ROMANIAN MATHEMATICAL MAGAZINE
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VECTORIAL GEOMETRY-II

COLLINEAR POINTS

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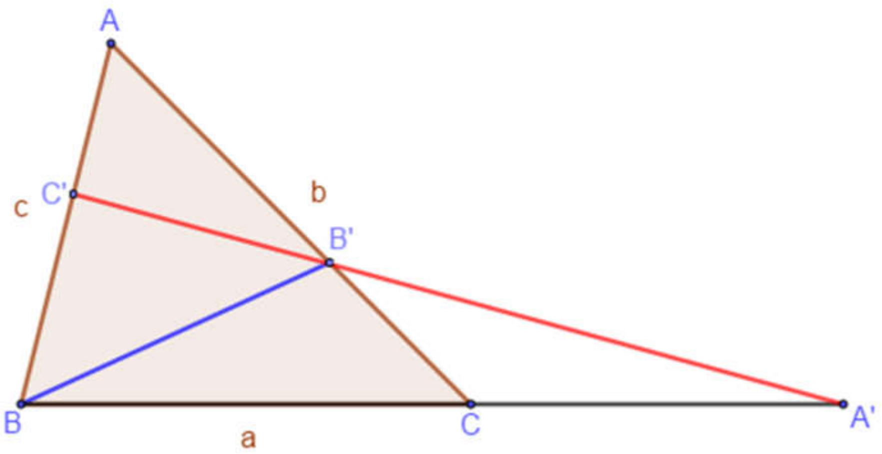
Abstract: *In this paper I was to present some applications about collinear points using vectorial geometry. This paper is dedicated to students who participate to Olympics and math competitions as well as young people passionate about geometry.*

Theorem (Menelaus)

In $\Delta ABC, A' \in BC, B' \in CA, C' \in AB$.If A', B', C' are collinear then,

$$\frac{\overline{A'B}}{\overline{A'C}} \cdot \frac{\overline{B'C}}{\overline{B'A}} \cdot \frac{\overline{C'A}}{\overline{C'B}} = 1$$

Proof.



Let us denote: $\frac{\overline{A'B}}{\overline{A'C}} = m$, $\frac{\overline{B'C}}{\overline{B'A}} = n$, $\frac{\overline{C'A}}{\overline{C'B}} = p$ then, $\overrightarrow{A'B} = m\overrightarrow{A'C}$, $\overrightarrow{B'C} = n\overrightarrow{B'A}, \overrightarrow{C'A} = p\overrightarrow{C'B}$

Now, the points A', B', C' are collinear if and only if exists $x, y \in \mathbb{R}$, with $x + y = 1$ such that

$$\overrightarrow{BB'} = x\overrightarrow{BA'} + y\overrightarrow{BC'}; (1)$$

Other, $\overrightarrow{B'C} = n\overrightarrow{B'A}$ then, $\overrightarrow{BB'} = \frac{1}{1-n}\overrightarrow{BC} - \frac{n}{1-n}\overrightarrow{BA}; (2)$.

$$\overrightarrow{BC} = \overrightarrow{BA'} + \overrightarrow{A'C} = \overrightarrow{BA'} + \frac{1}{m}\overrightarrow{A'B} = \left(1 - \frac{1}{m}\right)\overrightarrow{BA'}$$

$$\overrightarrow{BA} = \overrightarrow{BC'} + \overrightarrow{C'A} = \overrightarrow{BC'} + p\overrightarrow{C'B} = (1-p)\overrightarrow{BC'}$$

Hence, relation (2) becomes as:

$$\overrightarrow{BB'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{n(1-p)}{1-n}\overrightarrow{BC'};(3)$$

From (1),(3) it follows that:

$$x\overrightarrow{BA'}+y\overrightarrow{BC'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{n(1-p)}{1-n}\overrightarrow{BC'}$$

How, vectors $\overrightarrow{BA'}$ and $\overrightarrow{BC'}$ are not collinear, we get: $x=\frac{m-1}{m(1-n)}$, $y=-\frac{n(1-p)}{1-n}$ and because $x+y=1$ it follows that $mnp=1$. Therefore,

$$\frac{\overline{A'B}}{\overline{A'C}}\cdot\frac{\overline{B'C}}{\overline{B'A}}\cdot\frac{\overline{C'A}}{\overline{C'B}}=1$$

Theorem (Reciprocal Menelaus)

In $\Delta ABC, A'\in BC, B'\in CA, C'\in AB$. If $\frac{\overline{A'B}}{\overline{A'C}}\cdot\frac{\overline{B'C}}{\overline{B'A}}\cdot\frac{\overline{C'A}}{\overline{C'B}}=1$ then A',B',C' are collinear.

Proof.

Let us denote: $\frac{\overline{A'B}}{\overline{A'C}}=m$, $\frac{\overline{B'C}}{\overline{B'A}}=n$, $\frac{\overline{C'A}}{\overline{C'B}}=p$ then, $\overrightarrow{A'B}=m\overrightarrow{A'C}$, $\overrightarrow{B'C}=n\overrightarrow{B'A}, \overrightarrow{C'A}=p\overrightarrow{C'B}$

How $\overrightarrow{B'C}=n\overrightarrow{B'A}$ then, $\overrightarrow{BB'}=\frac{1}{1-n}\overrightarrow{BC}-\frac{n}{1-n}\overrightarrow{BA};(1)$.

$$\overrightarrow{BC}=\overrightarrow{BA'}+\overrightarrow{A'C}=\overrightarrow{BA'}+\frac{1}{m}\overrightarrow{A'B}=\left(1-\frac{1}{m}\right)\overrightarrow{BA'}$$

$$\overrightarrow{BA}=\overrightarrow{BC'}++\overrightarrow{C'A}=\overrightarrow{BC'}+p\overrightarrow{C'B}=(1-p)\overrightarrow{BC'}$$

So, (1) becomes as: $\overrightarrow{BB'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{n(1-p)}{1-n}\overrightarrow{BC'};(2)$ and how $mnp=1$, we get $p=\frac{1}{mn}$ and $\overrightarrow{BB'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{mn-1}{m(1-n)}\overrightarrow{BC'};(3)$.

