SOME SPECIAL DEFINITE INTEGRALS

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Abstract. In this paper we present some special definite integrals.

Problem 1.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1+\ln x)\sin x - x\cos x\ln x}{\sin^2 x + x^2\ln^2 x}dx = \arctan\frac{3\sqrt{3}+2\pi\ln\frac{\pi}{3}}{3\sqrt{3}-2\pi\ln\frac{\pi}{3}} - \arctan\frac{2\sqrt{2}+\pi\ln\frac{\pi}{4}}{2\sqrt{2}-\pi\ln\frac{\pi}{4}}$$

Proof.

$$\frac{(1+\ln x)\sin x - x\cos x\ln x}{\sin^2 x + x^2\ln^2 x} = \frac{(\cos x+\ln x+1)(\sin x - x\ln x)}{2\sin^2 x+2x^2\ln^2 x} -$$

$$-\frac{(\cos x+\ln x+1)(\sin x + x\ln x)}{2\sin^2 x+2x^2\ln^2 x} = \frac{(\frac{\sin x+x\ln x}{\sin x-x\ln x})'}{1+(\frac{\sin x+x\ln x}{\sin x-x\ln x})^2}$$

Hence,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1+\ln x)\sin x - x\cos x\ln x}{\sin^2 x + x^2\ln^2 x}dx = \left(\arctan\frac{\sin x+x\ln x}{\sin x - x\ln x}\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} =$$

$$= \arctan\frac{\frac{\sqrt{3}}{2}+\frac{\pi}{3}\ln\frac{\pi}{3}}{\frac{\sqrt{3}}{2}-\frac{\pi}{3}\ln\frac{\pi}{3}} - \arctan\frac{\frac{\sqrt{2}}{2}+\frac{\pi}{4}\ln\frac{\pi}{4}}{\frac{\sqrt{2}}{2}-\frac{\pi}{4}\ln\frac{\pi}{4}} =$$

$$= \arctan\frac{3\sqrt{3}+2\pi\ln\frac{\pi}{3}}{3\sqrt{3}-2\pi\ln\frac{\pi}{3}} - \arctan\frac{2\sqrt{2}+\pi\ln\frac{\pi}{4}}{2\sqrt{2}-\pi\ln\frac{\pi}{4}}$$

□

Problem 2.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1+\ln x)\cos x + x\sin x\ln x}{\cos^2 x + x^2\ln^2 x}dx = \arctan\frac{2\sqrt{2}+\pi\ln\frac{\pi}{4}}{2\sqrt{2}-\pi\ln\frac{\pi}{4}} - \arctan\frac{3+\pi\ln\frac{\pi}{6}}{3-\pi\ln\frac{\pi}{6}}$$

Proof.

$$\frac{(1+\ln x)\cos x - x\sin x\ln x}{\cos^2 x + x^2\ln^2 x} = \frac{(-\sin x+\ln x+1)(\cos x - x\ln x)}{2\cos^2 x+2x^2\ln^2 x} +$$

$$+\frac{(\sin x+\ln x+1)(\cos x + x\ln x)}{2\cos^2 x+2x^2\ln^2 x} = \frac{(\frac{\cos x+x\ln x}{\cos x-x\ln x})'}{1+(\frac{\cos x+x\ln x}{\cos x-x\ln x})^2}$$

Hence,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1+\ln x)\cos x + x\sin x\ln x}{\cos^2 x + x^2\ln^2 x}dx = \left(\arctan\frac{\cos x+x\ln x}{\cos x - x\ln x}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} =$$

$$= \arctan\frac{\frac{\sqrt{2}}{2}+\frac{\pi}{4}\ln\frac{\pi}{4}}{\frac{\sqrt{2}}{2}-\frac{\pi}{4}\ln\frac{\pi}{4}} - \arctan\frac{\frac{1}{2}+\frac{\pi}{6}\ln\frac{\pi}{6}}{\frac{1}{2}-\frac{\pi}{6}\ln\frac{\pi}{6}} =$$

$$= \arctan \frac{2\sqrt{2} + \pi \ln \frac{\pi}{4}}{2\sqrt{2} - \pi \ln \frac{\pi}{4}} - \arctan \frac{3 + \pi \ln \frac{\pi}{6}}{3 - \pi \ln \frac{\pi}{6}}$$

