, then:

$$\frac{1}{a^6(L_n b + L_{n+1}c)^2} + \frac{1}{b^6(L_n c + L_{n+1}a)^2} + \frac{1}{c^6(L_n a + L_{n+1}b)^2} \geq \frac{3}{L_{n+2}^2}, \forall n \in \mathbb{N}$$

Proof. By Bergström's inequality we have:

$$A_n = \frac{1}{a^6(L_n b + L_{n+1}c)^2} + \frac{1}{b^6(L_n c + L_{n+1}a)^2} + \frac{1}{c^6(L_n a + L_{n+1}b)^2} \geq$$

$$\geq \frac{1}{3}\left(\frac{1}{a^3(L_n b + L_{n+1}c)} + \frac{1}{b^3(L_n c + L_{n+1}a)} + \frac{1}{c^3(L_n a + L_{n+1}b)}\right)^2, \forall n \in \mathbb{N} \tag{1}$$

Also, we have:

$$B_n = \frac{1}{a^3(L_n b + L_{n+1}c)} + \frac{1}{b^3(L_n c + L_{n+1}a)} + \frac{1}{c^3(L_n a + L_{n+1}b)} =$$

$$= \frac{\frac{1}{a^2}}{L_n ab + L_{n+1}ac} + \frac{\frac{1}{b^2}}{L_n bc + L_{n+1}ab} + \frac{\frac{1}{c^2}}{L_n ac + L_{n+1}bc}, \forall n \in \mathbb{N}$$

where we apply again, Bergström's inequality and AM-GM inequality and we deduce that

$$B_n \geq \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{(L_n + L_{n+1})(ab + bc + ca)} = \frac{(ab + bc + ca)^2}{L_{n+2}(abc)^2(ab + bc + ca)} = \frac{ab + bc + ca}{L_{n+2}} \overset{\text{AM-GM}}{\geq}$$

$$\overset{\text{AM-GM}}{\geq} \frac{3 \cdot \sqrt[3]{(abc)^2}}{L_{n+2}} = \frac{3}{L_{n+2}}, \forall n \in \mathbb{N} \tag{2}$$

From (1) and (2) follows that:

$$A_n \geq \frac{1}{3}\left(\frac{3}{L_{n+2}}\right)^2 = \frac{3}{L_{n+2}^2}, \forall n \in \mathbb{N}$$

and we are done. □

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Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com