If $x=\frac{m-1}{m(1-n)}$, $y=-\frac{mn-1}{m(1-n)}$ then $x+y=\frac{m-1}{m(1-n)}-\frac{mn-1}{m(1-n)}=\frac{m(1-n)}{m(1-n)}=1$.

So, exists $x,y\in\mathbb{R}$, with $x+y=1$ such that $\overrightarrow{BB'}=x\overrightarrow{BA'}+y\overrightarrow{BC'}$ and then the points A',B',C' are collinear.

Application 1.

In $ABCD$ parallelogram, poits E,F are such that $2\overrightarrow{BE}=\overrightarrow{AB}$ and $\overrightarrow{AF}=3\overrightarrow{AD}$. Prove that E,F and C are collinear.

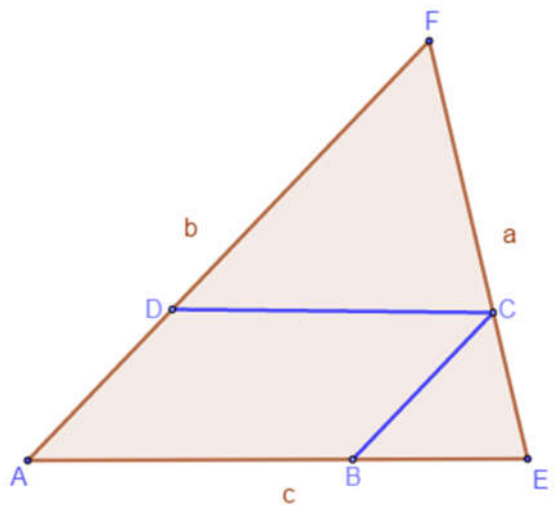
Solution.

How $2\overrightarrow{BE}=\overrightarrow{AB}$ and $\overrightarrow{AF}=3\overrightarrow{AD}$ then $\overrightarrow{CE}=\overrightarrow{CB}+\overrightarrow{BE}=\overrightarrow{DA}+\overrightarrow{BE}=-\frac{1}{3}\overrightarrow{AF}+\frac{1}{2}\overrightarrow{AB}$. Hence,

$$\overrightarrow{CE}=-\frac{1}{3}\overrightarrow{AF}+\frac{1}{2}\overrightarrow{AB};(1)$$

Other, $\overrightarrow{FC}=\overrightarrow{FA}+\overrightarrow{AC}=\overrightarrow{FA}+\overrightarrow{AD}+\overrightarrow{AB}=-\overrightarrow{AF}+\frac{1}{3}\overrightarrow{AF}+\overrightarrow{AB}=-\frac{2}{3}\overrightarrow{AF}+\overrightarrow{AB}$.

Thus, $\overrightarrow{FC}=-\frac{2}{3}\overrightarrow{AF}+\overrightarrow{AB}$; (2). From (1),(2) we have $\overrightarrow{FC}=2\overrightarrow{CE}$ and then, the points E,F and C are collinear.



Application 2.

In $\Delta ABC, E\in AB, F\in AC$ such that $EF\parallel BC, M\in EF, N\in BC$ such that

$$\frac{ME}{MF}=\frac{NB}{NC}=\lambda,\lambda>0$$

Prove that M,N and A are collinear.

Solution.

How $\frac{ME}{MF}=\frac{NB}{NC}=\lambda,\lambda>0$, we have:

$$\overrightarrow{ME}=-\lambda\overrightarrow{MF},\qquad \overrightarrow{NB}=-\lambda\overrightarrow{NC}\Rightarrow$$

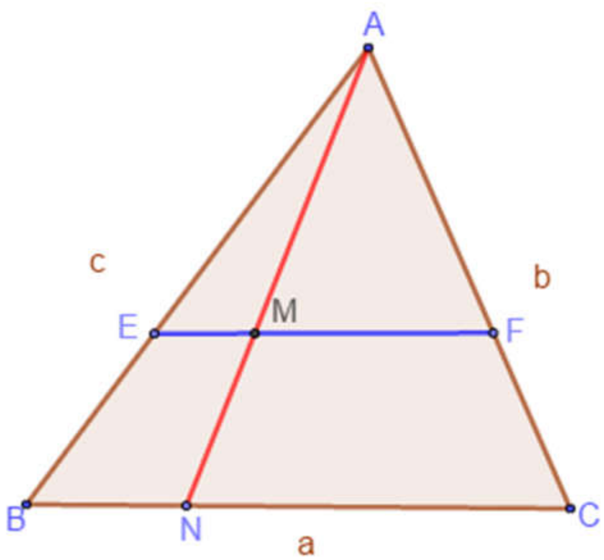
$$\overrightarrow{AM}=\frac{1}{1+\lambda}\overrightarrow{AE}+\frac{\lambda}{1+\lambda}\overrightarrow{AF};(1)$$

$$\overrightarrow{AN}=\frac{1}{1+\lambda}\overrightarrow{AB}+\frac{\lambda}{1+\lambda}\overrightarrow{AC};(2)$$

But $\Delta AEF\sim\Delta ABC$ then, $\frac{AE}{AB}=\frac{AF}{AC}=k$.