□

Problem 3.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1 + \ln x) \sin x + x \cos x \ln x}{\sin^2 x + x^2 \ln^2 x} dx = \arctan \frac{3\sqrt{3} + 2\pi \ln \frac{\pi}{3}}{3\sqrt{3} - 2\pi \ln \frac{\pi}{3}} - \arctan \frac{3 + \pi \ln \frac{\pi}{6}}{3 - \pi \ln \frac{\pi}{6}}$$

Proof.

$$\frac{(1 + \ln x) \sin x + x \cos x \ln x}{\sin^2 x + x^2 \ln^2 x} = \frac{(\cos x + \ln x + 1)(\sin x + x \ln x)}{2\cos^2 x + 2x^2 \ln^2 x} -$$

$$- \frac{(\cos x + \ln x + 1)(\sin x + x \ln x)}{2\sin^2 x + 2x^2 \ln^2 x} = \frac{(\frac{\sin x + x \ln x}{\sin x - x \ln x})'}{1 + (\frac{\sin x + x \ln x}{\sin x - x \ln x})^2}$$

Hence,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1 + \ln x) \sin x + x \cos \ln x}{\sin^2 x + x^2 \ln^2 x} dx = \Big(\arctan \frac{\sin x + x \ln x}{\sin x - x \ln x} \Big) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} =$$

$$= \arctan \frac{\frac{\sqrt{3}}{2} + \frac{\pi}{3} \ln \frac{\pi}{3}}{\frac{\sqrt{3}}{2} - \frac{\pi}{3} \ln \frac{\pi}{3}} - \arctan \frac{\frac{1}{2} + \frac{\pi}{6} \ln \frac{\pi}{6}}{\frac{1}{2} - \frac{\pi}{6} \ln \frac{\pi}{6}} =$$

$$= \arctan \frac{3\sqrt{3} + 2\pi \ln \frac{\pi}{3}}{3\sqrt{3} - 2\pi \ln \frac{\pi}{3}} - \arctan \frac{3 + \pi \ln \frac{\pi}{6}}{3 - \pi \ln \frac{\pi}{6}}$$

□

Problem 4.

If $f : \mathbb{R} \to \mathbb{R}$, satisfy $f(0) = 2019$ and $3f(x) = f(x + y) + 2f(x - y) + y$ for any $x, y \in \mathbb{R}$, then:

$$\int_e^{\pi} f(x)dx = \frac{\pi - e}{2}(\pi + e + 4038)$$

Proof.

$$\text{If we take } y = x,, \text{ then } 3f(x) = f(2x) + 2f(0) + x \tag{1}$$

$$\text{If we take } y = -x, \text{ then } 3f(x) = f(0) + 2f(x) - x \tag{2}$$

From (1) and (2) we obtain that $f(x) = x + f(0)$, so $f : \mathbb{R} \to \mathbb{R}, f(x) = x + 2019$. Then:

$$\int_e^{\pi} f(x)dx = \int_e^{\pi} (x + 2019)dx = \Big(\frac{x^2}{2} + 2019x \Big) \Big|_e^{\pi} = \frac{1}{2}(\pi - e)(\pi + e) + 2019(\pi - e) =$$

$$= \frac{\pi - e}{2}(\pi + e + 4038)$$

□

Problem 5.

If $f : \mathbb{R} \to \mathbb{R}$, satisfy $f(x + 2019) \le x \le f(x) + 2019$ for any $x \in \mathbb{R}$, then

$$\int_e^{\pi} f(x^2)dx = \frac{\pi - e}{3}(\pi^2 + \pi e + e^2 - 6057)$$

Proof.

(1) We have $f(x)+2019 \geq x \Leftrightarrow f(x) \geq x-2019, \forall x \in \mathbb{R}$

Also, we have $f(x+2019) \leq x$ where we take $x+2019=y$, thus

(2) $f(y) \leq y-2019, \forall y \in \mathbb{R}$, so $f(x) \leq x-2019, \forall x \in \mathbb{R}$

From (1) and (2) we obtain $f(x)=x-2019, \forall x \in \mathbb{R}$. Therefore,

$$\int_e^{\pi} f(x^2)dx = \int_e^{\pi}(x^2-2019)dx = \Big(\frac{x^3}{3}-2019x\Big)\Big|_e^{\pi} = \frac{1}{3}(\pi^2-e^3)-2019(\pi-e) =$$

$$= \frac{\pi-e}{3}(\pi^2+\pi e+e^2-6057)$$

□

Problem 6.

