NEW INEQUALITIES IN ACUTE TRIANGLES

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ABSTRACT. In this paper we present some new inequalities in acute triangles.

Problem 1.
In any acute triangle ABC holds:

$$\sum_{cyc}\left[\left(\frac{\sin A\sin B}{\sin C}\right)^2+\left(\frac{\cos A\cos B}{\cos C}\right)^2\right]\geq 3$$

Proof. We prove the following two inequalities:

$$\text{a) } \sum\left(\frac{\sin A\sin B}{\sin C}\right)^2\geq\frac{9}{4}; \qquad \text{b) } \sum\left(\frac{\cos A\cos B}{\cos C}\right)^2\geq\frac{3}{4}$$

Indeed
a) We have

$$\frac{\sin A\sin B}{\sin C}=\frac{\sin A\sin B}{\sin[\pi-(A+B)]}=\frac{\sin A\sin B}{\sin(A+B)}=\frac{\sin A\sin B}{\sin A\cos B+\sin B\cos A}=$$

$$=\frac{\sin A\sin B}{\sin A\sin B(\frac{\sin A\cos B}{\sin A\sin B}+\frac{\sin B\cos A}{\sin A\sin B})}=\frac{1}{\cot A+\cot B}$$

and analogous, we deduce that:

$$\sum\left(\frac{\sin A\sin B}{\sin C}\right)^2=\sum\frac{1}{(\cot A+\cot B)^2}$$

and if we take into account that

$$\sum\cot A\cot B=1$$

and if we make the substitutions

$$x=\cot A, y=\cot B, z=\cot C$$

then the inequality a) becomes

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\geq\frac{9}{4(xy+yz+zx)}, \text{ for any } x,y,z>0,$$

i.e. the inequality given to the Iran Olympiad, 1996. So, a) is proved.
b) We have

$$\frac{\cos A\cos B}{\cos C}=\frac{\sin C}{\cos C}\cdot\frac{\cos A\cos B}{\sin C}=\tan C\cdot\frac{\cos A\cos B}{\sin(A+B)}=$$

$$=\tan C\cdot\frac{\cos A\cos B}{\sin A\cos B+\sin B\cos A}=\frac{\tan C}{\tan B+\tan C}$$

and analogous by Nesbitt's inequality we have

$$\sum\frac{\tan C}{\tan B+\tan A}\geq\frac{3}{2}$$

By Cauchy-Schwarz's inequality and the above we obtain that

$$\sum\left(\frac{\cos A\cos B}{\cos C}\right)^2\geq\frac{1}{3}\left(\sum\frac{\cos A\cos B}{\cos C}\right)^2=\frac{1}{3}\left(\sum\frac{\tan C}{\tan B+\tan C}\right)^2\geq\frac{1}{3}\cdot\frac{9}{4}=\frac{3}{4}$$

So, b) is proved.
By adding the inequalities a) and b) we obtain the given inequality. □

Problem 2.
If $m\in\mathbb{R}_+$ then in any acute triangle ABC holds

$$\sum\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}\geq\frac{3}{2^{m+1}}$$

Proof.

$$\frac{\cos A\cos B}{\cos C}=\frac{\sin C}{\cos C}\cdot\frac{\cos A\cos B}{\sin C}=\tan C\cdot\frac{\cos A\cos B}{\sin(A+B)}=$$
$$=\tan C\cdot\frac{\cos A\cos B}{\sin A\cos B+\sin B\cos A}=\frac{\tan C}{\tan B+\tan C}$$

and by Nesbitt's inequality:

$$\sum\frac{\tan C}{\tan B+\tan A}\geq\frac{3}{2}$$

By J. Radon's inequality:

$$\sum\left(\frac{\cos A\cos B}{\cos C}\right)^{m+1}\geq\frac{1}{3^m}\left(\sum\frac{\cos A\cos B}{\cos C}\right)^{m+1}=\frac{1}{3^m}\left(\sum\frac{\tan C}{\tan B+\tan C}\right)^{m+1}\geq$$
$$\geq\frac{1}{3^m}\cdot\left(\frac{3}{2}\right)^{m+1}=\frac{3}{2^{m+1}}$$

□

Problem 3.
In any acute triangle ABC holds:

$$\sum\frac{\cos A\cos^2 B}{\cos^2 C}\geq\frac{9R}{4(R+r)}$$

Proof.

$$\frac{\cos A\cos B}{\cos C}=\frac{\sin C}{\cos C}\cdot\frac{\cos A\cos B}{\sin C}=\tan C\cdot\frac{\cos A\cos B}{\sin(A+B)}=$$
$$=\tan C\cdot\frac{\cos A\cos B}{\sin A\cos B+\sin B\cos A}=\frac{\tan C}{\tan B+\tan C}$$

and from Nesbitt $\sum\frac{\tan C}{\tan B+\tan A}\geq\frac{3}{2}$. It is well-known that $\sum\cos A=\frac{R+r}{R}$.
From H. Bergström's inequality and above:

$$\sum\frac{\cos A\cos^2 B}{\cos^2 C}=\sum\frac{\frac{\cos^2 A\cos^2 B}{\cos^2 C}}{\cos A}\geq\frac{(\sum\frac{\cos A\cos B}{\cos C})^2}{\sum\cos A}=$$
$$=\frac{(\sum\frac{\tan C}{\tan B+\tan A})^2}{\sum\cos A}\geq\frac{9R}{4(R+r)}$$

□

Problem 4.
If $m\in\mathbb{R}_+$, then in any acute triangle ABC holds:

$$\sum\frac{\cos A\cos^{m+1} B}{\cos^{m+1} C}\geq\frac{3^{m+1}R^m}{2^{m+1}(R+r)^m}$$

Proof.

$$\frac{\cos A\cos B}{\cos C}=\frac{\sin C}{\cos C}\cdot\frac{\cos A\cos B}{\sin C}=\tan C\cdot\frac{\cos A\cos B}{\sin(A+B)}=$$
$$=\tan C\cdot\frac{\cos A\cos B}{\sin A\cos B+\sin B\cos A}=\frac{\tan C}{\tan B+\tan C}$$

and by Nesbitt's inequality $\sum\frac{\tan C}{\tan B+\tan A}\geq\frac{3}{2}$ and from $\sum\cos A=\frac{R+r}{R}$ and using Bergström's inequality yields that:

$$\sum\frac{\cos A\cos^{m+1}B}{\cos^{m+1}C}=\sum\frac{\frac{\cos^{m+1}A\cos^{m+1}B}{\cos^{m+1}C}}{\cos^m A}\geq\frac{(\sum\frac{\cos A\cos B}{\cos C})^{m+1}}{(\sum\cos A)^m}=$$
$$=\frac{(\sum\frac{\tan C}{\tan B+\tan A})^{m+1}}{(\sum\cos A)^m}\geq\frac{3^{m+1}R^m}{2^{m+1}(R+r)^m}$$

□

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