Thus, $\overrightarrow{AE}=k\overrightarrow{AB},\overrightarrow{AF}=k\overrightarrow{AC};(3)$.

From (2),(3) relation (1) becomes as:



$$\overrightarrow{AM}=\frac{k}{1+\lambda}\overrightarrow{AB}+\frac{\lambda k}{1+\lambda}\overrightarrow{AC}=k\left(\frac{1}{1+\lambda}\overrightarrow{AB}+\frac{\lambda}{1+\lambda}\overrightarrow{AC}\right)=k\overrightarrow{AN}$$

Therefore, A,M and N are collinear.

Application 3.

In $\Delta ABC, BF, CE$ −symmedians from B and C respectively. If points E, F and I are collinear if and only if $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$.

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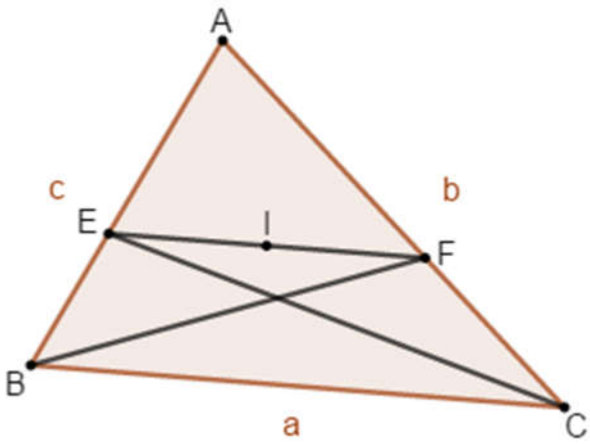
Solution.

From transversals theorem:

$$I \in EF \Leftrightarrow b \cdot \frac{EB}{EA} + c \cdot \frac{FC}{FA} = a; (1)$$



But, from Steiner;s theorem, we have:

$$\begin{cases} \frac{EB}{EA} = \frac{a^2}{b^2} \\ \frac{FC}{FA} = \frac{a^2}{c^2} \end{cases}; (2)$$

From (1),(2) it follows that: $b \cdot \frac{a^2}{b^2} + c \cdot \frac{a^2}{c^2} = a$

$$\Leftrightarrow \frac{a^2}{b} + \frac{a^2}{c} = a \Leftrightarrow \frac{a}{b} + \frac{a}{c} = 1 \Leftrightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a}$$

Application 3.

In $\Delta ABC, I \in Int(\Delta ABC)$. Prove that I −incentre if and only if $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \overrightarrow{0}$.

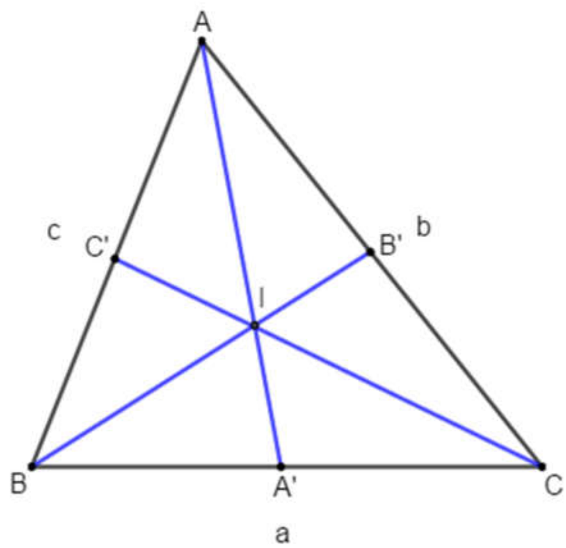
Solution.

Let $A' \in (BC), B' \in (CA), C' \in (AB)$.

Applying bisector theorem, we get:

$$\overrightarrow{BA'} = \frac{c}{b+c}\overrightarrow{BC},$$

$$\overrightarrow{AB'} = \frac{c}{a+c}\overrightarrow{AC}.$$



Thus,

$$\overrightarrow{AA'} = \frac{b}{b+c}\overrightarrow{AB} + \frac{c}{b+c}\overrightarrow{AC}$$

$$\overrightarrow{BB'} = -\overrightarrow{AB} + \frac{c}{a+c}\overrightarrow{AC}$$

How $I \in (AA')$, then exist $x \in (0,1)$ such that

$\overrightarrow{AI}=x\overrightarrow{AA'}$. It follows that: $\overrightarrow{BI}=\left(\frac{xb}{b+c}-1\right)\overrightarrow{AB}+\frac{xc}{b+c}\overrightarrow{AC}$.

How $\overrightarrow{BI}$ and $\overrightarrow{BB'}$ are collinear, we get: $\frac{\frac{xb}{b+c}-1}{-1}=\frac{\frac{xc}{b+c}}{\frac{c}{a+c}}$ and then $x=\frac{b+c}{a+b+c}$.

So, we have: $\overrightarrow{AI}=\frac{b}{a+b+c}\overrightarrow{AB}+\frac{c}{a+b+c}\overrightarrow{AC}$ $(and\ analogs)$. Adding, it follows that:

$$a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=(a+b+c)\overrightarrow{IA}+b\overrightarrow{AB}+c\overrightarrow{AC}=\left(-b\overrightarrow{AB}-c\overrightarrow{AC}\right)+b\overrightarrow{AB}+c\overrightarrow{AC}=0.$$

Therefore, I –incenter.

