SOME NEW INEQUALITIES IN CONVEXE POLYGONS

D.M. BĂTINEŢU-GIURGIU, MIHÁLY BENCZE, DANIEL SITARU, NECULAI STANCIU

Abstract. In this paper we present some new inequalities in convexe polygons.

Theorem 1.
If $A_1A_2\dots A_n, n\ge 3$ is a regular polygon, M a point on incircle and N a point on circumcircle of the polygon, then:

$$\sum_{k=1}^{n}\frac{MA_k^4}{NA_k^2}\ge\frac{1}{4}\left(3+\cos\frac{2\pi}{n}\right)\sum_{k=1}^{n}MA_k^2$$

Proof. Let O be the center of the polygon and xOy be the system of coordinates with $[Ox)=[OA_n)$. WLOG we assume that the circumradius is $R=1$.
We have $A_k(\cos\frac{2k\pi}{n},\sin\frac{2k\pi}{n}), k=\overline{1,n}$. Let t and u be the arguments of the afixes of the points M and N. We have that in radius is $r=\cos\frac{\pi}{n}$, so $M(\cos\frac{\pi}{n}\cos t,\cos\frac{\pi}{n}\sin t)$.
Therefore:

$$\sum_{k=1}^{n}MA_k^2=\sum_{k=1}^{n}\left(\cos\frac{\pi}{n}\cos t-\cos\frac{2k\pi}{n}\right)^2+\sum_{k=1}^{n}\left(\cos\frac{\pi}{n}\sin t-\sin\frac{2k\pi}{n}\right)^2=$$

(1)

$$=n\cos^2\frac{\pi}{n}+n-2\cos\frac{\pi}{n}\cos t\sum_{k=1}^{n}\cos\frac{2k\pi}{n}-2\cos\frac{\pi}{n}\sin t\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=n\left(1+\cos^2\frac{\pi}{n}\right)$$

In (1) we use the fact that the afixes $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, k=\overline{1,n}$ of the points A_k are the roots of the equation $x^n-1=0$, so that $\sum_{k=1}^{n}\varepsilon_k=0$, from where we deduce that:

$$\sum_{k=1}^{n}\cos\frac{2k\pi}{n}+i\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=0$$

Also, we have:

$$\sum_{k=1}^{n}NA_k^2=\sum_{k=1}^{n}\left(\cos u-\cos\frac{2k\pi}{n}\right)^2+\sum_{k=1}^{n}\left(\sin u-\sin\frac{2k\pi}{n}\right)^2=$$

$$=n+n-2\cos u\sum_{k=1}^{n}\cos\frac{2k\pi}{n}-2\sin u\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=2n \tag{2}$$

By (1) and (2) we deduce that:

$$\sum_{k=1}^{n}MA_k^2=\frac{1}{2}\left(1+\cos^2\frac{\pi}{n}\right)\sum_{k=1}^{n}NA_k^2=\frac{1}{4}\left(3+\cos\frac{2\pi}{n}\right)\sum_{k=1}^{n}NA_k^2 \tag{3}$$

By (3) and by Bergström's inequality we have:

$$\sum_{k=1}^{n}\frac{MA_k^4}{NA_k^2}\geq\frac{(\sum_{k=1}^{n}MA_k^2)^2}{\sum_{k=1}^{n}NA_k^2}=\frac{\sum_{k=1}^{n}MA_k^2}{\sum_{k=1}^{n}NA_k^2}\sum_{k=1}^{n}MA_k^2=\frac{1}{4}\Big(3+\cos\frac{2\pi}{n}\Big)\sum_{k=1}^{n}MA_k^2$$

and we are done. □

Theorem 2.
If $A_1A_2\ldots A_n, n\geq 3$ is a regular polygon, M a point on incircle and N a point on circumcircle of the polygon, then:

$$\sum_{k=1}^{n}\frac{MA_k^4}{NA_k^2}\geq\frac{1}{16}\Big(3+\cos\frac{2\pi}{n}\Big)^2\sum_{k=1}^{n}NA_k^2$$

