ABOUT AN INEQUALITY BY BOGDAN FUȘTEI –IX

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1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\geq\frac{1}{4}\left(\sum_{cyc}\frac{b+c}{a}+\sum_{cyc}\frac{m_b+m_c}{m_a}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution. **Lemma 1. 2) In ΔABC the following relationship holds:**

$$\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Proof. We have:

$$\frac{m_a}{h_a}\overset{Tereshin}{\geq}\frac{1}{h_a}\cdot\frac{b^2+c^2}{4R}=\frac{1}{h_a}\cdot\frac{1}{2}\cdot\frac{bc}{2R}\cdot\frac{b^2+c^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{h_a}\cdot\frac{1}{2}\cdot h_a\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Equality holds if and only if triangle is equilateral.

Lemma 2. 3) In ΔABC the following relationship holds:

$$\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$$

Proof. m_a, m_b, m_c can be length sides of a triangle with $F_m=\frac{3}{4}F, \overline{m_a}=\frac{3}{4}a$,

$\overline{h_a}=2\frac{\overline{F_m}}{\bar{a}}=\frac{2\cdot\frac{3}{4}F}{m_a}=\frac{3}{2}\cdot\frac{F}{m_a}$. Using **Lemma 1** in triangle with sides m_a, m_b, m_c we get,

$$\frac{\overline{m_a}}{\overline{h_a}}\overset{Lemma\ 1}{\geq}\frac{1}{2}\left(\frac{\bar{b}}{\bar{c}}+\frac{\bar{c}}{\bar{b}}\right)=\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)\Leftrightarrow\frac{\frac{3}{4}a}{\frac{3}{2}\cdot\frac{F}{m_a}}\overset{Lemma\ 1}{\geq}\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$$

$\Leftrightarrow\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$. Equality holds if and only if triangle is equilateral.

Let's get back to the main problem. Using up these **Lemmas**, it follows that,

$$2\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)\Leftrightarrow\frac{m_a}{h_a}\geq\frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{1}{4}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$$

$$\sum_{cyc}\frac{m_a}{h_a}\geq\frac{1}{4}\left(\sum_{cyc}\frac{b+c}{a}+\sum_{cyc}\frac{m_b+m_c}{m_a}\right)$$

Remark. In same class of the problem.

4) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\geq\frac{1}{2}\sum_{cyc}\frac{b+c}{a}$$

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Solution. **Lemma 1. 5) In ΔABC the following relationship holds:**

$$\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Proof. We have:

$$\frac{m_a}{h_a}\overset{Tereshin}{\geq}\frac{1}{h_a}\cdot\frac{b^2+c^2}{4R}=\frac{1}{h_a}\cdot\frac{1}{2}\cdot\frac{bc}{2R}\cdot\frac{b^2+c^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{h_a}\cdot\frac{1}{2}\cdot h_a\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Equality holds if and only if triangle is equilateral. Using **Lemma 1** and summing these inequality, it follows that:

$$\sum_{cyc}\frac{m_a}{h_a}\geq\sum_{cyc}\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\geq\frac{1}{2}\sum_{cyc}\frac{b+c}{a}$$

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\geq\frac{1}{2}\sum_{cyc}\frac{m_b+m_c}{m_a}$$

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Solution. **Lemma 1. 7) In ΔABC the following relationship holds:**

$$\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Proof. We have:

$$\frac{m_a}{h_a}\overset{Tereshin}{\geq}\frac{1}{h_a}\cdot\frac{b^2+c^2}{4R}=\frac{1}{h_a}\cdot\frac{1}{2}\cdot\frac{bc}{2R}\cdot\frac{b^2+c^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{h_a}\cdot\frac{1}{2}\cdot h_a\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Equality holds if and only if triangle is equilateral.

Lemma 2. 8) In ΔABC the following relationship holds:

$$\frac{m_a}{h_a}\geq\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$$

Proof. m_a, m_b, m_c can be length sides of a triangle with $F_m = \frac{3}{4}F, \overline{m_a} = \frac{3}{4}a$,

$\overline{h_a} = 2\frac{\overline{F_m}}{\bar{a}} = \frac{2\cdot\frac{3}{4}F}{m_a} = \frac{3}{2}\cdot\frac{F}{m_a}$. Using **Lemma 1** in triangle with sides m_a, m_b, m_c we get,

$$\frac{\overline{m_a}}{\overline{h_a}} \overset{Lemma\ 1}{\geq} \frac{1}{2}\left(\frac{\bar{b}}{\bar{c}} + \frac{\bar{c}}{\bar{b}}\right) = \frac{1}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right) \Leftrightarrow \frac{\frac{3}{4}a}{\frac{3}{2}\cdot\frac{F}{m_a}} \overset{Lemma\ 1}{\geq} \frac{1}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right)$$

$\Leftrightarrow \frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right)$. Equality holds if and only if triangle is equilateral.