Reverse, let $I'\in Int(\Delta ABC)$ who verify relation $a\overrightarrow{I'A}+b\overrightarrow{I'B}+c\overrightarrow{I'C}=\vec{0}$ and from $a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=\vec{0}$, we obtain: $(a+b+c)\overrightarrow{II'}=\vec{0}$ and know that $a+b+c\neq 0$, it follows that $I=I'$.

Application 4.

In ΔABC, BF, CE –symmedians from B and C respectively. If points E, F and O are collinear if and only if $\cot B+\cot C=\cot A$.

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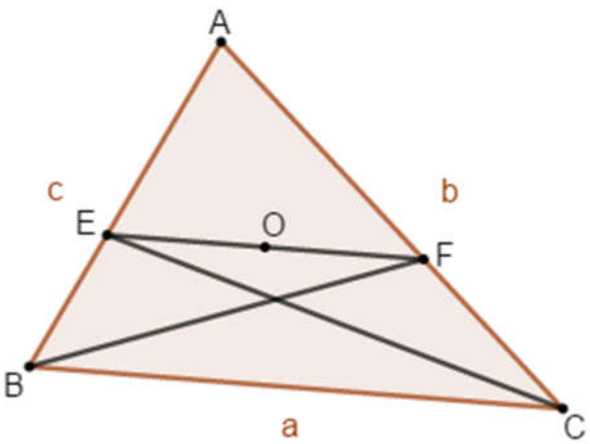
Solution.

From transversals theorem: $O\in EF\Leftrightarrow$

$\frac{EB}{EA}\cdot\sin 2B+\frac{FC}{FA}\cdot\sin 2C=\sin 2A$; (1)

From Steiner's theorem, we have:

$$\begin{cases}\frac{EB}{EA}=\frac{a^2}{b^2}\\ \frac{FC}{FA}=\frac{a^2}{c^2}\end{cases};(2)$$



From (1),(2) it follows that: $O\in EF\Leftrightarrow$

$$\frac{a^2}{b^2}\cdot\sin 2B+\frac{a^2}{c^2}\cdot\sin 2C=\sin 2A\Leftrightarrow$$

$$\frac{\sin^2 A}{\sin^2 B}\cdot 2\sin B\cos B+\frac{\sin^2 A}{\sin^2 C}\cdot 2\sin C\cos C=2\sin A\cos A\Leftrightarrow$$

$$\frac{\cos B}{\sin B}\cdot\sin A+\frac{\cos C}{\sin C}\cdot\sin A=\cos A\Leftrightarrow\cot B+\cot C=\cot A.$$

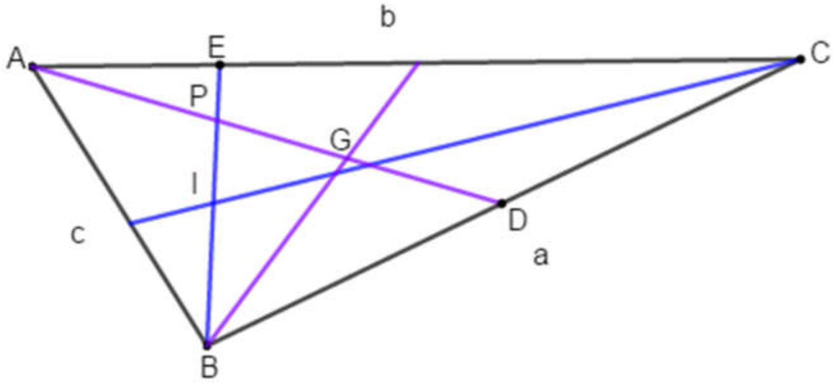
Application 5.

In $\Delta ABC, D$ –middle point of $(BC), G$ –centroid, BE –internal bisector, $\{P\} = AD \cap BE$.

Prove that $\overrightarrow{PG} = \overrightarrow{GD}$ if and only if $|\overrightarrow{BC}| = 4|\overrightarrow{AB}|$.

Solution.

Let us denote:



$$\alpha = \frac{AB}{BC} = \frac{AE}{EC}, \beta = \frac{AP}{PD}$$

We have:

$$\overrightarrow{BE} = \frac{\overrightarrow{BA} + \alpha\overrightarrow{BC}}{1+\alpha} =$$

$$= \frac{1}{1+\alpha}\overrightarrow{BA} + \frac{\alpha}{1+\alpha}\overrightarrow{BC}$$

$$\overrightarrow{BP} = \frac{\overrightarrow{BA} + \beta\overrightarrow{BD}}{1+\beta} = \frac{1}{1+\beta}\overrightarrow{BA} + \frac{1}{2}\cdot\frac{\beta}{1+\beta}\overrightarrow{BC}$$

How $\overrightarrow{BP}$ and $\overrightarrow{BE}$ are collinear, then $\frac{\frac{1}{1+\alpha}}{\frac{1}{1+\beta}} = \frac{\frac{\alpha}{1+\alpha}}{\frac{\beta}{1+\beta}}$. Therefore, $\beta = 2\alpha \Leftrightarrow \overrightarrow{BC} = 4\overrightarrow{AB}$.

Application 6.