Proof. Let O be the center of the polygon and xOy be the system of coordinates with $[Ox)=[OA_n)$. WLOG we assume that the circumradius is $R=1$.
We have $A_k(\cos\frac{2k\pi}{n},\sin\frac{2k\pi}{n}), k=\overline{1,n}$. Let t and u be the arguments of the afixes of the points M and N. We have that inradius is $r=\cos\frac{\pi}{n}$, so $M(\cos\frac{\pi}{n}\cos t,\cos\frac{\pi}{n}\sin t)$.
Therefore:

$$\sum_{k=1}^{n}MA_k^2=\sum_{k=1}^{n}\Big(\cos\frac{\pi}{n}\cos t-\cos\frac{2k\pi}{n}\Big)^2+\sum_{k=1}^{n}\Big(\cos\frac{\pi}{n}\sin t-\sin\frac{2k\pi}{n}\Big)^2=$$

(1)

$$=n\cos^2\frac{\pi}{n}+n-2\cos\frac{\pi}{n}\cos t\sum_{k=1}^{n}\cos\frac{2k\pi}{n}-2\cos\frac{\pi}{n}\sin t\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=n\Big(1+\cos^2\frac{\pi}{n}\Big)$$

In (1) we use the fact that the afixes $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, k=\overline{1,n}$ of the points A_k are the roots of the equation $x^n-1=0$, so that $\sum_{k=1}^{n}\varepsilon_k=0$, from where we deduce that:

$$\sum_{k=1}^{n}\cos\frac{2k\pi}{n}+i\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=0$$

Also, we have:

$$\sum_{k=1}^{n}NA_k^2=\sum_{k=1}^{n}\Big(\cos u-\cos\frac{2k\pi}{n}\Big)^2+\sum_{k=1}^{n}\Big(\sin u-\sin\frac{2k\pi}{n}\Big)^2=$$

$$=n+n-2\cos u\sum_{k=1}^{n}\cos\frac{2k\pi}{n}-2\sin u\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=2n \tag{2}$$

By (1) and (2) we deduce that:

$$\sum_{k=1}^{n}MA_k^2=\frac{1}{2}\Big(1+\cos^2\frac{\pi}{n}\Big)\sum_{k=1}^{n}NA_k^2=\frac{1}{4}\Big(3+\cos\frac{2\pi}{n}\Big)\sum_{k=1}^{n}NA_k^2 \tag{3}$$

By (3) and by Bergström's inequality we have:

$$\sum_{k=1}^{n}\frac{MA_k^4}{NA_k^2}\geq\frac{(\sum_{k=1}^{n}MA_k^2)^2}{\sum_{k=1}^{n}NA_k^2}=\frac{\frac{1}{16}(3+\cos\frac{2\pi}{n})^2(\sum_{k=1}^{n}NA_k^2)^2}{\sum_{k=1}^{n}NA_k^2}=\frac{1}{16}\Big(3+\cos\frac{2\pi}{n}\Big)^2\sum_{k=1}^{n}NA_k^2$$

and we are done. □

Theorem 3.
If $A_1A_2\dots A_n, n \geq 3$ is a convex polygon inscribed in the circle $C(O,R)$ and circumscribed in the circle $C(I,r)$ such that $O \in IntA_1A_2\dots A_n$. If s is the semiperimeter of the given polygon, then $2sr \leq nR^2$.