If $a \in \mathbb{R}, f: \mathbb{R} \to \mathbb{R}$, is a strictly increasing function and $g: \mathbb{R} \to \mathbb{R}$ be a function which satisfy $f(g(x+a)) \leq f(x) \leq f(g(x)+a)$ for any $x \in \mathbb{R}$, then:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} g(x)dx = \frac{\pi}{12}\Big(\frac{7\pi}{24}-a\Big)$$

Proof.

(1) We have $g(x)+a \geq x \Leftrightarrow g(x) \geq x-a, \forall x \in \mathbb{R}$

Also, we have $g(x+a) \leq x$, where we take $x+a=y$, thus

(2) $g(y) \leq y-a, \forall y \in \mathbb{R}$, so $g(x) \leq x-a, \forall x \in \mathbb{R}$

From (1) and (2) we obtain $g(x)=x-a, \forall x \in \mathbb{R}$.

$$\text{Hence, } \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} g(x)dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(x-a)dx = \Big(\frac{x^2}{2}-ax\Big)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{12}\Big(\frac{7\pi}{24}-a\Big)$$

□

Problem 7.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x+1-x^2}{(1+x\sin x)\sqrt{1-x^2}}dx = 2\arcsin\frac{2(\pi+2\sqrt{2})}{8+\sqrt{2}\pi}$$

Proof. We have

$$\frac{\cos x+1-x^2}{(1+x\sin x)\sqrt{1-x^2}} = \frac{\cos^2 x+(1-x^2)\cos x}{(1+x\sin x)\sqrt{1-x^2}\cdot\cos x} =$$

$$= \frac{1+x\sin x+\cos x+x\sin x\cos x-x\sin x-x^2\cos x-\sin^2 x-x\sin x\cos x}{(1+x\sin x)\sqrt{1+2x\sin x+x^2\sin^2 x-x^2-2x\sin x-\sin^2 x}} =$$

(1) $$= \frac{(1+\cos x)(1+x\sin x)-(x+\sin x)(\sin x+x\cos x)}{\sqrt{(1+x\sin x)^2-(x+\sin x)^2}} = \frac{(\frac{x+\sin x}{1+x\sin x})'}{\sqrt{1-(\frac{x+\sin x}{1+x\sin x})^2}}$$

$$\text{Also, we have } \Big|\frac{x+\sin x}{1+x\sin x}\Big| < 1 \Leftrightarrow (x+\sin x)^2 < (1+x\sin x)^2 \Leftrightarrow$$

(2) $\Leftrightarrow x^2+2x\sin x+x^2\sin^2 x < 1+2x\sin x+x^2\sin^2 x \Leftrightarrow x^2<1 \Leftrightarrow -1<x<1$

From (1) and (2) we obtain:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x + 1 - x^2}{(1 + x\sin x)\sqrt{1-x^2}}dx = \Big(\arcsin\frac{x+\sin x}{1+x\sin x}\Big)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} =$$

$$= \arcsin\frac{\frac{\pi}{4}+\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}} - \arcsin\frac{-\frac{\pi}{4}-\sin\frac{\pi}{4}}{1-\frac{\pi}{4}(\sin(-\frac{\pi}{4}))} =$$

$$= 2\arcsin\frac{\frac{\pi}{4}+\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}} = 2\arcsin\frac{2(\pi+2\sqrt{2})}{8+\sqrt{2}\pi}$$

□

Problem 8.
If $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(0) = 2019$ and there exists $t \in (0,1)$ such that $f(x) - f(tx) = x^2 + x$ for any real x. Then,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f(x)dx = \frac{\pi^3}{648(1-t^2)} + \frac{5\pi^2}{288(1-t)} + \frac{2019\pi}{12}$$

Proof. If we take $x = t^k y, y \in \mathbb{R}, k \in \mathbb{N}$, then by $f(x) - f(tx) = x^2 + x$ we obtain that