In $\Delta ABC, AD$ –internal bisector and $M \in AB, N \in AC$.

a) ***Find $y, z \in \mathbb{R}$ such that $\overrightarrow{AD} = y \cdot \overrightarrow{AB} + z \cdot \overrightarrow{AC}$.***
b) ***If $P_i \in (ABC)$ and $(x_i, y_i, z_i) \in \mathbb{R}^3 i = \overline{1,3}$ such that $x_i + y_i + z_i = 1, \forall i = \overline{1,3}$ and $\overrightarrow{OP_i} = x_i \cdot \overrightarrow{OA} + y_i \cdot \overrightarrow{OB} + z_i \cdot \overrightarrow{OC}, \forall O \in (ABC)$, then P_1, P_2, P_3 are collinear if and only if exists $u, v, w \in \mathbb{R}$ with property $ux_i + vy_i + wz_i = 0, \forall i = \overline{1,3}$.***
c) ***Prove that the points M, N, D are collinear if and only if $b \cdot \frac{BM}{AM} + c \cdot \frac{CN}{AN} = \frac{a^2}{b+c}$.***

Solution.

a) It is easy to prove that D middle point of $[II_a]$; $(usual\ notations)$ then,

$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AI} + \frac{1}{2}\overrightarrow{AI_a}$. We know the following relations:

$$\overrightarrow{AI} = \frac{b}{a+b+c}\cdot\overrightarrow{AB} + \frac{c}{a+b+c}\cdot\overrightarrow{AC};\ \ \overrightarrow{AI_a} = \frac{b}{-a+b+c}\cdot\overrightarrow{AB} + \frac{c}{-a+b+c}\cdot\overrightarrow{AC}$$

Hence,

$$\overrightarrow{AD} = \frac{b}{-a^2 + (b + c)^2} \cdot \overrightarrow{AB} + \frac{c}{-a^2 + (b + c)^2} \cdot \overrightarrow{AC}$$

Therefore, $y = \frac{b}{-a^2+(b+c)^2}$ and $z = \frac{c}{-a^2+(b+c)^2}$.

b) Let the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ then P_1, P_2, P_3 are collinear if and only if

$$\frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} = \frac{z_1 - z_2}{z_1 - z_3}; (1)$$

For "$\Rightarrow$" , we get $u = y_1z_2 - y_2z_1, v = z_1x_2 - z_2x_1, w = x_1y_2 - x_2y_1$.

For "$\Leftarrow$", if $ux_i + vy_i + wz_i = 0, \forall i = \overline{1,3}$ then $\frac{x_1-x_2}{y_1-y_2} = -\frac{v-w}{u-w} = \frac{x_1-x_3}{y_1-y_3}$ hence,

$$\frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} = \frac{z_1 - z_2}{z_1 - z_3}$$

c) Let $M(x_m, y_M, 0)$ hence, $\frac{x_M}{y_M} = \frac{\overrightarrow{BM}}{\overrightarrow{MA}}$ and for $N(x_N, 0, y_N)$ we have $\frac{x_N}{z_N} = \frac{\overrightarrow{CN}}{\overrightarrow{NA}}$.

$d_{MN}: ux + vy + wz = 0$, then $\begin{cases} \frac{v}{u} = -\frac{x_M}{y_M} = \frac{\overrightarrow{BM}}{\overrightarrow{AM}} \\ \frac{w}{u} = -\frac{x_n}{y_n} = \frac{\overrightarrow{CN}}{\overrightarrow{AN}} \end{cases}$ and using point a) it follows that

$$D\left(\frac{-a^2}{-a^2 + (b + c)^2}, \frac{b(b + c)}{-a^2 + (b + c)^2}, \frac{c(b + c)}{-a^2 + (b + c)^2}\right)$$

So, $D \in MN$ if and only if $b \cdot \frac{\overrightarrow{BM}}{\overrightarrow{AM}} + c \cdot \frac{\overrightarrow{CN}}{\overrightarrow{AN}} = \frac{a^2}{b+c}$.

Application 7.

In $\Delta ABC, N$ –Nagel's point, BF, CE –symmedians from B and C respectively. Prove that the points E, F and N are collinear if and only if $\frac{1}{b^2r_b} + \frac{1}{c^2r_c} += \frac{1}{a^2r_a}$.

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Solution.

From transversal's theorem: $\frac{PB}{PA} \cdot (s - b) + \frac{QC}{QA} \cdot (s - c) = s - a; (1)$

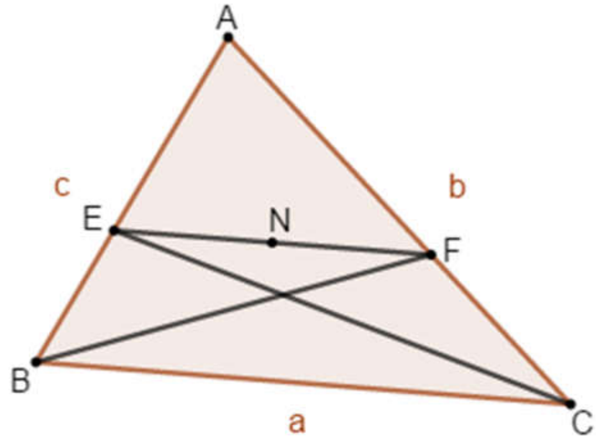
From Steiner's theorem, we have: $\begin{cases} \frac{PB}{PA} = \left(\frac{BC}{AC}\right)^2 = \frac{a^2}{b^2} \\ \frac{QC}{QA} = \left(\frac{BC}{AB}\right)^2 = \frac{a^2}{c^2} \end{cases}; (2)$

From (1),(2) it follows that: $\frac{a^2}{b^2} \cdot (s - b) + \frac{a^2}{c^2} \cdot (s - c) = s - a$

$$\frac{s-b}{b^2}+\frac{s-c}{c^2}=\frac{s-a}{a^2}$$

But, $r_a=\frac{F}{s-a}\Rightarrow s-a=\frac{F}{r_a}\Rightarrow$

$$\frac{1}{b^2r_b}+\frac{1}{c^2r_c}=\frac{1}{a^2r_a}$$



Application 8.

In ΔABC, BE, CF −internal bisectors and O −circumcenter. Prove that the points E, O and F are collinear if and only if $\cos A=\cos B+\cos C$.