Let's get back to the main problem. Using **Lemma 2** and summing, it follows that,

$$\sum_{cyc} \frac{m_a}{h_a} \geq \frac{1}{2}\sum_{cyc}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right) = \frac{1}{2}\sum_{cyc}\frac{m_b + m_c}{m_a}$$

Equality holds if and only if triangle is equilateral. **Remark.** The problem can be developed.

9) In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a}{h_a} \geq \frac{1}{2(1+\lambda)}\sum_{cyc}\frac{b+c}{a} + \frac{\lambda}{2(1+\lambda)}\sum_{cyc}\frac{m_b + m_c}{m_a}; \lambda \geq 0$$

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Solution. **Lemma 1. 10) In ΔABC the following relationship holds:**

$$\frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{b}{c} + \frac{c}{b}\right)$$

Proof. We have:

$$\frac{m_a}{h_a} \overset{Tereshin}{\geq} \frac{1}{h_a}\cdot\frac{b^2+c^2}{4R} = \frac{1}{h_a}\cdot\frac{1}{2}\cdot\frac{bc}{2R}\cdot\frac{b^2+c^2}{bc}\left(\frac{b}{c} + \frac{c}{b}\right) = \frac{1}{h_a}\cdot\frac{1}{2}\cdot h_a\left(\frac{b}{c} + \frac{c}{b}\right) = \frac{1}{2}\left(\frac{b}{c} + \frac{c}{b}\right)$$

Equality holds if and only if triangle is equilateral.

Lemma 2. 11) In ΔABC the following relationship holds:

$$\frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right)$$

Proof. m_a, m_b, m_c can be length sides of a triangle with $F_m = \frac{3}{4}F, \overline{m_a} = \frac{3}{4}a$,

$\overline{h_a} = 2\frac{\overline{F_m}}{\bar{a}} = \frac{2\cdot\frac{3}{4}F}{m_a} = \frac{3}{2}\cdot\frac{F}{m_a}$. Using **Lemma 1** in triangle with sides m_a, m_b, m_c we get,

$$\frac{\overline{m_a}}{\overline{h_a}} \overset{Lemma\ 1}{\geq} \frac{1}{2}\left(\frac{\bar{b}}{\bar{c}} + \frac{\bar{c}}{\bar{b}}\right) = \frac{1}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right) \Leftrightarrow \frac{\frac{3}{4}a}{\frac{3}{2}\cdot\frac{F}{m_a}} \overset{Lemma\ 1}{\geq} \frac{1}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right)$$

$\Leftrightarrow \frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right)$. Equality holds if and only if triangle is equilateral.

Let's get back to the main problem. Multiplying in **Lemma 1** with $x > 0$, in **Lemma 2** with $y > 0$ and adding result:

$$x\frac{m_a}{h_a} + y\frac{m_a}{h_a} \ge \frac{x}{2}\left(\frac{b}{c} + \frac{c}{b}\right) + \frac{y}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right) \Leftrightarrow$$

$$(x + y)\frac{m_a}{h_a} \ge \frac{x}{2}\left(\frac{b}{c} + \frac{c}{b}\right) + \frac{y}{2}\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right) \Leftrightarrow$$

$$\sum_{cyc}\frac{m_a}{h_a} \ge \frac{x}{2(x + y)}\sum_{cyc}\frac{b + c}{a} + \frac{y}{2(x + y)}\sum_{cyc}\frac{m_b + m_c}{m_a} \Leftrightarrow$$

$$\sum_{cyc}\frac{m_a}{h_a} \ge \frac{1}{2\left(\frac{y}{x} + 1\right)}\sum_{cyc}\frac{b + c}{a} + \frac{\frac{y}{x}}{2\left(1 + \frac{y}{x}\right)}\sum_{cyc}\frac{m_b + m_c}{m_a} \overset{\frac{y}{x}=\lambda}{=} \frac{1}{2(1 + \lambda)}\sum_{cyc}\frac{b + c}{a} + \frac{\lambda}{2(1 + \lambda)}\sum_{cyc}\frac{m_b + m_c}{m_a}; \lambda \ge 0$$

Equality holds if and only if triangle is equilateral.

Note. For $\lambda = 1$ it follows **Inequality in Triangle 2356** proposed by **Bogdan Fuștei-Romanian Mathematical Magazine-12/2020.**

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-Interactive Journal-www.ssmrmh.ro