$f(t^k y) - f(t^{k+1}y) = t^{2k}y + t^k y$. So, $\sum_{k=0}^{n-1}(f(t^k x) - f(t^{k+1}x)) = \sum_{k=0}^{n-1}(t^{2k}x^2 + t^k x) \Leftrightarrow$

$$\Leftrightarrow f(x) - f(t^n x) = x^2\cdot\frac{1-t^{2n}}{1-t^2} + x\cdot\frac{1-t^n}{1-t} \Leftrightarrow f(x) - \lim_{n\to\infty} f(t^n x) =$$

$$= x^2\lim_{n\to\infty}\frac{1-t^{2n}}{1-t^2} + x\lim_{n\to\infty}\frac{1-t^n}{1-t} \Leftrightarrow$$

$$\Leftrightarrow f(x) - f(0) = x^2\cdot\frac{1}{1-t^2} + x\cdot\frac{1}{1-t} \Leftrightarrow f(x) = x^2\cdot\frac{1}{1-t^2} + x\cdot\frac{1}{1-t} + f(0)$$

$$\text{Hence, } \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f(x)dx = \Big(\frac{1}{1-t^2}\cdot\frac{x^3}{3} + \frac{1}{1-t}\cdot\frac{x^2}{2} + 2019x\Big)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} =$$

$$= \frac{1}{3(1-t^2)}\Big(\frac{\pi^3}{4^4} - \frac{\pi^3}{6^3}\Big) + \frac{1}{2(1-t)}\Big(\frac{\pi^2}{4^2} - \frac{\pi^2}{6^2}\Big) + 2019\Big(\frac{\pi}{4} - \frac{\pi}{6}\Big) =$$

$$= \frac{\pi^3}{648(1-t^2)} + \frac{5\pi^2}{288(1-t)} + \frac{2019\pi}{12}$$

□

Problem 9.
If $f:(0,\pi)\to\mathbb{R}$ with $f'(x) = \frac{\cos 2020x}{\sin x}$ for any real $x \in (0,\pi)$, then

$$f(x) = \frac{2}{2019}\cos 2019x + \frac{2}{2017}\cos 2017x + \ldots + \frac{2}{3}\cos 3x + 2\cos x + \ln\Big|\tan\frac{x}{2}\Big| + C$$

Proof.

Let $f_n:(0,\pi)\to\mathbb{R}, f_n(x) = \int\frac{\cos nx}{\sin x}dx, \forall x \in (0,\pi)$. We have that

$f_{n+2}(x) - f_n(x) = \int\frac{\cos(n+2)x - \cos nx}{\sin x}dx = -2\int\sin(n+1)xdx = \frac{2}{n+1}\cos(n+1)x + C$, so

$$f_{k+2}(x) = f_k(x) + \frac{2}{k+1}\cos(k+1)x, \forall k \in \mathbb{N} \tag{1}$$

$$f_0(x) = \int \frac{1}{\sin x} dx = \ln\left|\tan\frac{x}{2}\right| + C \tag{2}$$

From (1) and (2) we obtain:

$$\sum_{k=0}^{2018}(f_{k+2}(x) - f_k(x)) = 2\sum_{k=0}^{2018}\frac{1}{k+1}\cos(k+1)x \Leftrightarrow f_{2020}(x) - f_0(x) =$$

$$= 2\sum_{k=0}^{2018}\frac{1}{k+1}\cos(k+1)x \Leftrightarrow$$

$$\Leftrightarrow f_{2020}(x) = f_0(x) + 2\sum_{k=0}^{2018}\frac{1}{k+1}\cos(k+1)x =$$

$$= \frac{2}{2019}\cos 2019x + \frac{2}{2017}\cos 2017x + \ldots + \frac{2}{3}\cos 3x + 2\cos x + \ln\left|\tan\frac{x}{2}\right| + C$$

Hence,

$$f(x) = \frac{2}{2019}\cos 2019x + \frac{2}{2017}\cos 2017x + \ldots + \frac{2}{3}\cos 3x + 2\cos x + \ln\left|\tan\frac{x}{2}\right| + C$$

where C is an arbitrary real constant. □

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