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Solution.

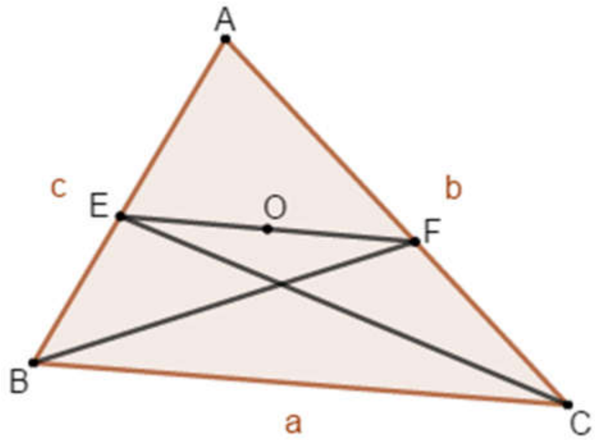
Applying transversals theorem, we have:

$O\in EF\Leftrightarrow$

$$\frac{EB}{EA}\cdot\sin 2B+\frac{FC}{FA}\cdot\sin 2C=\sin 2A;(1)$$

From bisector theorem, we have:

$$\begin{cases}\frac{FB}{FA}=\frac{a}{b}\\ \frac{EC}{EA}=\frac{a}{c}\end{cases};(2)$$



From (1),(2) it follows that: $\frac{a}{b}\cdot\sin 2B+\frac{a}{c}\cdot\sin 2C=\sin 2A\Leftrightarrow$

$$2\sin B\cos B\cdot\frac{\sin A}{\sin B}+2\sin C\cos C\cdot\frac{\sin A}{\sin C}=2\sin A\cos A\Leftrightarrow$$

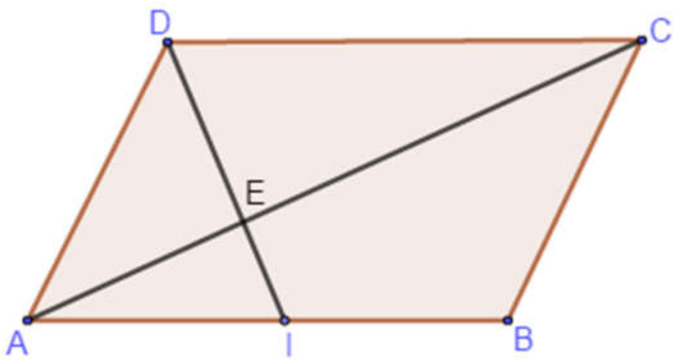
$$2\sin A\cos B+2\sin A\cos C=2\sin A\cos A\Leftrightarrow$$

$$\cos B+\cos C=\cos A$$

Application 9.

In $ABCD$ parallelogram, I –middle point of AB and $E \in ID$ such that $3\overrightarrow{IE} = \overrightarrow{ID}$. Prove that the points A, E and C are collinear.

Solution.



Because $3\overrightarrow{IE} = \overrightarrow{ID} \Rightarrow \frac{IE}{ED} = \frac{1}{2} \Rightarrow$

$$\overrightarrow{ED} = -2\overrightarrow{EI}$$

$$\overrightarrow{AE} = \frac{1}{1—2}\overrightarrow{AD}—21—2\overrightarrow{AI} =$$

$$= \frac{1}{3}\overrightarrow{AD} + \frac{2}{3}\overrightarrow{AI}$$

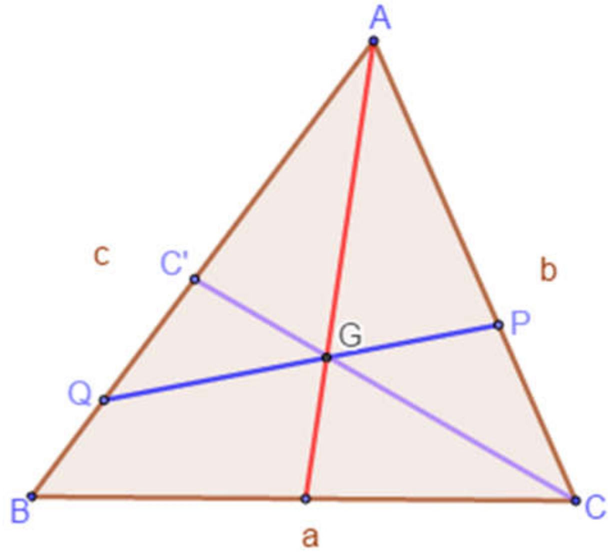
How $2\overrightarrow{AI} = \overrightarrow{AB}$ it follows that:

$\overrightarrow{AE} = \frac{1}{3}\overrightarrow{AD} + \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\overrightarrow{AD} + \overrightarrow{AB}) = \frac{1}{3}\overrightarrow{AC} \Rightarrow A, E$ and C are collinear.

Application 10.

In ΔABC, G –centroid and $P \in AC, Q \in BC$ such that $\frac{CP}{PA} + \frac{BQ}{QA} = 1$. Then prove that the points P, Q and G are collinear.

Solution.