Proof. We have that:

$$\Delta_k = area(OA_kA_{k+1}) = \frac{OA_k \cdot OA_{k+1}}{2}\sin A_kOA_{k+1} \leq \frac{OA_k \cdot OA_{k+1}}{2}, k = \overline{1,n}.$$

$$\text{Hence, } sr = \Delta = \sum_{k=1}^{n}\Delta_k \leq \frac{1}{2}\sum_{k=1}^{n} OA_k \cdot OA_{k+1} = \frac{1}{2}\cdot n \cdot R^2 \Leftrightarrow 2sr \leq nR^2$$

and the proof is complete. □

Theorem 4.
If $A_1A_2\dots A_n, n \geq 3$ be a convex polygon circumscribed to circle $C(I,r)$ and $M \in IntA_1A_2\dots A_n$. If r_k are the inradius of the triangles MA_kA_{k+1}, $k = \overline{1,n}, A_{n+1} = A_1$ then $\sum_{k=1}^{n} r_k > r$.

Proof. We denote by Δ the area of the given polygon and by s the semiperimeter of the polygon. Denoting Δ_k the area of triangle MA_kA_{k+1} and s_k, the semiperimeter of triangle $MA_kA_{k+1}k = \overline{1,n}$. We have that:

$$\sum_{k=1}^{n} r_k = \sum_{k=1}^{n}\frac{\Delta_k}{s_k} > \sum_{k=1}^{n}\frac{\Delta_k}{s} = \frac{1}{s}\sum_{k=1}^{n}\Delta_k = \frac{\Delta}{s} = \frac{s \cdot r}{s} = r,$$

and we are done. □

Theorem 5.
If $A_1A_2\dots A_n (n \geq 3)$ be a convex polygon with the semiperimeter p which is circumscribed to circle $C(I,r), M \in IntA_1A_2\dots A_n$ and r_k be the inradius of the triangle MA_kA_{k+1}, with the area $S_k(k = \overline{1,n})A_{n+1} = A_1$ then $\sum_{k=1}^{n}\frac{r_k^2}{S_k} > \frac{r}{p}$

Proof. Let S be the area of the given polygon and p_k be the semiperimeter of triangle $MA_kA_{k+1}(k = \overline{1,n})$. By Bergström's inequality we have:

$$(1) \qquad U_n = \sum_{k=1}^{n}\frac{r_k^2}{S_k} \geq \frac{(\sum_{k=1}^{n} r_k)^2}{\sum_{k=1}^{n} S_k} = \frac{(\sum_{k=1}^{n} r_k)^2}{S}$$

Evidently we have $p_k < p, \forall k = \overline{1,n}$, so

$$(2) \quad r_1 + r_2 + \ldots + r_n = \frac{S_1}{p_1} + \frac{S_2}{p_2} + \ldots + \frac{S_n}{p_n} > \frac{S_1}{p} + \frac{S_2}{p} + \ldots + \frac{S_n}{p} = \frac{S}{p} = \frac{rp}{p} = r$$

From (1) and (2) we obtain that $U_n > \frac{r^2}{S} = \frac{r^2}{pr} = \frac{r}{p}$, and we are done. □

Theorem 6.
If $A_1A_2\dots A_n, n \geq 3$ be a convex n - gon with a_k the length of the side $[A_kA_{k+1}]$, $k = \overline{1,n}$, and s the semiperimeter of the polygon, then:

$$\sum_{k=1}^{n}\Big(\frac{a_k}{s-a_k}\Big)^{m+1} \geq n\cdot\Big(\frac{2}{n-2}\Big)^{m+1}$$

Proof. By J. Radon's inequality we have:

$$U_n=\sum_{k=1}^{n}\Big(\frac{a_k}{s-a_k}\Big)^{m+1}\geq\frac{1}{n^m}\left(\sum_{k=1}^{n}\frac{a_k}{s-a_k}\right)^{m+1}=\frac{1}{n^m}\cdot S_n^{m+1}\tag{1}$$

where

$$S_n=\sum_{k=1}^{n}\frac{a_k}{s-a_k}=\sum_{k=1}^{n}\frac{a_k-s+s}{s-a_k}=-n+s\cdot\sum_{k=1}^{n}\frac{1}{s-a_k}$$

and by H. Bergström's inequality we deduce that:

$$S_n\geq -n+\frac{n^2s}{s(n-2)}=-n+\frac{n^2}{n-2}=\frac{2n}{n-2}\tag{2}$$

By (1) and (2) we are done. □

Theorem 7.
If $A_1A_2\ldots A_n, n\geq 3$ is a convex polygon, and $M\in Int(A_1A_2\ldots A_n)$, with $pr_{A_kA_{k+1}}M=T_k\in[A_kA_{k+1}]$, for any $k\in\{1,2,\ldots,n\}, A_{n+1}\equiv A_1$, then:

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}\geq 2n\tan\frac{\pi}{n}$$

Proof. We denote $\forall$ = for any and $k=\overline{1,n}\Leftrightarrow k\in\{1,2,\ldots,n\}$. We first prove the Lemma. Let $A,B,A\neq B$ be points in plane and $M\notin AB, T=pr_{AB}M$, then:

$$\frac{AB}{MT}=\tan u+\tan v$$

where $u=\mu(\angle AMT), v=\mu(\angle TMB)$, are the measures in radians of angles $\angle AMT$ and $\angle TMB$.

Proof of the lemma. We have the cases:
i) $T\in(AB)$. We have: $\tan u=\frac{AT}{MT}$ and $\tan v=\frac{BT}{MT}$, so $\tan u+\tan v=\frac{AB}{MT}$.
ii) $T\equiv A$. We have $\tan u=\frac{AT}{MT}=\frac{AA}{MT}=0$ and $\tan v=\frac{BT}{MT}$, so $\tan u+\tan v=\frac{AB}{MT}$.
iii) $T\equiv B$. We have $\tan u=\frac{AB}{MT}$ and $\tan v=\frac{BT}{MT}=\frac{BB}{MT}=0$, so $\tan u+\tan v=\frac{AB}{MT}$.
From lemma we have:
$\frac{A_kA_{k+1}}{MT_k}=\tan u_k+\tan v_k, \forall k=\overline{1,n}$, where $u_k=\mu(\angle A_kMT_k), v_k=\mu(\angle T_kMA_{k+1})$, $\forall k=\overline{1,n}$ and then

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}=\sum_{k=1}^{n}(\tan u_k+\tan v_k)$$

Since the function $f:[0,\frac{\pi}{2})\to[0,\infty), f(x)=\tan x$ is convex on $[0,\frac{\pi}{2})$ we can apply Jensen's inequality and we obtain that:

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}=\sum_{k=1}^{n}(\tan u_k+\tan v_k)\geq 2n\tan\Big(\frac{1}{2n}\sum_{k=1}^{n}(u_k+v_k)\Big)$$

Because $\sum_{k=1}^{n}(u_k+v_k)=2\pi$ we deduce that

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}\geq 2n\tan\frac{2\pi}{2n}=2n\tan\frac{\pi}{n}\text{ and we are done.}$$

Observation 1. If $A_1A_2\dots A_n$ is circumscribed on the circle $C(I;r)$ and $M\equiv I$, we have $MT_k=r,\forall k=\overline{1,n}$, and the given inequality becomes:

$$\frac{1}{r}\sum_{k=1}^{n}A_kA_{k+1}=\frac{2s}{r}\geq 2n\tan\frac{\pi}{n}\Leftrightarrow s\geq nr\tan\frac{\pi}{n} \tag{*}$$

The inequality (*) is a generalization of Mitrinovic's inequality

$$s\geq 3r\sqrt{3} \tag{M}$$

Observation 2. If $A_1A_2A_3$ is a triangle, then the given inequality becomes:

$$\frac{A_1A_2}{MT_1}+\frac{A_2A_3}{MT_2}+\frac{A_3A_1}{MT_3}\geq 6\tan\frac{\pi}{3}=6\sqrt{3} \tag{**}$$

For $M\equiv I$, we obtain (M). □

□

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Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com