Let us denote $\frac{CP}{PA} = m,\ \frac{BQ}{QA} = n$ and let C' middle point of AB. Because $\overrightarrow{GC} = -2\overrightarrow{GC'}$ we have:

$$\overrightarrow{AG} = \frac{1}{1—2}\overrightarrow{AC} + \frac{-2}{1—2}\overrightarrow{AC'} =$$

$$= \frac{1}{3}\overrightarrow{AC} + \frac{2}{3}\overrightarrow{AC'}; (1)$$

From $\frac{CP}{PA} = m,\ \frac{BQ}{QA} = n$ it follows that

$$\overrightarrow{CP} = m\overrightarrow{PA}, \qquad \overrightarrow{BQ} = n\overrightarrow{QA} \Rightarrow$$

$\overrightarrow{AC} = (m+1)\overrightarrow{AP},\ \overrightarrow{AC'} = \frac{n+1}{2}\overrightarrow{AQ}$ and relation (1) becomes as:

$$\overrightarrow{AG} = \frac{m+1}{3}\overrightarrow{AP} + \frac{n+1}{3}\overrightarrow{AQ}; (2)$$

Let $x = \frac{m+1}{3}, y = \frac{n+1}{3}$ and from $m + n = 1$ we get $x + y = 1$. So, exists $x, y \in \mathbb{R}$ such that $x + y = 1$ and $\overrightarrow{AG} = x\overrightarrow{AP} + y\overrightarrow{AQ}$, namely the points P, Q and G are collinear.

Application 11.

In ΔABC, G –centroid and $M \in AB, N \in AC$ such that $\frac{MB}{MA} + \frac{NC}{NA} = k$. Prove that the points M, N and G are collinear if and only if $k = 1$.

Solution.

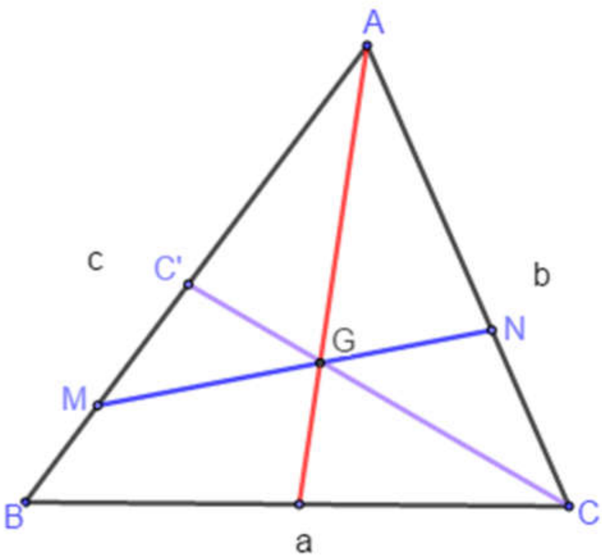
Let us denote $\frac{MB}{MA} = \alpha, \frac{NC}{NA} = \beta$ then,

$\overrightarrow{MA} = -\frac{1}{\alpha+1}\overrightarrow{AB},\ \overrightarrow{AN} = \frac{1}{\beta+1}\overrightarrow{AC}$. We have:

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = -\frac{1}{\alpha+1}\overrightarrow{AB} + \frac{1}{\beta+1}\overrightarrow{AC}$$

$$\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{AG} = -\frac{1}{\alpha+1}\overrightarrow{AB} + \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC})$$

$$= \left(\frac{1}{3} - \frac{1}{\alpha+1}\right)\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$$



We observe that the vectors $\overrightarrow{MG}$ and $\overrightarrow{MN}$ have same direction then,

$$\frac{\frac{1}{3} - \frac{1}{\alpha+1}}{-\frac{1}{\alpha+1}} = \frac{\frac{1}{3}}{\frac{1}{\beta+1}} \Leftrightarrow \alpha + \beta = 1$$

Application 12.

In plane of ΔABC let be the points D, M, S and T such that $5\overrightarrow{AT} = 3\overrightarrow{AB}, 2\overrightarrow{SA} + \overrightarrow{SC} = \vec{0}$, $35\overrightarrow{AD} = 18\overrightarrow{AB}$ and $34\overrightarrow{MA} + 36\overrightarrow{MB} + 5\overrightarrow{MC} = \vec{0}$.

a) Find $x, y \in \mathbb{R}$ such that $x\overrightarrow{MT} + y\overrightarrow{MS} = \vec{0}$.

b) Prove that the points C, M and D are collinear.

Solution.

a) From $5\overrightarrow{AT} = 3\overrightarrow{AB} \Rightarrow \overrightarrow{AT} = \frac{3}{2}\overrightarrow{TB} \Rightarrow \overrightarrow{MT} = \frac{2}{5}\overrightarrow{MA} + \frac{3}{5}\overrightarrow{MB}$ and from $\overrightarrow{AS} = \frac{1}{2}\overrightarrow{SC}$ we get:

$$\overrightarrow{MS} = \frac{2}{3}\overrightarrow{MA} + \frac{1}{3}\overrightarrow{MC} = \frac{2}{3}\overrightarrow{MA} + \frac{1}{3}\left(-\frac{34}{5}\overrightarrow{MA} - \frac{36}{5}\overrightarrow{MB}\right) =$$

$$= -\frac{8}{5}\overrightarrow{MA} - \frac{12}{5}\overrightarrow{MB} = -4\overrightarrow{MT}$$

So, $\overrightarrow{MS} + 4\overrightarrow{MT} = \vec{0}$ and we can choose $x = 4, y = 1$.

b) $\overrightarrow{AD} = \frac{18}{35}\overrightarrow{AB} \Rightarrow \overrightarrow{AD} = \frac{18}{17}\overrightarrow{DB} \Rightarrow \overrightarrow{MD} = \frac{17}{35}\overrightarrow{MA} + \frac{18}{35}\overrightarrow{MB}$ and from

$34\overrightarrow{MA} + 36\overrightarrow{MB} + 5\overrightarrow{MC} = \vec{0}$ we get $-\frac{1}{14}\overrightarrow{MC} = \frac{17}{35}\overrightarrow{MA} + \frac{18}{35}\overrightarrow{MB}$.

So, it follows $\overrightarrow{MD} = -\frac{1}{14}\overrightarrow{MC}$ and then, the points M, D, C are collinear.

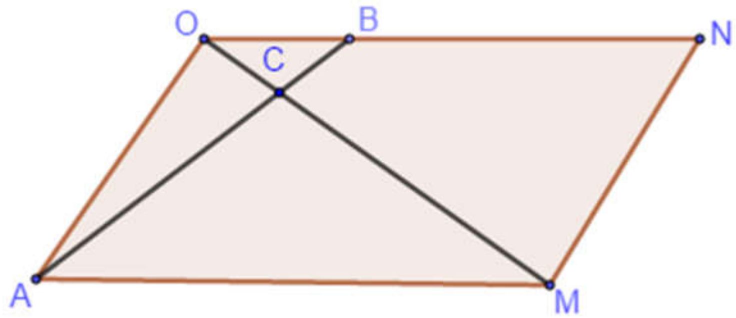
Application 13.

In $AMNO$ parallelogram the points B, C are such that $\overrightarrow{OB} = \frac{1}{n}\overrightarrow{ON}, \overrightarrow{OC} = \frac{1}{n+1}\overrightarrow{OM}$, where $n \in \mathbb{N}^*, n \geq 2$. Prove that the points A, B, C are collinear.

Solution.

We must to prove that exist $\alpha \in \mathbb{R}$ such that $\overrightarrow{AC} = \alpha\overrightarrow{AB}$.

How, $\overrightarrow{OM} = (n+1)\overrightarrow{OC}$ we have $\overrightarrow{CM} = -n\overrightarrow{CO}$.



It follows that:

$$\overrightarrow{AC} = \frac{1}{1—n}\overrightarrow{AM}—n1—n\overrightarrow{AO} = \frac{1}{n+1}\overrightarrow{AM} + \frac{n}{n+1}\overrightarrow{AO}$$

Because $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{AO} + \frac{1}{n}\overrightarrow{ON} = \overrightarrow{AO} + \frac{1}{n}\overrightarrow{AM}$ then, $\overrightarrow{AC} = \alpha\overrightarrow{AB} \Leftrightarrow$

$$\frac{n}{n+1}\overrightarrow{AO} + \frac{1}{n+1}\overrightarrow{AM} = \alpha\overrightarrow{AO} + \frac{\alpha}{n}\overrightarrow{AM}$$

How the vectors $\overrightarrow{AO}$ and $\overrightarrow{AM}$ are not collinear, we have $\frac{n}{n+1} = \alpha, \frac{1}{n+1} = \frac{\alpha}{n}$.

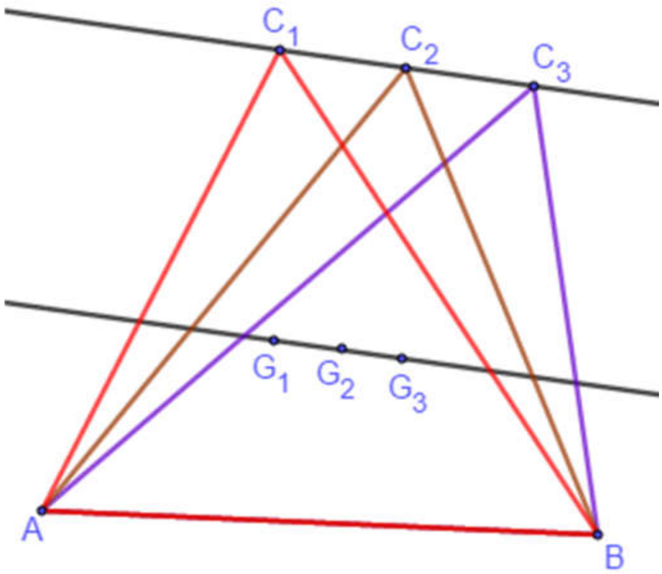
So, $\alpha = \frac{n}{n+1}, \overrightarrow{AC} = \frac{n}{n+1}\overrightarrow{AB}$ and then the points A, B, C are collinear.

Application.

In $\Delta ABC_1, \Delta ABC_2, \Delta ABC_3, G_1, G_2, G_3$ —centroids. Prove that the points G_1, G_2, G_3 are collinear if and only if the points C_1, C_2, C_3 are collinear.

Solution.

Let O in plane of that triangles.



From Leinbiz relation, we have:

$$\overrightarrow{OG_1}=\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)$$

$$\overrightarrow{OG_2}=\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC_2}\right)$$

$$\overrightarrow{OG_3}=\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC_3}\right)$$

It follows that $\overrightarrow{G_1G_2}=\overrightarrow{OG_2}-\overrightarrow{OG_1}=$ $\frac{1}{3}\left(\overrightarrow{OC_2}-\overrightarrow{OC_1}\right)=\frac{1}{3}\overrightarrow{C_1C_2}$ and similarly, $\overrightarrow{G_1G_3}=\frac{1}{3}\overrightarrow{C_1C_3}$.

The points G_1, G_2, G_3 are collinear if and only if exist $\alpha\in\mathbb{R}$ such that $\overrightarrow{G_1G_2}=\alpha\overrightarrow{G_1G_3}\Leftrightarrow$ $\frac{1}{3}\overrightarrow{C_1C_2}=\frac{\alpha}{3}\overrightarrow{C_1C_3}\Leftrightarrow\overrightarrow{C_1C_2}=\alpha\overrightarrow{C_1C_3}\Leftrightarrow C_1, C_2, C_3$ are collinear.

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